Math 181

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Regression

y_i $(i = 1, 2, 3, \cdots, n)$ are the given experimental data, and the corresponding curve may be approximated by function characterized based on (x_i, y_i). Assume the function, which represents the curve of (x_i, y_i) precisely, is $f(x) = y$. The approximation function is $y = \phi(x)$. Due to the observation error, the equality $s_i = \phi(x_i) - f(x_i) = 0$ does not have to be true. The reachable target is minimize the error, which is defined as

$$||\delta||_2^2 = \sum_{i=1}^{n} |\phi(x_i) - y_i|^2.$$

The problem

$$\min ||\delta||_2^2$$

is called **at least square** problem (data fitting).

Generally, one choose $\phi(x) \in \Phi = span\{\phi_1, \phi_2, \cdots, \phi_n\}$, where all of the $\phi_i(x)$ are base functions. To figure out the best approximation is equivalent to minimize the multivalue function

$$F(a_1, a_2, \cdots, a_n) = \sum \Big\{ \sum a_j \phi_j(x) - f(x) \Big\}^2 \rho_i$$

where ρ_i is some weight and usually set to equal 1 in our case. To have the minimizer, we need use higher dimensional derivative theory, which will be covered in later chapters. The required normal equation is

$$\sum (\phi_j, \phi)k) a_j = (f, \phi_k), \ k = 1, 2, \cdots, n,$$

where $(\cdot, \cdot)$ is inner product defined as

$$\begin{cases} (\phi_j, \phi_k) = \sum \phi_j(x_i)\phi_k(x_i) \\ (f, \phi_k) = \sum f(x_i)\phi_k(x_i) \end{cases}.$$

Remark 0.1. To have the above equation solvable, one has to have the base functions be linear independent on the sets of points $\{x_i\}$, and satisfy some other conditions (called Haar condition).

If the solvability is enjoyed, to approximate by polynomials (linear, quadratic, 3rd order, etc.) we have a formula

$$\begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix},$$

where $y = a_0 + a_1 x$ (note, $\phi_0 = 1, \phi_1 = x$). For quadratic equation

$$\begin{pmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix},$$

where $y = a_0 + a_1 x + a_2 x^2$ (note, $\phi_0 = 1, \phi_1 = x$, $\phi_2 = x^2$). For higher order polynomial,

$$\begin{pmatrix} n & \sum x_i & \sum x_i^2 & \cdots \sum x_i^n \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \cdots \sum x_i^{n+1} \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \cdots \sum x_i^{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum x_i^n & \sum x_i^{n+1} & \sum x_i^{n+2} & \cdots \sum x_i^{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \vdots \\ \sum x_i^n y_i \end{pmatrix},$$

where $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ (note, $\phi_0 = 1, \phi_1 = x$, $\phi_2 = x^2$, $\cdots$, $\phi_n = x^n$).

Example: