

**THE CLASSICAL RAMSEY NUMBER $R(3, 3, 3, 3)$ IS
NO GREATER THAN 62**

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ABSTRACT. In this paper we show that $R(3, 3, 3, 3) \leq 62$, that is, any edge coloring of a complete graph on 62 vertices with four colors must contain a monochromatic triangle.

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1. INTRODUCTION

An edge coloring of a complete graph is called *good* provided that there do not exist monochromatic triangles. There are, up to isomorphism, exactly two good edge colorings with three colors on the complete graph with 16 vertices. (See [6].) These are called the untwisted and twisted colorings. The automorphism group on each of these colorings acts transitively on the vertices. By removing a single vertex from each of these colorings (along with all edges incident with the removed vertex), we get two non-isomorphic good edge colorings with three colors on the

complete graph with 15 vertices. We refer to these as the untwisted and twisted colorings on 15 vertices. There are no others, up to isomorphism. (See [5].)

All of the arguments in this paper are based on our intimate knowledge of the untwisted and twisted colorings on 15 and 16 vertices with three colors, together with the knowledge that these are the only such good edge colorings possible. If we knew the good edge colorings on 16 vertices, but not on 15 vertices, then the arguments in this paper (with trivial and obvious modifications) would suffice only to prove that $R(3, 3, 3, 3) \leq 64$, instead of $R(3, 3, 3, 3) \leq 62$, as we prove here.

In this paper, we improve the known upper bound for the classical Ramsey number $R(3, 3, 3, 3)$. It is trivial to see that

$$\begin{aligned} R(3, 3, 3, 3) &\leq 4 \cdot (R(3, 3, 3; 2) - 1) + 1 + 1 \\ &= 4 \cdot (17 - 1) + 1 + 1 \\ &= 66. \end{aligned}$$

(See [4].) In Folkman [3], it is shown that $R(3, 3, 3, 3) \leq 65$. In Sanchez-Flores [9], it is shown that $R(3, 3, 3, 3) \leq 64$. In this paper, we improve this to read $R(3, 3, 3, 3) \leq 62$. That is, we show that for any coloring of a complete graph with 62 vertices using four colors, there must exist a monochromatic triangle. The best known lower bound for $R(3, 3, 3, 3)$ was provided by Chung [1], who constructed two non-isomorphic monochromatic triangle free edge colorings, from the untwisted and twisted good colorings of K_{16} with three colors, using four colors of the complete graph with 50 vertices, thus showing that $R(3, 3, 3, 3) \geq 51$.

The paper is organized as follows. As noted below, the reader who wishes to get a feel for the global structure of the proof without reading the entire paper is advised to begin reading at section 4.

Section 2 contains the basic definitions needed to read this paper. Also included are some lemmas about the good edge colorings on the complete graph on 15 or 16 vertices with three colors, some without proof. Lemmas 9 through 17 are needed only in subsections 3.10, 3.11, and 3.12. Some easy propositions about good edge colorings on the complete graph on 62 vertices with four colors are also proved here.

Section 3 is responsible for most of the length of this paper, and is best read with a good supply of colored pencils. It contains all of the local arguments. Its only purpose is to prove Theorem 14 at the beginning of section 4. The reader who believes Theorem 14 may skip this section. Suppose that we are given a good edge coloring, with four colors, on the complete graph on 62 vertices. Let u and v be two distinct vertices and let δ be any color. Suppose that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. (According to Definition 1 in section 2, we define the set $\|S_\delta(u)\|$ to be the set of all vertices x such that the edge from u to x is of color δ .) Then the set $S_\delta(u) \cap S_\delta(v)$ is referred to as an *attaching set*. The structure of a potential attaching set is quite limited. Our discussion of a potential attaching set is split into subsections according to the cardinality of the attaching set. With only one exception, the subsections of section 3 are independent of each other. The single exception is Theorem 12 in subsection 3.12, which uses Proposition 18 from subsection 3.10.

Each subsection culminates in a single theorem, which, except for subsection 3.11, prohibits the existence of the attaching set in question. In subsection 3.11, the possibility of the existence of attaching sets of cardinality 5 is still allowed, but only under severely limited conditions. The longest and most complex subsections of section 3 are subsection 3.11, which deals with attaching sets of cardinality 5, and subsection 3.12, which deals with attaching sets of cardinality 4. Except for Proposition 22, which makes use of Proposition 21, the propositions in subsection 3.11 are all independent of each other. Similarly, except for Proposition 28, which makes use of Proposition 27, the propositions in section 3.12 are all independent of each other.

Section 4 contains all of the global arguments, that is, arguments which consider more than one attaching set at a time. The reader who wants to get a feel for the global structure of the proof, without reading the entire paper, is strongly advised to begin reading here, accepting Theorem 14 on faith and referring to the definitions and propositions of section 2 only as needed. (Not much is needed here.) Theorem 14 states that all attaching sets in a good edge coloring on the complete graph on 62 vertices with four colors must have cardinalities 0, 1, 2, or 5. Furthermore, attaching sets of cardinality 5 are severely restricted. As stated above, the sole purpose of section 3 is to prove this theorem. From this theorem alone, with the help of rather trivial results from section 2, we prove our main result, Theorem 18, that $R(3, 3, 3, 3) \leq 62$.

The results of this paper were the subject of a semester long series of talks given in the Graph Theory Seminar at Iowa State University during the spring semester of 1994.

2. PRELIMINARIES

Notation. For convenience, we define

$$[i_1, \dots, i_n] = \{ (i_{f(1)}, \dots, i_{f(n)}) \mid f \text{ is a permutation on } \{1, \dots, n\} \}.$$

We also write $u \overset{\alpha}{-} v$, where u and v are vertices in some edge colored graph and α is a color, to indicate that the edge connecting u and v is of color α .

Definition 1. Let V be the vertex set of an edge colored complete graph. Let α be a color and let $v \in V$. Then we define

$$S_\alpha(v) = \{ x \in V \mid x \overset{\alpha}{-} v \}.$$

Definition 2. Let v_0, \dots, v_{15} be vertices of an edge coloring of a complete graph, and let α, β , and γ be colors. We define the following predicates:

- (1) $P_{\beta,\alpha}(v_1, \dots, v_5)$ iff_{df}
 $v_1 \overset{\beta}{-} v_2 \overset{\beta}{-} v_3 \overset{\beta}{-} v_4 \overset{\beta}{-} v_5 \overset{\beta}{-} v_1$ and $v_1 \overset{\alpha}{-} v_3 \overset{\alpha}{-} v_5 \overset{\alpha}{-} v_2 \overset{\alpha}{-} v_4 \overset{\alpha}{-} v_1$.
- (2) $M_{\beta,\alpha,\gamma}^0(v_1, \dots, v_{10})$ iff_{df}
 $P_{\beta,\alpha}(v_1, \dots, v_5)$ and $P_{\beta,\alpha}(v_6, \dots, v_{10})$ and $v_3 \overset{\gamma}{-} v_6, v_7, v_9, v_{10}$ and $v_8 \overset{\gamma}{-} v_6, v_7, v_9, v_{10}$.

v_1, v_2, v_4, v_5 and $v_1 \xrightarrow{\gamma} v_6$ and $v_2 \xrightarrow{\gamma} v_7$ and $v_3 \xrightarrow{\gamma} v_8$ and $v_4 \xrightarrow{\gamma} v_9$ and $v_5 \xrightarrow{\gamma} v_{10}$ and $v_1 \xrightarrow{\beta} v_7 \xrightarrow{\beta} v_4 \xrightarrow{\beta} v_{10}$ and $v_6 \xrightarrow{\beta} v_2 \xrightarrow{\beta} v_9 \xrightarrow{\beta} v_5$ and $v_2 \xrightarrow{\alpha} v_{10} \xrightarrow{\alpha} v_1 \xrightarrow{\alpha} v_9$ and $v_7 \xrightarrow{\alpha} v_5 \xrightarrow{\alpha} v_6 \xrightarrow{\alpha} v_4$.

(3) $M_{\beta, \alpha, \gamma}^1(v_1, \dots, v_{10})$ iff_{df}

$P_{\beta, \alpha}(v_1, \dots, v_5)$ and $P_{\beta, \alpha}(v_6, \dots, v_{10})$ and $v_4 \xrightarrow{\gamma} v_6, v_8, v_9, v_{10}$ and $v_7 \xrightarrow{\gamma} v_1, v_2, v_3, v_5$ and $v_2 \xrightarrow{\gamma} v_9$ and $v_4 \xrightarrow{\gamma} v_7$ and $v_1 \xrightarrow{\gamma} v_6$ and $v_3 \xrightarrow{\gamma} v_8$ and $v_5 \xrightarrow{\gamma} v_{10}$ and $v_1 \xrightarrow{\beta} v_{10}$ and $v_2 \xrightarrow{\beta} v_8 \xrightarrow{\beta} v_5 \xrightarrow{\beta} v_6 \xrightarrow{\beta} v_3 \xrightarrow{\beta} v_9$ and $v_3 \xrightarrow{\alpha} v_{10} \xrightarrow{\alpha} v_2 \xrightarrow{\alpha} v_6$ and $v_5 \xrightarrow{\alpha} v_9 \xrightarrow{\alpha} v_1 \xrightarrow{\alpha} v_8$.

(4) $M_{\beta, \alpha, \gamma}^2(v_1, \dots, v_{10})$ iff_{df}

$P_{\beta, \alpha}(v_1, \dots, v_5)$ and $P_{\beta, \alpha}(v_6, \dots, v_{10})$ and $v_5 \xrightarrow{\gamma} v_7, v_8, v_9, v_{10}$ and $v_6 \xrightarrow{\gamma} v_1, v_2, v_3, v_4$ and $v_1 \xrightarrow{\gamma} v_{10}$ and $v_5 \xrightarrow{\gamma} v_6$ and $v_2 \xrightarrow{\gamma} v_7$ and $v_3 \xrightarrow{\gamma} v_8$ and $v_4 \xrightarrow{\gamma} v_9$ and $v_2 \xrightarrow{\beta} v_{10} \xrightarrow{\beta} v_4 \xrightarrow{\beta} v_8$ and $v_3 \xrightarrow{\beta} v_7 \xrightarrow{\beta} v_1 \xrightarrow{\beta} v_9$ and $v_4 \xrightarrow{\alpha} v_7$ and $v_1 \xrightarrow{\alpha} v_8 \xrightarrow{\alpha} v_2 \xrightarrow{\alpha} v_9 \xrightarrow{\alpha} v_3 \xrightarrow{\alpha} v_{10}$.

(5) $N_{\beta, \alpha, \gamma}^0(v_1, \dots, v_8)$ iff_{df}

$P_{\gamma, \alpha}(v_1, v_2, v_3, v_4, v_5)$ and $P_{\gamma, \alpha}(v_5, v_6, v_7, v_8, v_1)$ and $v_7 \xrightarrow{\gamma} v_3$ and $v_8 \xrightarrow{\gamma} v_4$ and $v_6 \xrightarrow{\gamma} v_2$ and $v_8 \xrightarrow{\beta} v_2 \xrightarrow{\beta} v_7 \xrightarrow{\beta} v_4 \xrightarrow{\beta} v_6 \xrightarrow{\beta} v_3 \xrightarrow{\beta} v_8$.

(6) $N_{\beta, \alpha, \gamma}^1(v_1, \dots, v_8)$ iff_{df}

$P_{\gamma, \alpha}(v_1, v_2, v_3, v_4, v_5)$ and $P_{\gamma, \alpha}(v_5, v_6, v_7, v_8, v_1)$ and $v_7 \xrightarrow{\gamma} v_3$ and $v_8 \xrightarrow{\gamma} v_4$ and $v_8 \xrightarrow{\beta} v_2 \xrightarrow{\beta} v_7 \xrightarrow{\beta} v_4 \xrightarrow{\beta} v_6 \xrightarrow{\beta} v_3 \xrightarrow{\beta} v_8$ and $v_6 \xrightarrow{\alpha} v_2$.

(7) $N_{\beta, \alpha, \gamma}^2(v_1, \dots, v_8)$ iff_{df}

$P_{\gamma, \alpha}(v_1, v_2, v_3, v_4, v_5)$ and $P_{\gamma, \alpha}(v_5, v_6, v_7, v_8, v_1)$ and $v_8 \xrightarrow{\gamma} v_4$ and $v_7 \xrightarrow{\gamma} v_2$ and $v_6 \xrightarrow{\gamma} v_3$ and $v_8 \xrightarrow{\beta} v_2 \xrightarrow{\beta} v_6 \xrightarrow{\beta} v_4 \xrightarrow{\beta} v_7 \xrightarrow{\beta} v_3 \xrightarrow{\beta} v_8$.

(8) $A_{\alpha, \beta, \gamma}^0(v_1, \dots, v_{10})$ iff_{df}

$P_{\beta, \alpha}(v_1, v_2, v_3, v_4, v_5)$ and $P_{\gamma, \beta}(v_6, v_7, v_8, v_9, v_{10})$ and $v_1 \xrightarrow{\beta} v_6$ and $v_2 \xrightarrow{\beta} v_7$ and $v_3 \xrightarrow{\beta} v_8$ and $v_4 \xrightarrow{\beta} v_9$ and $v_5 \xrightarrow{\beta} v_{10}$ and $v_1 \xrightarrow{\alpha} v_8 \xrightarrow{\alpha} v_5 \xrightarrow{\alpha} v_7 \xrightarrow{\alpha} v_4 \xrightarrow{\alpha} v_6 \xrightarrow{\alpha} v_3 \xrightarrow{\alpha} v_{10} \xrightarrow{\alpha} v_2 \xrightarrow{\alpha} v_9 \xrightarrow{\alpha} v_1$ and $v_1 \xrightarrow{\gamma} v_7 \xrightarrow{\gamma} v_3 \xrightarrow{\gamma} v_9 \xrightarrow{\gamma} v_5 \xrightarrow{\gamma} v_6 \xrightarrow{\gamma} v_2 \xrightarrow{\gamma} v_8 \xrightarrow{\gamma} v_4 \xrightarrow{\gamma} v_{10} \xrightarrow{\gamma} v_1$.

(9) $A_{\alpha, \beta, \gamma}^1(v_1, \dots, v_{10})$ iff_{df}

$P_{\beta, \alpha}(v_1, v_2, v_3, v_4, v_5)$ and $P_{\gamma, \beta}(v_6, v_7, v_8, v_9, v_{10})$ and $v_1 \xrightarrow{\beta} v_6$ and $v_2 \xrightarrow{\beta} v_7$ and $v_3 \xrightarrow{\beta} v_8$ and $v_4 \xrightarrow{\beta} v_9$ and $v_5 \xrightarrow{\beta} v_{10}$ and $v_1 \xrightarrow{\alpha} v_8 \xrightarrow{\alpha} v_5 \xrightarrow{\alpha} v_7 \xrightarrow{\alpha} v_4 \xrightarrow{\alpha} v_6 \xrightarrow{\alpha} v_3 \xrightarrow{\alpha} v_{10} \xrightarrow{\alpha} v_2 \xrightarrow{\alpha} v_9 \xrightarrow{\alpha} v_1$ and $v_1 \xrightarrow{\gamma} v_7 \xrightarrow{\gamma} v_3 \xrightarrow{\gamma} v_9 \xrightarrow{\gamma} v_5 \xrightarrow{\gamma} v_6 \xrightarrow{\gamma} v_2 \xrightarrow{\gamma} v_8 \xrightarrow{\gamma} v_4 \xrightarrow{\gamma} v_{10} \xrightarrow{\gamma} v_1$.

(10) $C_{\alpha, \beta, \gamma}^0(v_1, \dots, v_{15})$ iff_{df}

$A_{\alpha, \beta, \gamma}^0(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$ and $A_{\beta, \gamma, \alpha}^0(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ and $A_{\gamma, \alpha, \beta}^0(v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})$.

(11) $C_{\alpha, \beta, \gamma}^1(v_1, \dots, v_{15})$ iff_{df}

$A_{\alpha, \beta, \gamma}^1(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$ and $A_{\beta, \gamma, \alpha}^1(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ and $A_{\gamma, \alpha, \beta}^1(v_6, v_7, v_8, v_9, v_{10}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11})$.

- (12) $B_{\alpha,\beta,\gamma}^0(v_0, \dots, v_{15})$ iff_{df}
 $C_{\alpha,\beta,\gamma}^0(v_1, \dots, v_{15})$ and $v_0 \xrightarrow{\alpha} v_1, v_2, v_3, v_4, v_5$ and $v_0 \xrightarrow{\beta} v_6, v_7, v_8, v_9, v_{10}$ and
 $v_0 \xrightarrow{\gamma} v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$.
- (13) $B_{\alpha,\beta,\gamma}^1(v_0, \dots, v_{15})$ iff_{df}
 $C_{\alpha,\beta,\gamma}^1(v_1, \dots, v_{15})$ and $v_0 \xrightarrow{\alpha} v_1, v_2, v_3, v_4, v_5$ and $v_0 \xrightarrow{\beta} v_6, v_7, v_8, v_9, v_{10}$ and
 $v_0 \xrightarrow{\gamma} v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$.

Lemma 1. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . If $\|U\| = 16$, then there exist $x_0, \dots, x_{15} \in U$ and some $i \in \{0, 1\}$ such that $B_{\alpha,\beta,\gamma}^i(x_0, \dots, x_{15})$*

Proof. See Kalbfleisch and Stanton [6]. □

Lemma 2. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . If $\|U\| = 15$, then there exist $x_1, \dots, x_{15} \in U$ and some $i \in \{0, 1\}$ such that $C_{\alpha,\beta,\gamma}^i(x_1, \dots, x_{15})$*

Proof. See Heinrich [5]. □

Remark. In Lemmas 1 and 2, $i = 0$ if the coloring is untwisted, and $i = 1$ if the coloring is twisted.

Lemma 3. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and δ . Let $A, B \subseteq U$ both contain no edges of color δ . Suppose that $\|U\| \geq 15$ and $\|A\| = \|B\| = 5$. Then $\|A \cap B\| \in \{0, 2, 5\}$.*

Proof. Clearly, $\|U\| \leq R(3, 3, 3; 2) - 1 = 17 - 1 = 16$ so that the conclusion follows from Lemmas 1 and 2 by inspection. □

Proposition 1. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $x \in V$. Then*

$$\begin{aligned} & (\|S_\alpha(x)\|, \|S_\beta(x)\|, \|S_\gamma(x)\|, \|S_\delta(x)\|) \\ & \in [16, 16, 16, 13] \cup [16, 16, 15, 14] \cup [16, 15, 15, 15]. \end{aligned}$$

Proof. It is clear that

$$\begin{aligned} 62 &= \|V\| \\ &= \|\{x\} \uplus S_\alpha(x) \uplus S_\beta(x) \uplus S_\gamma(x) \uplus S_\delta(x)\| \\ &= 1 + \|S_\alpha(x)\| + \|S_\beta(x)\| + \|S_\gamma(x)\| + \|S_\delta(x)\|. \end{aligned}$$

Also, for any $\eta \in \{\alpha, \beta, \gamma, \delta\}$, we see that the induced good edge coloring on the complete graph with vertex set $S_\eta(x)$ cannot contain any edges of color η , since otherwise we would have a monochromatic triangle of color η in V , contradicting the goodness of the original coloring. Thus, we have

$$\|S_\alpha(x)\|, \|S_\beta(x)\|, \|S_\gamma(x)\|, \|S_\delta(x)\| \leq R(3, 3, 3; 2) - 1 = 17 - 1 = 16.$$

The proposition follows. \square

Proposition 2. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u \in V$ with $\|S_\delta(u)\| = 16$. Then for any $x \in S_\delta(u)$ we have*

$$\|S_\delta(u) \cap S_\alpha(x)\| = \|S_\delta(u) \cap S_\beta(x)\| = \|S_\delta(u) \cap S_\gamma(x)\| = 5.$$

Proof. Note that $S_\delta(u) \cap S_\delta(x) = \emptyset$, since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} 16 &= \|S_\delta(u)\| \\ &= \|\{x\} \uplus (S_\delta(u) \cap S_\alpha(x)) \uplus (S_\delta(u) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\gamma(x))\| \\ &= 1 + \|S_\delta(u) \cap S_\alpha(x)\| + \|S_\delta(u) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\gamma(x)\|. \end{aligned}$$

But, since $S_\delta(u) \cap S_\eta(x)$ has no edges of colors δ or η whenever $\eta \in \{\alpha, \beta, \gamma\}$, we see that

$$\|S_\delta(u) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

for any $\eta \in \{\alpha, \beta, \gamma\}$. The proposition follows. \square

Proposition 3. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. If $\eta \in \{\alpha, \beta, \gamma\}$ and $x \in S_\delta(u) \cap S_\delta(v)$ with $\|S_\eta(x)\| \geq 15$, then*

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\}.$$

Proof. Without loss of generality, we assume that $\eta = \gamma$. Let $U = S_\gamma(x)$. Let $A = S_\delta(u) \cap S_\gamma(x)$ and $B = S_\delta(v) \cap S_\gamma(x)$. Note that $A, B \subseteq U$. Also, the complete graph with vertex set U has no edges of color γ under the induced good edge coloring, and thus is colored with the colors α , β , and δ . Also, neither A nor B has any edges of color δ . Furthermore, $\|U\| \geq 15$. By Proposition 2, we see that $\|A\| = \|B\| = 5$. We may now apply Lemma 3 to see that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = \|A \cap B\| \in \{0, 2, 5\},$$

as desired. \square

Proposition 4. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\|\{\eta \in \{\alpha, \beta, \gamma\} \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\}\}\| \geq 2.$$

Proof. First note that

$$\begin{aligned} &\{\eta \in \{\alpha, \beta, \gamma\} \mid \|S_\eta(x)\| \geq 15\} \\ &\subseteq \{\eta \in \{\alpha, \beta, \gamma\} \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{0, 2, 5\}\} \end{aligned}$$

by Proposition 3. But

$$\|\{\eta \in \{\alpha, \beta, \gamma\} \mid \|S_\eta(x)\| \geq 15\}\| \geq 2$$

by Proposition 1. The proposition follows. \square

Lemma 4. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . Let $\|U\| = 16$ and let $a \in U$. Then $\|\{x \in U \mid x \overset{\alpha}{-} a\}\| = 5$. Furthermore, there exists $w_0, w_1, w_2, w_3, w_4 \in U$ such that $\{w_0, w_1, w_2, w_3, w_4\} = \{x \in U \mid x \overset{\alpha}{-} a\}$ and $P_{\beta, \gamma}(w_0, w_1, w_2, w_3, w_4)$. Moreover, if $x, y \in U$ satisfy $a \overset{\alpha}{-} x \overset{\beta}{-} y \overset{\alpha}{-} a$, we may assume that $x = w_0$ and $y = w_4$.*

Proof. We omit the proof. \square

Lemma 5. *Let U be the vertex set of a complete graph with a good edge coloring with the pairwise distinct colors α , β , and γ , with $\|U\| = 16$. Let $a, b \in U$ with $a \overset{\alpha}{-} b$. Then there exist $x_0, \dots, x_{15} \in U$ such that one of the following holds:*

- (i) $C_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ with $a = x_0$ and $b = x_3$.
- (ii) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ with $a = x_0$ and $b = x_3$.
- (iii) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ with $a = x_0$ and $b = x_2$.
- (iv) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ with $a = x_0$ and $b = x_1$.

Proof. We omit the proof. \square

Remark. In Lemma 5, i applies if the coloring is untwisted, and either ii, iii, or iv applies if the coloring is twisted.

Lemma 6. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . Let $\|U\| = 16$ and let $a, b \in U$ with $a \overset{\gamma}{-} b$. Then, we have the following:*

- (1) $\|\{x \in U \mid a \overset{\alpha}{-} x \overset{\alpha}{-} b\}\| = 2$
- (2) $\|\{x \in U \mid a \overset{\alpha}{-} x \overset{\beta}{-} b\}\| = 1$
- (3) $\|\{x \in U \mid a \overset{\alpha}{-} x \overset{\gamma}{-} b\}\| = 2$

Proof. We omit the proof. \square

Lemma 7. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . Suppose that $\|U\| = 16$ and $\eta \in \{\alpha, \beta, \gamma\}$. If $a, b, c, d, e \in U$ are pairwise distinct and satisfy $a \overset{\eta}{-} b \overset{\eta}{-} c$ and $d \overset{\eta}{-} e$, then there exists some $x \in \{a, b, c\}$ and some $y \in \{d, e\}$ such that $x \overset{\eta}{-} y$.*

Proof. Suppose not. Then there exist distinct $x_0, x_1 \in U$ with $b \overset{\eta}{-} x_0, x_1 \overset{\eta}{-} d$ and there exist distinct $y_0, y_1 \in U$ with $b \overset{\eta}{-} y_0, y_1 \overset{\eta}{-} e$. But $\{x_0, x_1\} \cap \{y_0, y_1\} = \emptyset$, since otherwise we would have a monochromatic triangle. Clearly $a, c \notin \{x_0, x_1, y_0, y_1\}$, since we cannot have any η colored edges from $\{a, c\}$ to $\{d, e\}$. thus, we have pairwise distinct elements $a, c, x_0, x_1, y_0, y_1 \in U$ with $b \overset{\eta}{-} a, c, x_0, x_1, y_0, y_1$, which is impossible, by Lemma 3. \square

Lemma 8. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , and γ . Suppose that $\|U\| = 16$ and*

$\eta \in \{\alpha, \beta, \gamma\}$. If $a, b, c, d \in U$ are four pairwise distinct elements which satisfy $a \overset{\eta}{-} b \overset{\eta}{-} c \overset{\eta}{-} d \overset{\eta}{-} a$, then for any $u \in U \sim \{a, b, c, d\}$ there exists a unique $v \in \{a, b, c, d\}$ such that $u \overset{\eta}{-} v$.

Proof. By lemma 4, there exist pairwise distinct $x_0, x_1, x_2 \in U \sim \{d, b\}$ such that $a \overset{\eta}{-} b, d, x_0, x_1, x_2$. Similarly, there exist pairwise distinct $y_0, y_1, y_2 \in U \sim \{a, c\}$ such that $b \overset{\eta}{-} a, c, y_0, y_1, y_2$, pairwise distinct $z_0, z_1, z_2 \in U \sim \{b, d\}$ such that $c \overset{\eta}{-} b, d, z_0, z_1, z_2$, and $w_0, w_1, w_2 \in U \sim \{c, a\}$ such that $d \overset{\eta}{-} c, a, w_0, w_1, w_2$.

Note that the elements $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, w_0, w_1, w_2$ must be pairwise distinct, since otherwise we would either get a monochromatic triangle, or else a violation of Lemma 6(1). Thus, we have

$$U = \{a, b, c, d, x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, w_0, w_1, w_2\}.$$

The lemma follows. \square

Lemma 9. Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:

$$\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} = \emptyset$$

$$P_{\beta, \alpha}(u_3, u_4, u_0, u_1, u_2)$$

$$P_{\beta, \alpha}(v_3, v_4, v_0, v_1, v_2)$$

$$u_3 \overset{\gamma}{-} v_3$$

$$u_4 \overset{\gamma}{-} v_4$$

$$u_1 \overset{\gamma}{-} v_1$$

$$u_2 \overset{\gamma}{-} v_2$$

Then $u_0 \overset{\gamma}{-} v_0$. Furthermore, one of the following 25 cases must hold:

- (i) $M_{\beta, \alpha, \gamma}^0(u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4)$
- (ii) $M_{\beta, \alpha, \gamma}^0(u_1, u_2, u_3, u_4, u_0, v_1, v_2, v_3, v_4, v_0)$
- (iii) $M_{\beta, \alpha, \gamma}^0(u_2, u_3, u_4, u_0, u_1, v_2, v_3, v_4, v_0, v_2)$
- (iv) $M_{\beta, \alpha, \gamma}^0(u_3, u_4, u_0, u_1, u_2, v_3, v_4, v_0, v_1, v_2)$
- (v) $M_{\beta, \alpha, \gamma}^0(u_4, u_0, u_1, u_2, u_3, v_4, v_0, v_1, v_2, v_3)$
- (vi) $M_{\beta, \alpha, \gamma}^1(u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4)$
- (vii) $M_{\beta, \alpha, \gamma}^1(u_1, u_2, u_3, u_4, u_0, v_1, v_2, v_3, v_4, v_0)$
- (viii) $M_{\beta, \alpha, \gamma}^1(u_2, u_3, u_4, u_0, u_1, v_2, v_3, v_4, v_0, v_1)$
- (ix) $M_{\beta, \alpha, \gamma}^1(u_3, u_4, u_0, u_1, u_2, v_3, v_4, v_0, v_1, v_2)$
- (x) $M_{\beta, \alpha, \gamma}^1(u_4, u_0, u_1, u_2, u_3, v_4, v_0, v_1, v_2, v_3)$
- (xi) $M_{\beta, \alpha, \gamma}^1(u_4, u_3, u_2, u_1, u_0, v_4, v_3, v_2, v_1, v_0)$
- (xii) $M_{\beta, \alpha, \gamma}^1(u_3, u_2, u_1, u_0, u_4, v_3, v_2, v_1, v_0, v_4)$
- (xiii) $M_{\beta, \alpha, \gamma}^1(u_2, u_1, u_0, u_4, u_3, v_2, v_1, v_0, v_4, v_3)$
- (xiv) $M_{\beta, \alpha, \gamma}^1(u_1, u_0, u_4, u_3, u_2, v_1, v_0, v_4, v_3, v_2)$
- (xv) $M_{\beta, \alpha, \gamma}^1(u_0, u_4, u_3, u_2, u_1, v_0, v_4, v_3, v_2, v_1)$

- (xvi) $M_{\beta, \alpha, \gamma}^2(u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4)$
- (xvii) $M_{\beta, \alpha, \gamma}^2(u_1, u_2, u_3, u_4, u_0, v_1, v_2, v_3, v_4, v_0)$
- (xviii) $M_{\beta, \alpha, \gamma}^2(u_2, u_3, u_4, u_0, u_1, v_2, v_3, v_4, v_0, v_1)$
- (xix) $M_{\beta, \alpha, \gamma}^2(u_3, u_4, u_0, u_1, u_2, v_3, v_4, v_0, v_1, v_2)$
- (xx) $M_{\beta, \alpha, \gamma}^2(u_4, u_0, u_1, u_2, u_3, v_4, v_0, v_1, v_2, v_3)$
- (xxi) $M_{\beta, \alpha, \gamma}^2(u_4, u_3, u_2, u_1, u_0, v_4, v_3, v_2, v_1, v_0)$
- (xxii) $M_{\beta, \alpha, \gamma}^2(u_3, u_2, u_1, u_0, u_4, v_3, v_2, v_1, v_0, v_4)$
- (xxiii) $M_{\beta, \alpha, \gamma}^2(u_2, u_1, u_0, u_4, u_3, v_2, v_1, v_0, v_4, v_3)$
- (xxiv) $M_{\beta, \alpha, \gamma}^2(u_1, u_0, u_4, u_3, u_2, v_1, v_0, v_4, v_3, v_2)$
- (xxv) $M_{\beta, \alpha, \gamma}^2(u_0, u_4, u_3, u_2, u_1, v_0, v_4, v_3, v_2, v_1)$

Moreover, if the coloring is untwisted, then one of the cases (i), (ii), (iii), (iv), or (v) must hold.

Remark. Note that the question of which of the 25 cases is determined by a choice of $i, j \in \{0, 1, 2, 3, 4\}$ such that $u_i \xrightarrow{\gamma} v_0, v_1, v_2, v_3, v_4$ and $v_j \xrightarrow{\gamma} u_0, u_1, u_2, u_3, u_4$.

Proof. We omit the proof. \square

Lemma 10. Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:

$$\begin{aligned} \{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} &= \emptyset \\ P_{\beta, \alpha}(u_3, u_4, u_0, u_1, u_2) \\ P_{\beta, \alpha}(v_3, v_4, v_0, v_1, v_2) \\ u_3 &\xrightarrow{\gamma} v_3 \\ u_0 &\xrightarrow{\gamma} v_0 \\ u_2 &\xrightarrow{\gamma} v_2 \end{aligned}$$

(Then either both $u_1 \xrightarrow{\gamma} v_1$ and $u_4 \xrightarrow{\gamma} v_4$ hold or else one of the following 10 cases must hold:

- (i) $M_{\beta, \alpha, \gamma}^1(u_4, u_0, u_1, u_2, u_3, v_2, v_3, v_4, v_0, v_1)$
- (ii) $M_{\beta, \alpha, \gamma}^1(u_1, u_0, u_4, u_3, u_2, v_3, v_2, v_1, v_0, v_4)$
- (iii) $M_{\beta, \alpha, \gamma}^0(u_1, u_2, u_3, u_4, u_0, v_4, v_3, v_2, v_1, v_0)$
- (iv) $M_{\beta, \alpha, \gamma}^0(u_0, u_1, u_2, u_3, u_4, v_0, v_4, v_3, v_2, v_1)$
- (v) $M_{\beta, \alpha, \gamma}^2(u_3, u_4, u_0, u_1, u_2, v_2, v_1, v_0, v_4, v_3)$
- (vi) $M_{\beta, \alpha, \gamma}^2(u_2, u_1, u_0, u_4, u_3, v_3, v_4, v_0, v_1, v_2)$
- (vii) $M_{\beta, \alpha, \gamma}^2(u_1, u_0, u_4, u_3, u_2, v_0, v_1, v_2, v_3, v_4)$
- (viii) $M_{\beta, \alpha, \gamma}^2(u_4, u_0, u_1, u_2, u_3, v_0, v_4, v_3, v_2, v_1)$
- (ix) $M_{\beta, \alpha, \gamma}^2(u_4, u_3, u_2, u_1, u_0, v_2, v_3, v_4, v_0, v_1)$
- (x) $M_{\beta, \alpha, \gamma}^2(u_1, u_2, u_3, u_4, u_0, v_3, v_2, v_1, v_0, v_4)$

Moreover, if the coloring is untwisted, then either both $u_1 \xrightarrow{\gamma} v_1$ and $u_4 \xrightarrow{\gamma} v_4$ hold or else one of the cases (iii) or (iv) must hold.

Remark. Note that under the hypotheses of Lemma 10, in case we have both $u_1 \overset{\gamma}{-} v_1$ and $u_4 \overset{\gamma}{-} v_4$ holding, we see that the hypotheses of Lemma 9 are satisfied. Thus, we see that under the hypotheses of Lemma 10, either one of the 10 cases of Lemma 10 holds or else one of the 25 cases of Lemma 9 holds.

Proof. We omit the proof. \square

Lemma 11. *Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:*

$$\begin{aligned} \{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} &= \emptyset \\ P_{\beta, \alpha}(u_3, u_4, u_0, u_1, u_2) \\ P_{\beta, \alpha}(v_3, v_4, v_0, v_1, v_2) \\ u_4 &\overset{\gamma}{-} v_4 \\ u_0 &\overset{\gamma}{-} v_0 \\ u_1 &\overset{\gamma}{-} v_1 \end{aligned}$$

Then either both $u_2 \overset{\gamma}{-} v_2$ and $u_3 \overset{\gamma}{-} v_3$ hold or else one of the following 10 cases must hold:

- (i) $M_{\beta, \alpha, \gamma}^2(u_0, u_4, u_3, u_2, u_1, v_4, v_3, v_2, v_1, v_0)$
- (ii) $M_{\beta, \alpha, \gamma}^2(u_0, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_0)$
- (iii) $M_{\beta, \alpha, \gamma}^0(u_4, u_0, u_1, u_2, u_3, v_1, v_0, v_4, v_3, v_2)$
- (iv) $M_{\beta, \alpha, \gamma}^0(u_2, u_3, u_4, u_0, u_1, v_3, v_2, v_1, v_0, v_4)$
- (v) $M_{\beta, \alpha, \gamma}^1(u_3, u_4, u_0, u_1, u_2, v_2, v_1, v_0, v_4, v_3)$
- (vi) $M_{\beta, \alpha, \gamma}^1(u_2, u_1, u_0, u_4, u_3, v_3, v_4, v_0, v_1, v_2)$
- (vii) $M_{\beta, \alpha, \gamma}^1(u_1, u_2, u_3, u_4, u_0, v_1, v_0, v_4, v_3, v_2)$
- (viii) $M_{\beta, \alpha, \gamma}^1(u_4, u_3, u_2, u_1, u_0, v_4, v_0, v_1, v_2, v_3)$
- (ix) $M_{\beta, \alpha, \gamma}^1(u_2, u_3, u_4, u_0, u_1, v_0, v_4, v_3, v_2, v_1)$
- (x) $M_{\beta, \alpha, \gamma}^1(u_3, u_2, u_1, u_0, u_4, v_0, v_1, v_2, v_3, v_4)$

Moreover, if the coloring is untwisted, then either both $u_2 \overset{\gamma}{-} v_2$ and $u_3 \overset{\gamma}{-} v_3$ hold or else one of the cases (iii) or (iv) must hold.

Remark. Note that under the hypotheses of Lemma 11, in case we have both $u_2 \overset{\gamma}{-} v_2$ and $u_3 \overset{\gamma}{-} v_3$ holding, we see that the hypotheses of Lemma 9 are satisfied. Thus, we see that under the hypotheses of Lemma 11, either one of the 10 cases of Lemma 11 holds or else one of the 25 cases of Lemma 9 holds.

Proof. We omit the proof. \square

Lemma 12. *Let $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:*

$$\{u_2, u_3, u_4\} \cap \{u_6, u_7, u_8\} = \emptyset$$

$$P_{\gamma,\beta}(u_1, u_2, u_3, u_4, u_5)$$

$$P_{\gamma,\beta}(u_5, u_6, u_7, u_8, u_1)$$

Then one of the following 5 cases must hold:

- (i) $N_{\alpha,\beta,\gamma}^0(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$
- (ii) $N_{\alpha,\beta,\gamma}^1(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$
- (iii) $N_{\alpha,\beta,\gamma}^1(u_1, u_8, u_7, u_6, u_5, u_4, u_3, u_2)$
- (iv) $N_{\alpha,\beta,\gamma}^2(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$
- (v) $N_{\alpha,\beta,\gamma}^2(u_1, u_8, u_7, u_6, u_5, u_4, u_3, u_2)$

Moreover, if the coloring is untwisted, then case (i) must hold.

Proof. We omit the proof. \square

Lemma 13. Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:

$$\begin{aligned} \{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} &= \emptyset \\ P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2) \\ P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2) \\ u_0 &\overset{\gamma}{-} v_0, v_1, v_2, v_3, v_4 \\ v_0 &\overset{\gamma}{-} u_0, u_1, u_2, u_3, u_4 \end{aligned}$$

Then one of the following 10 cases must hold:

- (i) $M_{\beta,\alpha,\gamma}^0(u_3, u_4, u_0, u_1, u_2, v_3, v_4, v_0, v_1, v_2)$
- (ii) $M_{\beta,\alpha,\gamma}^0(u_3, u_4, u_0, u_1, u_2, v_2, v_1, v_0, v_4, v_3)$
- (iii) $M_{\beta,\alpha,\gamma}^1(u_2, u_3, u_4, u_0, u_1, v_4, v_0, v_1, v_2, v_3)$
- (iv) $M_{\beta,\alpha,\gamma}^1(u_2, u_3, u_4, u_0, u_1, v_1, v_0, v_4, v_3, v_2)$
- (v) $M_{\beta,\alpha,\gamma}^1(u_3, u_2, u_1, u_0, u_4, v_4, v_0, v_1, v_2, v_3)$
- (vi) $M_{\beta,\alpha,\gamma}^1(u_3, u_2, u_1, u_0, u_4, v_1, v_0, v_4, v_3, v_2)$
- (vii) $M_{\beta,\alpha,\gamma}^2(u_1, u_2, u_3, u_4, u_0, v_0, v_1, v_2, v_3, v_4)$
- (viii) $M_{\beta,\alpha,\gamma}^2(u_1, u_2, u_3, u_4, u_0, v_0, v_4, v_3, v_2, v_1)$
- (ix) $M_{\beta,\alpha,\gamma}^2(u_4, u_3, u_2, u_1, u_0, v_0, v_1, v_2, v_3, v_4)$
- (x) $M_{\beta,\alpha,\gamma}^2(u_4, u_3, u_2, u_1, u_0, v_0, v_4, v_3, v_2, v_1)$

Moreover, if the coloring is untwisted, then one of the cases (i) or (ii) must hold.

Proof. We omit the proof. \square

Lemma 14. Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:

$$\begin{aligned} \{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\} &= \emptyset \\ P_{\beta,\alpha}(u_3, u_4, u_0, u_1, u_2) \\ P_{\beta,\alpha}(v_3, v_4, v_0, v_1, v_2) \end{aligned}$$

Then there exists some $x \in \{u_0, u_1, u_2, u_3, u_4\}$ and some $y \in \{v_0, v_1, v_2, v_3, v_4\}$ such that

$$x \overset{\gamma}{-} v_0, v_1, v_2, v_3, v_4 \quad \text{and} \quad y \overset{\gamma}{-} u_0, u_1, u_2, u_3, u_4.$$

Furthermore, if $u_i \overset{\eta}{-} x$ for some $\eta \in \{\alpha, \beta\}$, then

$$\|\{j \in \{0, 1, 2, 3, 4\} \mid u_i \overset{\eta}{-} v_j\}\| = \|\{j \in \{0, 1, 2, 3, 4\} \mid u_i \overset{\gamma}{-} v_j\}\| = 2.$$

Similarly, if $v_i \overset{\eta}{-} y$ for some $\eta \in \{\alpha, \beta\}$, then

$$\|\{j \in \{0, 1, 2, 3, 4\} \mid v_i \overset{\eta}{-} u_j\}\| = \|\{j \in \{0, 1, 2, 3, 4\} \mid v_i \overset{\gamma}{-} u_j\}\| = 2.$$

Proof. We omit the proof. \square

Lemma 15. Let $u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4$ be vertices in some good edge coloring of a complete graph on either 15 or 16 vertices, colored with the three pairwise distinct colors α, β, γ . Suppose that the following hold:

$$P_{\beta, \alpha}(u_3, u_4, u_0, u_1, u_2)$$

$$P_{\beta, \alpha}(v_3, v_4, v_0, v_1, v_2)$$

Then

$$\|\{u_0, u_1, u_2, u_3, u_4\} \cap \{v_0, v_1, v_2, v_3, v_4\}\| \in \{0, 2, 5\}.$$

Proof. We omit the proof. \square

Lemma 16. Let U be the vertex set of a complete graph with a good edge coloring, colored with the colors α, β , and γ . Let $\|U\| \in \{15, 16\}$ and suppose that $A, B \subseteq U$ each have no edges of color γ in the induced colorings, where $\|A\| = \|B\| = 5$ and $A \cap B = \emptyset$. Then, there exist $x_0, x_1, x_2, x_3, x_4 \in A$ and $y_0, y_1, y_2, y_3, y_4 \in B$ such that $M_{\beta, \alpha, \gamma}^i(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3, y_4)$ holds for some $i \in \{0, 1, 2\}$.

Proof. We omit the proof. \square

Lemma 17. Let U be the vertex set of a complete graph with a good edge coloring, colored with the colors α, β , and γ . Let $\|U\| = 16$ and suppose that $a, b, c \in U$ with $a \overset{\gamma}{-} b \overset{\beta}{-} c \overset{\gamma}{-} a$. Then there exists a unique $d \in U$ such that $d \overset{\alpha}{-} a, b, c$.

Proof. We omit the proof. \square

3. THE LOCAL ARGUMENTS

3.1. Attaching Sets of Cardinality 15 or 16.

Theorem 1. Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $u \neq v$ and $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \leq 14$.

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| \geq 15$. Since $S_\delta(u) \cap S_\delta(v) \neq \emptyset$, we must have $u \overset{\eta}{-} v$ for some $\eta \in \{\alpha, \beta, \gamma\}$. Without loss of generality, we may assume that $u \overset{\alpha}{-} v$. Thus, we have $S_\alpha(u) \cap S_\alpha(v) = \emptyset$, so that

$$S_\alpha(u) = \{v\} \uplus (S_\alpha(u) \cap S_\beta(v)) \uplus (S_\alpha(u) \cap S_\gamma(v)) \uplus (S_\alpha(u) \cap S_\delta(v)).$$

It is clear that

$$\|S_\alpha(u) \cap S_\beta(v)\|, \|S_\alpha(u) \cap S_\gamma(v)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5.$$

Also, we have $\|S_\alpha(u)\| \geq 13$ by Proposition 1, so that

$$13 \leq 1 + 5 + 5 + \|S_\alpha(u) \cap S_\delta(v)\|.$$

Thus, we have

$$\|S_\alpha(u) \cap S_\delta(v)\| \geq 2.$$

Now, note that

$$S_\delta(v) = (S_\beta(u) \cap S_\delta(v)) \uplus (S_\gamma(u) \cap S_\delta(v)) \uplus (S_\alpha(u) \cap S_\delta(v)) \uplus (S_\delta(u) \cap S_\delta(v)),$$

so that

$$16 \geq \|S_\beta(u) \cap S_\delta(v)\| + \|S_\gamma(u) \cap S_\delta(v)\| + 2 + 15,$$

which gives the desired contradiction. The theorem is proved. \square

3.2. Attaching Sets of Cardinality 14.

Proposition 5. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 14$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 3]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5,$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 14 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 3] \cup [5, 4, 4]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 2. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 14$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 14$. Let $S_\delta(u) \sim S_\delta(v) = \{a, b\}$. Without loss of generality, we may assume that $a \overset{\gamma}{-} b$. By Lemma 6(2), there exists some $x \in S_\delta(u)$ with $a \overset{\alpha}{-} x \overset{\beta}{-} b$. Clearly, we have $x \in S_\delta(u) \cap S_\delta(v)$. Note that

$$\begin{aligned} (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \uplus \{a\} &= S_\delta(u) \cap S_\alpha(x), \\ (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus \{b\} &= S_\delta(u) \cap S_\beta(x), \end{aligned}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = S_\delta(u) \cap S_\gamma(x).$$

But $\|S_\delta(u) \cap S_\eta(x)\| = 5$ for $\eta \in \{\alpha, \beta, \gamma\}$ by Lemma 4. Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ = (4, 4, 5) \notin [5, 5, 3], \end{aligned}$$

which contradicts Proposition 5. The theorem is proved. \square

3.3. Attaching Sets of Cardinality 13.

Proposition 6. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 13$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) &= \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ &\quad \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 13 = \|S_\delta(u) \cap S_\delta(v)\| &= 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &\quad + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 2] \cup [5, 4, 3] \cup [4, 4, 4]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 3. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 13$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 13$. Let $S_\delta(u) \sim S_\delta(v) = \{a, b, c\}$. For any $x \in S_\delta(u) \cap S_\delta(v)$, we must have $x \overset{\eta}{-} a, b, c$ for some $\eta \in \{\alpha, \beta, \gamma\}$, by Proposition 2 and Proposition 6. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \biguplus_{\eta \in \{\alpha, \beta, \gamma\}} (S_\delta(u) \cap S_\delta(v) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)).$$

This will give the contradiction

$$13 \leq 1 + 1 + 1$$

once we show that

$$\|S_\delta(u) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)\| \leq 1$$

for any $\eta \in \{\alpha, \beta, \gamma\}$. Suppose not. Then we may find distinct $x, y \in S_\delta(u) \cap S_\eta(a) \cap S_\eta(b) \cap S_\eta(c)$. But Lemma 6(1) implies that

$$\|\{z \in S_\delta(u) \mid x \overset{\eta}{-} z \overset{\eta}{-} y\}\| = 2,$$

which is impossible since $a, b, c \in \{z \in S_\delta(u) \mid x \overset{\eta}{-} z \overset{\eta}{-} y\}$. The theorem is proved. \square

3.4. Attaching Sets of Cardinality 12.

Proposition 7. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 12$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 1] \cup [5, 4, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5,$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$12 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|.$$

Thus, we have

$$(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 1] \cup [5, 4, 2] \cup [5, 3, 3] \cup [4, 4, 3].$$

The proposition now follows by an application of Proposition 4. \square

Theorem 4. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 12$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 12$. Let $S_\delta(u) \sim (S_\delta(u) \cap S_\delta(v)) = \{a_0, a_1, a_2, a_3\}$. For each $i \in \{0, 1, 2, 3\}$, we define

$$U_i = \bigcup_{\eta \in \{\alpha, \beta, \gamma\}} \bigcap_{j \neq i} (S_\delta(u) \cap S_\eta(a_j)).$$

By Proposition 2 and Proposition 7, we see that

$$S_\delta(u) \cap S_\delta(v) \subseteq U_0 \cup U_1 \cup U_2 \cup U_3.$$

This will give the contradiction

$$12 \leq 1 + 1 + 1 + 1,$$

once we show that $\|U_i\| \leq 1$ for each $i \in \{0, 1, 2, 3\}$. Suppose not. Then we may find distinct $x, y \in U_i$ for some fixed i . Without loss of generality, we may assume that $i = 3$. Then there exist $\eta, \xi \in \{\alpha, \beta, \gamma\}$ such that $x \xrightarrow{\eta} a_0, a_1, a_2$ and $y \xrightarrow{\xi} a_0, a_1, a_2$. But Lemma 6 implies that

$$\|\{z \in S_\delta(u) \mid x \xrightarrow{\eta} z \xrightarrow{\xi} y\}\| \leq 2,$$

which is impossible since $a_0, a_1, a_2 \in \{z \in S_\delta(u) \mid x \xrightarrow{\eta} z \xrightarrow{\xi} y\}$. The theorem is proved. \square

3.5. Attaching Sets of Cardinality 11.

Proposition 8. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 11$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 0] \cup [5, 3, 2].$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5,$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) &= \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ &\quad \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 11 &= \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &\quad + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 5, 0] \cup [5, 4, 1] \cup [5, 3, 2] \cup [4, 4, 2] \cup [4, 3, 3]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 5. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 11$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 11$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \{x \in S_\delta(u) \cup S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5\}.$$

By Proposition 8, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 4$ for some η . Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges. The elements of U_η all have degree 5 in this graph. Clearly, not all of the edges in U_η can be of color η . Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η . By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But $\|S_\delta(u) \cap S_\eta(x)\| \leq 5 = \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\|$, since $x \in U_\eta$, so that $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$. Thus, we actually have

$$\begin{aligned} \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| &= \|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| \\ &= 2. \end{aligned}$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2, z\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{-} w_j$ or $v_i \overset{\eta}{-} w_j$ for any $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $w_0 \overset{\eta}{-} w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) \subseteq \{x, y, z\}$ for all $i \in \{0, 1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2, z\}$ with $\|U_\eta\| \geq 4$, we must have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0, 1, 2\}$. Without loss of generality, we may assume that $u_0 \in U_\eta$. We cannot have $u_0 \overset{\eta}{-} u_i$ for any $i \in \{1, 2\}$, since otherwise we would have a monochromatic triangle. But then we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, z, v_0, v_1, v_2\}$, so that, since $u_0 \in U_\eta$, we must have $u_0 \overset{\eta}{-} v_0, v_1, v_2$, which contradicts Lemma 6(1) since $y \overset{\eta}{-} v_0, v_1, v_2$. \square

3.6. Attaching Sets of Cardinality 10.

Proposition 9. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 10$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 4, 0] \cup [5, 2, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 10 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 4, 0] \cup [5, 3, 1] \cup [5, 2, 2] \cup [4, 4, 2] \cup [4, 3, 2]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 6. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 10$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 10$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \{x \in S_\delta(u) \cup S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5\}.$$

By Proposition 9, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 4$ for some η . Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges. The elements of U_η all have degree 5 in this graph.

Clearly, not all of the edges in U_η can be of color η . Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η . By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But $\|S_\delta(u) \cap S_\eta(x)\| \leq 5 = \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\|$, since $x \in U_\eta$, so that $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$. Thus, we actually have

$$\begin{aligned} \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| &= \|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| \\ &= 2. \end{aligned}$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{-} w_j$ or $v_i \overset{\eta}{-} w_j$ for any $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $w_0 \overset{\eta}{-} w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) = \{x, y\}$ for all $i \in \{0, 1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2\}$ with $\|U_\eta\| \geq 4$, we must have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0, 1, 2\}$. Without loss of generality, we may assume that $u_0 \in U_\eta$. We cannot have $u_0 \overset{\eta}{-} u_i$ for any $i \in \{1, 2\}$, since otherwise we would have a monochromatic triangle. But now we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, v_0, v_1, v_2\}$, which contradicts the fact that $u_0 \in U_\eta$. \square

3.7. Attaching Sets of Cardinality 9.

Proposition 10. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 9$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 3, 0] \cup [5, 2, 1] \cup [4, 2, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 9 = \|S_\delta(u) \cap S_\delta(v)\| &= 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &+ \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 3, 0] \cup [5, 2, 1] \cup [4, 4, 0] \cup [4, 3, 1] \cup [4, 2, 2] \cup [3, 3, 2]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 7. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 9$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 9$. For any $\eta \in \{\alpha, \beta, \gamma\}$ we define

$$U_\eta = \{x \in S_\delta(u) \cup S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \geq 4\}$$

and

$$U'_\eta = \{x \in S_\delta(u) \cup S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 5\}$$

By Proposition 10, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \cup U_\beta \cup U_\gamma,$$

so that we must have $\|U_\eta\| \geq 3$ for some η . Fix such an $\eta \in \{\alpha, \beta, \gamma\}$. We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges.

The elements of U_η all have degree either 4 or 5 and the elements of $U'_\eta \subseteq U_\eta$ all have degree 5 in this graph.

Clearly, not all of the edges in U_η can be of color η . Choose distinct $x, y \in U_\eta$ such that the edge from x to y is not of color η .

First, we show that either $x \notin U'_\eta$ or $y \notin U'_\eta$. Suppose not. Then $x, y \in U'_\eta$. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But $\|S_\delta(u) \cap S_\eta(x)\| \leq 5 = \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\|$, since $x \in U_\eta$, so that $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$. Thus, we actually have

$$\begin{aligned} \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| &= \|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| \\ &= 2. \end{aligned}$$

Thus, we may suppose that there are pairwise distinct

$$x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1, v_2 \in S_\delta(u) \cap S_\delta(v),$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0, w_1\}.$$

But this is clearly impossible, since $S_\delta(u) \cap S_\delta(v)$ has only 9 elements. Thus, we have shown that either $x \notin U'_\eta$ or $y \notin U'_\eta$, as desired.

Next, we show that $x, y \notin U'_\eta$. Suppose not. Then we may assume, without loss of generality, that $x \in U'_\eta$ and $y \in U_\eta \sim U'_\eta$. By Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But $\|S_\delta(u) \cap S_\eta(x)\| \leq 5 = \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\|$, since $x \in U_\eta$, so that $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = S_\delta(u) \cap S_\eta(x)$. Thus, we actually have

$$\begin{aligned} \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| &= \|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| \\ &= 2. \end{aligned}$$

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, w_1, v_0, v_1\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, w_0, w_1\}.$$

Clearly, we cannot have $u_i \overset{\eta}{-} w_j$ for any $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $v_i \overset{\eta}{-} w_j$ for any $i \in \{0, 1\}$ and $j \in \{0, 1\}$, since we would then have a monochromatic triangle. Likewise, we cannot have $w_0 \overset{\eta}{-} w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) = \{x, y\}$ for all $i \in \{0, 1\}$, so that $w_0, w_1 \notin U_\eta$. Now, since $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1\}$ with

$\|U_\eta\| \geq 3$, we must have either $u_i \in U_\eta$ for some $i \in \{0, 1, 2\}$ or $v_i \in U_\eta$ for some $i \in \{0, 1\}$. But, for any $i \in \{0, 1, 2\}$, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_i) \subseteq \{x, v_0, v_1\}$, so that $u_i \notin U_\eta$. Thus, we must have $v_i \in U_\eta$ for some $i \in \{0, 1\}$. Without loss of generality, we may assume that $v_0 \in U_\eta$. But then we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(v_0) \subseteq \{y, u_0, u_1, u_2\}$ so that, since $v_0 \in U_\eta$, we must have $v_0 \stackrel{\eta}{\sim} u_0, u_1, u_2$, which contradicts Lemma 6(1) since $x \stackrel{\eta}{\sim} u_0, u_1, u_2$. Thus, we have shown that $x, y \notin U'_\eta$, as desired.

Now, by Lemma 6(1), we see that

$$\|S_\delta(u) \cap S_\eta(x) \cap S_\eta(y)\| = 2.$$

But, $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = \|S_\delta(u) \cap S_\eta(x)\| - 1$, since $\|S_\delta(u) \cap S_\eta(x)\| = 5$ by Proposition 2 and $y \in U_\eta \sim U'_\eta$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| \in \{1, 2\}.$$

Next, we show that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2$. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 1$, so that we may assume that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, u_2, w_0, v_0, v_1, v_2\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, u_2, w_0\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, v_2, w_0\}.$$

We cannot have $w_0 \stackrel{\eta}{\sim} u_i$ or $w_0 \stackrel{\eta}{\sim} v_i$ for any $i \in \{0, 1, 2\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0) = \{x, y\}$, so that $U_\eta \subseteq \{x, y, u_0, u_1, u_2, v_0, v_1, v_2\}$. Since $\|U_\eta\| \geq 3$, we have either $u_i \in U_\eta$ or $v_i \in U_\eta$ for some $i \in \{0, 1, 2\}$. Without loss of generality, we may suppose that $u_0 \in U_\eta$. We cannot have $u_0 \stackrel{\eta}{\sim} u_i$ for any $i \in \{1, 2\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(u_0) \subseteq \{x, v_0, v_1, v_2\}$, which implies that $u_0 \stackrel{\eta}{\sim} v_0, v_1, v_2$ since $u_0 \in U_\eta$. But this is impossible, by Lemma 6(1), since $y \stackrel{\eta}{\sim} v_0, v_1, v_2$. Thus, we have shown that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) \cap S_\eta(y)\| = 2,$$

as desired.

Thus, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{x, y, u_0, u_1, w_0, w_1, v_0, v_1, z\},$$

where

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(x) = \{u_0, u_1, w_0, w_1\}$$

and

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(y) = \{v_0, v_1, w_0, w_1\}.$$

Clearly, we cannot have $u_i \stackrel{\eta}{\sim} w_j$ for any $i \in \{0, 1\}$ and $j \in \{0, 1\}$, since otherwise we would have a monochromatic triangle. Similarly, we cannot have $v_i \stackrel{\eta}{\sim} w_j$

for any $i \in \{0, 1\}$ and $j \in \{0, 1\}$, since we would then have a monochromatic triangle. Likewise, we cannot have $w_0 \overset{\eta}{\sim} w_1$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_i) \subseteq \{x, y, z\}$ for all $i \in \{0, 1\}$, so that $w_0, w_1 \notin U_\eta$. Since $x \overset{\eta}{\sim} w_0 \overset{\eta}{\sim} y \overset{\eta}{\sim} w_1 \overset{\eta}{\sim} x$, we see by Lemma 8 that either $z \overset{\eta}{\sim} w_0$ or $z \overset{\eta}{\sim} w_1$ but not both. Without loss of generality, we may suppose that $z \overset{\eta}{\sim} w_0$. Thus, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0) = \{x, y, z\},$$

so that, by Proposition 10, we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_0)\| = 5$$

for some $\xi \in \{\alpha, \beta, \gamma\} \sim \{\eta\}$. But, $S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_0) \subseteq \{u_0, u_1, w_1, v_0, v_1\}$, so that we must have

$$w_1, v_0, v_1 \in S_\delta(u) \cap S_\eta(y) \cap S_\xi(w_0),$$

which is impossible, since

$$\|S_\delta(u) \cap S_\eta(y) \cap S_\xi(w_0)\| \leq R(3; 2) - 1 = 3 - 1 = 2.$$

The proof is complete. □

3.8. Attaching Sets of Cardinality 8.

Proposition 11. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 8$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 2, 0] \cup [3, 2, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 8 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} & (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ & \in [5, 2, 0] \cup [5, 1, 1] \cup [4, 3, 0] \cup [4, 2, 1] \cup [3, 3, 1] \cup [3, 2, 2]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Proposition 12. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 8$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} & (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ & \in [3, 2, 2]. \end{aligned}$$

Proof. Suppose not. Then, by Proposition 11, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3, w_4\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y_0, y_1\}$.

For each $i \in \{0, 1, 2, 3, 4\}$, we define

$$n_i = \|\{j \in \{0, 1\} \mid w_i \overset{\alpha}{-} y_j\}\|.$$

For each $j \in \{0, 1\}$, we define

$$m_j = \|\{i \in \{0, 1, 2, 3, 4\} \mid w_i \overset{\alpha}{-} y_j\}\|.$$

Note that we cannot have $w_i \overset{\alpha}{-} w_{i'}$ for any distinct $i, i' \in \{0, 1, 2, 3, 4\}$, since otherwise we would have a monochromatic triangle. Thus, we have $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i) \subseteq \{x, y_0, y_1\}$ for any $i \in \{0, 1, 2, 3, 4\}$. By Proposition 11, we see that, for any $i \in \{0, 1, 2, 3, 4\}$, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i)\| \in \{0, 2, 3, 5\}$, so that $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(w_i)\| \geq 2$, since $w_i \overset{\alpha}{-} x$. Thus, for any $i \in \{0, 1, 2, 3, 4\}$, we have $w_i \overset{\alpha}{-} y_j$ for some $j \in \{0, 1\}$, so that $n_i \geq 1$.

