Math 181
Name:
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Exam # I

1. (10 points) Consider the line l described by $\dfrac{x}{2} + \dfrac{y}{3} = 1$.

   (a) What is the slope?
   (b) What is the y-intercept and what is the x-intercept?
   (c) If a line l_1, which is parallel to l, intersects with **x-axis** at $x = 1$. What is the equation of l_1?
   Solution:
   (a) Rewrite the equation of the straight line l as

   $$3x + 2y = 6 \Rightarrow y = -\frac{3}{2}x + 3.$$

   Then the slope of l is $m = -\dfrac{3}{2}$. (b) The y-intercept is given as $x = 0$, which is $y = 3$. The x-intercept is given as $y = 0$, which is $x = 2$.
   (c) As l_1 is parallel to l, so they must have same slope. Hence, $m_{l_1} = m = -\dfrac{3}{2}$. So, by point-slope formula, noticing that the x-intercept is $x = 1$ as $y = 0$, the equation is

   $$y = m_{l_1}x + b \Rightarrow y = -\frac{3}{2}x + b \Rightarrow y = -\frac{3}{2}x + \frac{3}{2}.$$

2. (8 points) Solve quadratic equations:

   $$1 - \frac{5}{x} + \frac{6}{x^2} = 0$$

   Solution:
   First, we claim $x \neq 0$. Otherwise, the equation reads

   $$1 - \frac{5}{0} + \frac{6}{0^2} = 1 - \infty + \infty = \text{undefined} \neq 0$$

   which yields contradiction. Hence, $x \neq 0$ is true.
   Then, multiplying each side by x^2, so that

   $$x^2 - 5x + 6 = 0.$$

   By factorization,

   $$(x-2)(x-3) = 0 \Rightarrow x = 2,\ x = 3.$$

3. (12 points) Find the following limits:

(i) $\lim_{x\to 0} \dfrac{x^2+x-6}{x^2+6x-16}$
Solution:

$$\lim_{x\to 0} \frac{x^2+x-6}{x^2+6x-16} = \frac{0^2+0-6}{0^2+0-16} = \frac{6}{16} = \frac{3}{8}.$$

(ii) $\lim_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right)$
Solution:

$$\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1}\right) = \lim_{x\to 1} \frac{x+1-2}{x^2-1} = \lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

4. (20 points) Find the following derivatives, read each problem carefully!

(i) $f(x) = 4x^8 + \cos^2 x - \sqrt[3]{x} + \sin^2 x$
Solution:
Note $f = 4x^8 - \sqrt[3]{x} + 1$ since $\cos^2\theta + \sin^2\theta = 1$. So, by power rule

$$f' = 32x^7 - \frac{1}{3}x^{-2/3}.$$

(ii) $f(x) = (\tan^2 x)(x^4 - 2x)(\cos^2 x)$
Solution:
Note $\tan x \cos x = \sin x$, so $f(x) = (x^4 - 2x)\sin^2 x$ gives

$$\begin{aligned} f' &= (x^4-2x)' \sin^2 x + (\sin^2 x)'(x^4 - 2x) \\ &= (4x^3 - 2)\sin^2 x + 2\sin x \cos x (x^4 - 2x) \\ &= (4x^3-2)\sin^2 x + \sin 2x(x^4 - 2x). \end{aligned}$$

The last equality are derived from the lemma we discussed in class that

$$(f^2)' = 2ff'$$

provided f is differentiable.

5. (20 points) Some large birds cannot start flight on land, but instead must begin flight

by diving off a cliff. A bird dives so that its height H (in meters) above the water below the cliff at time t (in seconds) after beginning the dive is given by the following data:

t	0	1	3
H	200	180	44

The height is given by a quadratic equation of the form $H = at^2 + bt + c$.
(i) What is the formula for the height of the bird in terms of time t?
(ii) How much time does the bird have to begin flying, before hitting the water?

Solution:
(i) Substituting the data into the formula, we have

$$\begin{cases} 200 = a \times 0^2 + b \times 0 + c \\ 180 = a + b + c \\ 44 = 9a + 3b + c \end{cases} \Rightarrow c = 200, \text{ and } \begin{cases} a + b = -20 \\ 9a + 3b = -156. \end{cases}$$

Hence, $a = -16$ and $b = -4$ so that

$$H(t) = -16t^2 - 4t + 200.$$

(ii) If the bird hit the water level, the height of the bird is 0. So, to measure the flying time, one needs to solve

$$0 = H(t) = -16t^2 - 4t + 200 \Rightarrow t = \frac{-1 \pm \sqrt{801}}{8}.$$

Note the time must be nonnegative, the flying time is therefore $t = (-1 + \sqrt{801})/8 \approx 3.41$ seconds.

6. (30 points) A person is being medicated. The concentration of a certain drug t hours

after the start of treatment is given by

$$C(t) = \frac{t+1}{t^2+1}\ mg/l.$$

(i) Find $C'(t)$.
(ii) What is instantaneous rate of change of the concentration when $t = 1$?
(iii) What is the equation of the line tangent to the curve when $t = 1$?
(iv) When $t = 1$, is the concentration increasing or decreasing?
Solution:
(i) By quotient rule

$$C'(t) = \frac{(t^2+1) - 2t(t+1)}{(t^2+1)^2} = \frac{-t^2 - 2t + 1}{(t^2+1)^2}.$$

(ii) The instantaneous rate of change of the concentration is the derivative of C, so that

$$C'(1) = \frac{-1 - 2 + 1}{4} = -\frac{1}{2}\ mg/l/hours.$$

(iii) Note, if $t = 1$ the corresponding $C(1) = 1\ mg/l$. Using the slope-point formula, we have

$$y - y_0 = C'(1)(x - x_0) \Rightarrow y - 1 = -\frac{1}{2}(x - 1).$$

(iv) Note the derivative of function is the slope of the tangent line of function, and the slope of line characterizes the trend of the line. Since $C'(1) < 0$, so the tangent line of $C(t)$ at $t = 1$ is downstream. Therefore, $C(t)$ is decreasing at $t = 1$.

7. (Bonus problem: 20 points) **Use DEFINITION TO DERIVE** the derivative of function

$$f(x) = \cos(nx),$$

where n is an arbitrary natural number, such that $n = 1, 2, \cdots$.

Solution:
By definition,

$$\begin{aligned}
f' &= \lim_{h\to 0} \frac{\cos(n(x+h)) - \cos(nx)}{h} \\
&= \lim_{h\to 0} \frac{\cos(nx)\cos(nh) - \sin(nx)\sin(nh) - \cos(nx)}{h} \\
&= \lim_{h\to 0} \frac{(\cos(nx)\cos(nh) - \cos(nx)) - \sin(nx)\sin(nh)}{h} \\
&= \lim_{h\to 0} - \sin(nx)\sin(nh)h \\
&= - \sin(nx) \lim_{h\to 0} \frac{\sin(nh)}{h} \\
&= - \sin(nx) \lim_{y\to 0} n\frac{\sin y}{y}, \quad y = nh \\
&= -n\sin(nx)
\end{aligned}$$

where we employing identity

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$