Math 151 - Exam 1A - solutions

Problem 1 [10 pts] Solve for $x$: $4^{x+3} = \frac{1}{16}$.
Solution Take logarithms with base 4 to get $x + 3 = -2$, so $x = -5$.

Problem 2 [10 pts] Suppose $\log_b(Y) = 4$ and $\log_b(Z) = -5$. Use properties of logarithms to find the value of
$$\log_b(\sqrt[3]{YZ}).$$
Solution
$$\log_b(\sqrt[3]{YZ}) = \frac{1}{3} \log_b(YZ) = \frac{1}{3}(\log_b Y + \log_b Z) = -\frac{1}{3}.$$

Problem 3 [12 pts] Find all solutions: $\log_2(t^2 - 2t) = 3$.
Solution Take exponentials with base 2 to get $t^2 - 2t = 2^3 = 8$, so $t^2 - 2t - 8 = 0$. This gives the solutions $t = -2, t = 4$.

Problem 4 [10 pts] Find the interest earned on $60,000 invested for 5 years at 4.8% interest, compounded quarterly. Round to the nearest cent.
Solution The whole amount you have in the bank is
$$60,000 \cdot 1.012^{20} \approx 76166.06.$$ For the interest earned, subtract the capital. So the answer is
$$76166.06 - 60,000 = 16166.06.$$

Problem 5 [15 pts] Consider the function $f(x) = 3x^2 - 2x + 4$.
a) Find the average rate of change $f_{2,4}$ for $f$ between 2 and 4.
b) Write $f_{2,b}$ for the average rate of change for $f$ between 2 and $b$. Find $\lim_{b \to 2} f_{2,b}$.
Solution a) We use $f(2) = 12$ and $f(4) = 44$. So
$$f_{2,4} = \frac{44 - 12}{4 - 2} = 16.$$
b) First,

\[ f_{2,b} = \frac{f(b) - f(2)}{b - 2} = \frac{3b^2 - 2b - 8}{b - 2}. \]

Factor the numerator, cancel out \( b - 2 \), and you get that the instantaneous rate of change is

\[ \lim_{b \to 2} \frac{3b^2 - 2b - 8}{b - 2} = \lim_{b \to 2} \frac{3b + 4}{1} = 10. \]

**Problem 6** [10 pts] Find \( \lim_{z \to \infty} \frac{5z^2 + 2z - 1}{3z + 1} \).

**Solution** The degree of the numerator (= 2) is higher than that of the denominator (= 1), so the limit is \( \infty \). Alternatively,

\[ \lim_{z \to \infty} \frac{5z^2 + 2z - 1}{3z + 1} = \lim_{z \to \infty} \frac{5z + 2 - 1/z}{3 + 1/z} = \lim_{z \to \infty} \frac{5z}{3} = \infty. \]

**Problem 7** [8 pts] Let \( f(x) = \begin{cases} 
3x - 5 & \text{if } x < 2 \\
2x & \text{if } 2 \leq x < 6 \\
18 - x & \text{if } 6 \leq x 
\end{cases} \)

Find the following limits. Write DNE for a limit that does not exist.

a) \( \lim_{x \to 2^-} f(x) = 1 \)
b) \( \lim_{x \to 2^+} f(x) = 4 \)

c) \( \lim_{x \to 2^2} f(x) = \text{DNE} \)

d) Find all points \( x \) where \( f(x) \) is not continuous.

**Solution** Plotting this function may help to see what is going on, see above.

d) \( f(x) \) is continuous at all points \( x \) except at \( x = 2 \). Note that for \( x = 6 \), both one-sided limits agree with \( f(6) = 12 \), so \( f(x) \) is continuous there. All other \( x \)-values besides 2 and 6 are using only one (polynomial) formula for \( f(x) \), so \( f(x) \) is continuous there.

**Problem 8** [15 pts] Blood alcohol content \( A(t) \) can be modeled with an exponential law,

\[
A(t) = Ce^{kt}.
\]

Suppose \( A(0) = 10 \) (in mg/L) at time \( t = 0 \) hours. Two hours after \( t = 0 \), \( A(t) \) has decreased to 6 mg/L.

a) Find the values of the constants \( C \) and \( k \). Give exact answers (no decimal fractions).

b) When will the blood alcohol content be 2 mg/L if \( A(t) \) continues to follow this law? Round your answer to one digit after the decimal point.

**Solution** a) We get \( C = 10 \) from \( t = 0 \). Now substitute \( t = 2 \) and \( A(2) = 6 \). Solve this for \( k \) by taking logarithms to get

\[
k = \frac{\ln(6/10)}{2} = \frac{\ln 0.6}{2}.
\]

b) We need to use the values for \( C \) and \( k \) from part a) and solve \( A(t) = 2 \) for \( t \). This gives

\[
2 = 10e^{kt}
\]

so after dividing by 10, we take logarithms to get \( \ln 0.2 = kt \). Divide by \( k \) to get

\[
t = \frac{\ln 0.2}{k} = \frac{2 \ln 0.2}{\ln 0.6} \approx 6.3
\]

(in hours).

**Problem 9** [10 pts] Consider the function

\[
P(t) = 4^t.
\]

Approximate the instantaneous rate of change of \( P(t) \) at \( t = 2 \). Round the answer to four digits after the decimal point.
Solution  Here is a table of values of average rates of change $P_{2,t}$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P_{2,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.999</td>
<td>22.16534238353703</td>
</tr>
<tr>
<td>1.9999</td>
<td>22.179172399307937</td>
</tr>
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<tr>
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</tr>
<tr>
<td>2.01</td>
<td>22.196091381352147</td>
</tr>
</tbody>
</table>

The $t$-values closest to 2 give the best estimate of the limit of the average rates of change, so we estimate

$$P'(2) \approx 22.1807.$$  

The other values of $t$ tell us that this is not affected by roundoff errors too much.