April 4, 2008
Exam II of Math 181 Name: ______________________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible; when you do use your calculator, sketch all relevant graphs and write down all relevant mathematics. You have 50 minutes to take this exam.

1. (*15 points*) Find the derivative of the following functions:
(i) $g(t) = 3\sin(2t+1)$

Solution: By chain rule, $g' = 6\cos(2t+1)$

(ii) $f(x) = \frac{e^x}{x}$

Solution: By quotient rule: $f' = \frac{(e^x)'x - e^x(x')}{x^2} = \frac{xe^x - e^x}{x^2}$

(iii) $h(x) = (\ln(2x+1))^3$

Solution: By chain rule, $h'(x) = 3(\ln(2x+1))^2 \cdot \frac{1}{2x+1} \cdot 2 = \frac{6(\ln(2x+1))^2}{2x+1}$.

2. (*15 points*) Solve the following equations exactly (NOT the decimal number):
(i) $6\log(x) + 2 = \log(x) - 8$

Solution: Collecting the constants on one side such that

$$5\log(x) = -10 \Rightarrow \log(x) = -2 \Rightarrow x = 10^{-2} = 0.01.$$

(ii) $2e^{2x} - e^{3x} = 0$

Solution: Define $e^x = u$, one has

$$2u^2 - u^3 = 0 \Rightarrow u^2(2-u) = 0 \Rightarrow u = 0 \ \textit{or} \ u = 2.$$

Since $e^x > 0$, we have $e^x = u = 2$ so that $x = \ln(2)$.

3. (*15 points*) Find the minimum and maximum of the function $f(x) = 4x^3 - 3x^2 + 7$ on $x \in [0, 2]$. (You must show your solution by very rigorous mathematical arguments, if you only have solution or results from calculator listed, you will lost all of the point!).

Solution: First, notice that the function $f(x)$ is a polynomial so that it is smooth (infinitely many times differentiable), then there does not exist any singularities for the function f and its derivative. So, the only candidates for critical points are only stationary points and ending points.
The stationary points are the points such that $f' = 12x^2 - 6x = 0 \Rightarrow x = 0,\ x = 1/2$. The

ending points are $x = 0, 2$. Then, evaluating the functions at the critical points gives

$$f(0) = 7,\ f(1/2) = \frac{1}{2} - \frac{3}{4} + 7 = \frac{27}{4},\ f(2) = 32 - 12 + 7 = 27.$$

Thus, $f(2)$ is the maximum and $f(1/2)$ is the minimum in the domain $[0, 2]$. (Actually, $x = 0$ is the local maximum and global minimum.)

4. (*10 points*) Find the equation of the line tangent to the graph of $y = e^{2x}$ at the point where $x = 1$.

Solution: The equation for the tangent line is

$$y - f(x_0) = f'(x_0)(x - x_0), \quad where\ x_0 = 1.$$

So, by noticing that $f' = 2e^{2x}$, the tangent line is

$$y - e^2 = 2e^2(x - 1) \Rightarrow y = 2e^2x - e^2.$$

5. (*15 points*) Assume $g(x) = \dfrac{2x^2 + 3}{4x + 1}$. Find out where the function is increasing?

Solution: To determining the domain that the function $g(x)$ is increasing, it is sufficient to find the interval that $g' > 0$. Note

$$g' = \frac{4x(4x + 1) - (2x^2 + 3)4}{(4x + 1)^2} = \frac{8x^2 + 4x - 12}{(4x + 1)^2},$$

so $g' > 0$ is equivalent to

$$2x^2 + x - 3 > 0$$

since $(4x + 1)^2 > 0$ for all x. Then, we have for $x > 1$ and $x < -\frac{3}{2}$, $g(x)$ is increasing.

6. (*15 points*) Find the maximal sustainable harvest and the corresponding breeding population under that harvest where the growth function of P is given by

$$F(P) = 1.4P - 0.00004P^2.$$

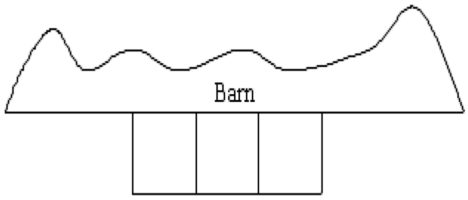
Solution: Recall $H(p) = F(p) - p = 0.4p - 0.00004P^2$, the critical point is $H' = 0$ so that $0.4 = 0.00008p$, which implies $p = 5000$. Then the maximal harvest is $H(p) = 0.4 \times 5000 - 0.00004 \times (5000)^2 = 2000 - 1000 = 1000$. Actually, note that

$$F(p) = p + kp(1 - \alpha p) = p + 0.4p(1 - 0.0001p)$$

so that

$$H_{max} = \frac{k}{4\alpha} = \frac{0.4}{4 \times 0.0001} = 1000.$$

7. (*15 points*) A farmer has 80 feet of fence with which he plans to enclose a rectangular pen along one side of his 100-foot barn as shown below. What are the dimensions of the pen that



make the area of the pens as large as possible?
Solution: Set the length is y and the width is x, so that the total length is $L = 4x + y = 80$ while the total area is $A = xy$. Then, we need to maximize $A = x(80 - 4x)$. The critical values are

$$A' = 0 \Rightarrow 80 - 8x = 0 \Rightarrow x = 10.$$

And, $y = 40$ by the definition of L. Note, $A" = -8 < 0$ which implies the geometry of the area function is concave. So, the total dimension is maximize to be $A = 400$.