Thus, we have

$$m_0 + m_1 = n_0 + n_1 + n_2 + n_3 + n_4 \geq 1 + 1 + 1 + 1 + 1 = 5,$$

so that we must have $m_j \geq 3$ for some $j \in \{0, 1\}$. Without loss of generality, we may suppose that $y_0 \overset{\alpha}{-} w_0, w_1, w_2$. But this is impossible, by Lemma 6(1), since $x \overset{\alpha}{-} w_0, w_1, w_2$. The proposition follows. \square

Theorem 8. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 8$.*

Proof. By Proposition 12, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \in \{2, 3\}$ for any $x \in S_\delta(u) \cap S_\delta(v)$ and any $\eta \in \{\alpha, \beta, \gamma\}$. For each $\eta \in \{\alpha, \beta, \gamma\}$, we define

$$U_\eta = \{x \in S_\delta(u) \cap S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 3\}.$$

By Proposition 12, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \uplus U_\beta \uplus U_\gamma.$$

Note that the number of edges of color $\eta \in \{\alpha, \beta, \gamma\}$ in $S_\delta(u) \cap S_\delta(v)$ is

$$\frac{3 \cdot \|U_\eta\| + 2 \cdot (8 - \|U_\eta\|)}{2}.$$

It follows that $\|U_\eta\|$ is even for every $\eta \in \{\alpha, \beta, \gamma\}$. Thus, we have

$$(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|) \in [8, 0, 0] \cup [6, 2, 0] \cup [4, 4, 0] \cup [4, 2, 2].$$

Thus, there exists some $\eta \in \{\alpha, \beta, \gamma\}$ such that

$$\|U_\eta\| \in \{0, 2\}.$$

Fix such an η . We will consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the η colored edges.

If $\|U_\eta\| = 0$, then the graph under consideration is a triangle free graph with 8 vertices which all have degree 2. Such a graph has exactly 8 edges. There are two such possibilities. The first possibility is

$$w_0 \overset{\eta}{-} w_1 \overset{\eta}{-} w_2 \overset{\eta}{-} w_3 \overset{\eta}{-} w_4 \overset{\eta}{-} w_5 \overset{\eta}{-} w_6 \overset{\eta}{-} w_7 \overset{\eta}{-} w_0$$

where $S_\delta(u) \cap S_\delta(v) = \{w_0, \dots, w_7\}$, which fails by Lemma 7 by consideration of $w_0 \overset{\eta}{-} w_1 \overset{\eta}{-} w_2$ and $w_4 \overset{\eta}{-} w_5$. The second possibility is

$$x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} x_2 \overset{\eta}{-} x_3 \overset{\eta}{-} x_0$$

and

$$y_0 \overset{\eta}{-} y_1 \overset{\eta}{-} y_2 \overset{\eta}{-} y_3 \overset{\eta}{-} y_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}$, which fails by Lemma 7 by consideration of $x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} x_2$ and $y_0 \overset{\eta}{-} y_1$. Thus, we cannot have $\|U_\eta\| = 0$.

Therefore, we must have $\|U_\eta\| = 2$. Thus, the graph under consideration is a triangle free graph with 8 vertices, 2 of which have degree 3, and 6 of which have degree 2. Such a graph has exactly 9 edges. Let $U_\eta = \{a, b\}$.

If $a \overset{\eta}{-} b$, there are two possibilities for the remaining 8 edges. The first possibility is

$$a \overset{\eta}{-} x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} y_0 \overset{\eta}{-} y_1 \overset{\eta}{-} y_2 \overset{\eta}{-} y_3 \overset{\eta}{-} b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, y_0, y_1, y_2, y_3\}$, which fails by Lemma 7, by consideration of $a \overset{\eta}{-} x_0 \overset{\eta}{-} x_1$ and $y_1 \overset{\eta}{-} y_2$. The second possibility is

$$a \overset{\eta}{-} x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} x_2 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} y_0 \overset{\eta}{-} y_1 \overset{\eta}{-} y_2 \overset{\eta}{-} b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, x_2, y_0, y_1, y_2\}$, which fails by Lemma 7 by consideration of $x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} x_2$ and $y_0 \overset{\eta}{-} y_1$.

Thus, we cannot have $a \overset{\eta}{-} b$. Here there are three possibilities. The first possibility is

$$a \overset{\eta}{-} x_0 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} y_0 \overset{\eta}{-} y_1 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} z_0 \overset{\eta}{-} z_1 \overset{\eta}{-} z_2 \overset{\eta}{-} b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, y_0, y_1, z_0, z_1, z_2\}$, which fails by Lemma 7 by consideration of $a \overset{\eta}{-} y_0 \overset{\eta}{-} y_1$ and $z_1 \overset{\eta}{-} z_2$. The second possibility is

$$a \overset{\eta}{-} x_0 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} y_0 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} z_0 \overset{\eta}{-} z_1 \overset{\eta}{-} z_2 \overset{\eta}{-} z_3 \overset{\eta}{-} b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, y_0, z_0, z_1, z_2, z_3\}$, which fails by Lemma 7 by consideration of $z_0 \overset{\eta}{-} z_1 \overset{\eta}{-} z_2$ and $x_0 \overset{\eta}{-} b$. The third possibility is

$$a \overset{\eta}{-} x_0 \overset{\eta}{-} x_1 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} y_0 \overset{\eta}{-} y_1 \overset{\eta}{-} b$$

and

$$a \overset{\eta}{-} z_0 \overset{\eta}{-} z_1 \overset{\eta}{-} b,$$

where $S_\delta(u) \cap S_\delta(v) = \{a, b, x_0, x_1, y_0, y_1, z_0, z_1\}$, which fails by Lemma 7 by consideration of $x_0 \overset{\eta}{-} a \overset{\eta}{-} y_0$ and $z_1 \overset{\eta}{-} b$.

The proof is complete \square

3.9. Attaching Sets of Cardinality 7.

Proposition 13. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 1, 0] \cup [4, 2, 0] \cup [2, 2, 2]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) &= \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ &\quad \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 7 = \|S_\delta(u) \cap S_\delta(v)\| &= 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &\quad + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 1, 0] \cup [4, 2, 0] \cup [4, 1, 1] \cup [3, 3, 0] \cup [3, 2, 1] \cup [2, 2, 2]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Proposition 14. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [4, 2, 0] \cup [2, 2, 2]. \end{aligned}$$

Proof. Suppose not. Then, by Proposition 13, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 1,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3, w_4\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y\}$.

By Lemma 6(3), we see that

$$\|S_\delta(u) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3, 4\}$ and $\eta \in \{\beta, \alpha\}$. But $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = S_\delta(u) \cap S_\alpha(x)$ since $\|S_\delta(u) \cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3, 4\}$ and $\eta \in \{\beta, \gamma\}$. Since $x \overset{\alpha}{\sim} w_i$ for all $i \in \{0, 1, 2, 3, 4\}$, we must have $w_i \overset{\alpha}{\sim} y$, by Proposition 13. But now we have $y \overset{\alpha}{\sim} w_0, w_1, w_2, w_3, w_4$, which is impossible by Lemma 6(3), since $x \overset{\alpha}{\sim} w_0, w_1, w_2, w_3, w_4$.

The proof is complete \square

Proposition 15. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 7$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 2, 2]. \end{aligned}$$

Proof. Suppose not. Then, by Proposition 14, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 4,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

Let $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{w_0, w_1, w_2, w_3\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{y_0, y_1\}$.

By Lemma 6(3), we see that

$$\|S_\delta(u) \cap S_\alpha(x) \cap S_\eta(w_i)\| = 2$$

for any $i \in \{0, 1, 2, 3\}$ and $\eta \in \{\beta, \alpha\}$.

But $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = \|S_\delta(u) \cap S_\alpha(x)\| - 1$ since $\|S_\delta(u) \cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 4$. Thus, we actually have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) \cap S_\eta(w_i)\| \in \{1, 2\}$$

for any $i \in \{0, 1, 2, 3\}$ and $\eta \in \{\beta, \gamma\}$. Since $x \overset{\alpha}{\sim} w_i$ for all $i \in \{0, 1, 2, 3\}$, we must have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_i)\| \neq 0$$

for all $i \in \{0, 1, 2, 3\}$ and all $\xi \in \{\alpha, \beta, \gamma\}$. Thus, by Proposition 14, we see that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\xi(w_i)\| = 2$$

for all $i \in \{0, 1, 2, 3\}$ and all $\xi \in \{\alpha, \beta, \gamma\}$.

Now, note that we cannot have $y_0 \overset{\beta}{\sim} y_1$, since then we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \subseteq \{w_0, w_1, w_2, w_3, x\}$$

for all $i \in \{0, 1\}$. Now, suppose that $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| \neq 2$ for some $i \in \{0, 1\}$. Then, since $y_i \overset{\beta}{\sim} x$, we must have, by Proposition 14, $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| = 4$. But then $S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \cap S_\alpha(x) = 3$. But, this is

impossible, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i) \cap S_\alpha(x)\| \leq R(3; 2) - 1 = 3 - 1 = 2$. Thus, we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(y_i)\| = 2.$$

In fact, we have now shown that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(z)\| = 2$$

for all $z \in S_\delta(u) \cap S_\delta(v)$.

Now, we consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the β colored edges. This graph is a triangle free graph on 7 vertices, all of which have degree 7. Such a graph has exactly 7 edges. There is only one possibility. We must have

$$z_0 \xrightarrow{\beta} z_1 \xrightarrow{\beta} z_2 \xrightarrow{\beta} z_3 \xrightarrow{\beta} z_4 \xrightarrow{\beta} z_5 \xrightarrow{\beta} z_6 \xrightarrow{\beta} z_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{z_0, \dots, z_6\}$, which fails, by Lemma 7, by consideration of $z_0 \xrightarrow{\beta} z_1 \xrightarrow{\beta} z_2$ and $z_4 \xrightarrow{\beta} z_5$. The proof is complete. \square

Theorem 9. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 7$.*

Proof. Note that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(z)\| = 2$$

for all $z \in S_\delta(u) \cap S_\delta(v)$, by Proposition 15. Consider the graph with vertex set $S_\delta(u) \cap S_\delta(v)$ whose edges are the β colored edges. This graph is a triangle free graph on 7 vertices, all of which have degree 2. Such a graph has exactly 7 edges. There is only one possibility. We must have

$$z_0 \xrightarrow{\beta} z_1 \xrightarrow{\beta} z_2 \xrightarrow{\beta} z_3 \xrightarrow{\beta} z_4 \xrightarrow{\beta} z_5 \xrightarrow{\beta} z_6 \xrightarrow{\beta} z_0,$$

where $S_\delta(u) \cap S_\delta(v) = \{z_0, \dots, z_6\}$, which fails, by Lemma 7, by consideration of $z_0 \xrightarrow{\beta} z_1 \xrightarrow{\beta} z_2$ and $z_4 \xrightarrow{\beta} z_5$.

The proof is complete. \square

3.10. Attaching Sets of Cardinality 6.

Proposition 16. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 6$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 0, 0] \cup [3, 2, 0] \cup [2, 2, 1]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) &= \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ &\quad \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 6 &= \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &\quad + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [5, 0, 0] \cup [4, 1, 0] \cup [3, 2, 0] \cup [2, 2, 1]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Proposition 17. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 6$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [3, 2, 0] \cup [2, 2, 1]. \end{aligned}$$

Proof. Suppose not. Then, by Proposition 16, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\begin{aligned} \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| &= 5, \\ \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| &= 0, \end{aligned}$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

By Proposition 1, either $\|S_\beta(x)\| \geq 15$ or $\|S_\gamma(x)\| \geq 15$. Without loss of generality, we may suppose that $\|S_\gamma(x)\| \geq 15$. By Proposition 2, we have $\|S_\delta(u) \cap S_\gamma(x)\| = 5$ and $\|S_\delta(v) \cap S_\gamma(x)\| = 5$. Thus, by Lemma 16, we see that there exist $x_0, x_1, x_2, x_3, x_4 \in S_\delta(u) \cap S_\gamma(x)$ and $y_0, y_1, y_2, y_3, y_4 \in S_\delta(v) \cap S_\gamma(x)$ such that

$$M_{\beta, \alpha, \delta}^i(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3, y_4)$$

for some $i \in \{0, 1, 2\}$. Note also that $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = S_\delta(u) \cap S_\alpha(x)$, since $\|S_\delta(u) \cap S_\alpha(x)\| = 5$ by Proposition 2 and $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 5$.

First, suppose that $i = 0$. Note that $x \xrightarrow{\gamma} x_0 \xrightarrow{\beta} x_4 \xrightarrow{\gamma} x$ with $x, x_0, x_4 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \xrightarrow{\alpha} x, x_0, x_4$. Since $w \xrightarrow{\alpha} x_0 \xrightarrow{\alpha} y_3, y_4$ and $w \xrightarrow{\alpha} x_4 \xrightarrow{\alpha} y_0, y_1$, we have $S_\delta(v) \cap S_\gamma(x) \cap$

$S_\alpha(w) \subseteq \{y_2\}$. But, since $w \in S_\delta(u) \cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w)\| \geq 2$, which is impossible. Thus, $i \neq 0$.

Next, suppose that $i = 1$. Note that $x \xrightarrow{\gamma} x_0 \xrightarrow{\beta} x_1 \xrightarrow{\gamma} x$ with $x, x_0, x_1 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \xrightarrow{\alpha} x, x_0, x_1$. Since $w \xrightarrow{\alpha} x_0 \xrightarrow{\alpha} y_2, y_3$ and $w \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} y_0, y_4$, we have $S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w) \subseteq \{y_1\}$. But, since $w \in S_\delta(u) \cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w)\| \geq 2$, which is impossible. Thus, $i \neq 1$.

First, suppose that $i = 2$. Note that $x \xrightarrow{\gamma} x_2 \xrightarrow{\beta} x_3 \xrightarrow{\gamma} x$ with $x, x_2, x_3 \in S_\delta(u)$. Thus, by Lemma 17, there exists some $w \in S_\delta(u)$ with $w \xrightarrow{\alpha} x, x_2, x_3$. Since $w \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} y_3, y_4$ and $w \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} y_1$, we have $S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w) \subseteq \{y_0, y_2\}$. But, since $w \in S_\delta(u) \cap S_\alpha(x) \subseteq S_\delta(v)$, we see by Lemma 6(3), that $\|S_\delta(v) \cap S_\gamma(x) \cap S_\alpha(w)\| \geq 2$, which implies that $w \xrightarrow{\alpha} y_0, y_2$. But this gives a monochromatic triangle, since $y_0 \xrightarrow{\alpha} y_2$. Thus, $i \neq 2$.

The proof is complete. \square

Proposition 18. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose that $a, b, c, d \in S_\delta(u) \cap S_\delta(v)$. $a \xrightarrow{\alpha} b \xrightarrow{\alpha} c \xrightarrow{\alpha} d \xrightarrow{\alpha} a$ and $a \xrightarrow{\beta} c$ and $b \xrightarrow{\beta} d$. Then $\|S_\delta(u) \cap S_\delta(v)\| \notin \{4, 6\}$*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| \in \{4, 6\}$.

First, we show that there exists some $x \in \{a, b, c, d\}$ with $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 2$. Suppose not. Then there exist $a', b', c', d' \in S_\delta(u) \cap S_\delta(v) \sim \{a, b, c, d\}$ such that $a \xrightarrow{\alpha} a', b \xrightarrow{\alpha} b', c \xrightarrow{\alpha} c',$ and $d \xrightarrow{\alpha} d'$. By Lemma 8, we have $\|\{a, b, c, d, a', b', c', d'\}\| = 8$, which contradicts the fact that $\|S_\delta(u) \cap S_\delta(v)\| \in \{4, 6\}$. Thus, there exists some $x \in \{a, b, c, d\}$ with $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 2$, as desired.

Without loss of generality, we may suppose that $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 2$. By Lemma 4, there exist $x_0, x_1, x_2 \in S_\delta(u) \cap S_\alpha(a)$ and $y_0, y_1, y_2 \in S_\delta(v) \cap S_\alpha(a)$ such that both $P_{\beta, \gamma}(b, x_0, x_1, x_2, d)$ and $P_{\beta, \gamma}(b, y_0, y_1, y_2, d)$ hold.

Now, we show that $c \xrightarrow{\beta} x_0, x_2, y_0, y_2$. Suppose not. Without loss of generality, we may suppose that $c \xrightarrow{\beta} x_2$ fails. Since $c \xrightarrow{\alpha} b, d$, we must have $S_\delta(u) \cap S_\alpha(a) \cap S_\beta(c) \subseteq \{x_0, x_1\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\alpha(a) \cap S_\beta(c)\| = 2$, so that $c \xrightarrow{\beta} x_0, x_1$. But this gives a monochromatic triangle, since $x_0 \xrightarrow{\beta} x_1$. Thus, we have

$$c \xrightarrow{\beta} x_0, x_2, y_0, y_2,$$

as desired.

Now, we note the following:

Either $x_0 \xrightarrow{\gamma} y_2$ or $x_0 \xrightarrow{\delta} y_2$ since $x_0, y_2 \in S_\alpha(a) \cap S_\beta(c)$.

Either $x_2 \xrightarrow{\gamma} y_0$ or $x_2 \xrightarrow{\delta} y_0$ since $x_2, y_0 \in S_\alpha(a) \cap S_\beta(c)$.

Note further that $b, x_0, x_1, x_2, d, y_2, y_1, y_0 \in S_\alpha(a)$. Note also that the following hold:

$$\begin{aligned} \{x_0, x_1, x_2\} \cap \{y_2, y_1, y_0\} &= \emptyset \\ P_{\beta, \gamma}(b, x_0, x_1, x_2, d) \\ P_{\beta, \gamma}(d, y_2, y_1, y_0, b) \end{aligned}$$

Thus, we may apply Lemma 1212 with

$$(u_1, \dots, u_8) = (b, x_0, x_1, x_2, d, y_2, y_1, y_0)$$

to see that $\|S_\alpha(a)\| \leq 14$.

But this is easily seen to be impossible. If $\|S_\delta(u) \cap S_\delta(v)\| = 4$, then $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$, so that $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(a)\| = 1$ so that, by Proposition 3, we have $\|S_\beta(a)\| \leq 14$, which is impossible, by Proposition 4, since $\|S_\alpha(a)\| \leq 14$. If, on the other hand, we have $\|S_\delta(u) \cap S_\delta(v)\| = 6$, then we have, by Proposition 17, that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(a)\| \in \{1, 3\}$ for some $\eta \in \{\beta, \gamma\}$, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 2$. Thus, we have $\|S_\eta(a)\| \leq 14$ for some $\eta \in \{\beta, \gamma\}$, which is impossible, by Proposition 4, since $\|S_\alpha(a)\| \leq 14$.

The proof is complete. \square

Proposition 19. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose that $a, b, c, d \in S_\delta(u) \cap S_\delta(v)$. If $a \overset{\gamma}{-} b \overset{\beta}{-} c \overset{\beta}{-} d \overset{\beta}{-} a$ and $a \overset{\alpha}{-} c$ and $b \overset{\alpha}{-} d$, then $\|S_\delta(u) \cap S_\delta(v)\| \neq 6$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 6$. Since $a \overset{\beta}{-} d \overset{\beta}{-} c \overset{\beta}{-} b$, we have, by Proposition 17,

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| \in \{2, 3\}.$$

First, we show that either $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2$ or $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 2$. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 3$. Thus, there exists $x, y \in S_\delta(u) \cap S_\delta(v) \sim \{a, b, c, d\}$ with $S_\delta(u) \cap S_\delta(v) \cap S_\beta(c) = \{b, d, x\}$ and $S_\delta(u) \cap S_\delta(v) \cap S_\beta(d) = \{a, c, y\}$. Since $x \overset{\beta}{-} c \overset{\beta}{-} d \overset{\beta}{-} y$, we must have $x \neq y$. Thus, we have $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, x, y\}$. Since $c \overset{\beta}{-} x, b, d$ and $c \overset{\alpha}{-} a$, we see, by Proposition 17, that $c \overset{\alpha}{-} y$. Since $d \overset{\beta}{-} y, a, c$ and $d \overset{\alpha}{-} b$, we see, by Proposition 17, that $d \overset{\alpha}{-} x$. Note that $x \overset{\gamma}{-} b$ since $x, b \in S_\delta(u) \cap S_\alpha(d) \cap S_\beta(c)$, and that $y \overset{\gamma}{-} a$ since $y, a \in S_\delta(u) \cap S_\alpha(c) \cap S_\beta(d)$. We cannot have $b \overset{\beta}{-} y$, since, if we did, we would get a contradiction with Proposition 18, by consideration of $y \overset{\beta}{-} d \overset{\beta}{-} c \overset{\beta}{-} b \overset{\beta}{-} y$ and $d \overset{\alpha}{-} b$ and $y \overset{\alpha}{-} c$. Thus, we have $b \overset{\alpha}{-} y$, since $b, y \in S_\delta(u) \cap S_\gamma(a)$. Also, we cannot have $x \overset{\beta}{-} a$, since, if we did, we would get a contradiction with Proposition 18, by consideration of $x \overset{\beta}{-} c \overset{\beta}{-} d \overset{\beta}{-} a \overset{\beta}{-} x$ and $c \overset{\alpha}{-} a$ and $x \overset{\alpha}{-} d$. Thus, we have $a \overset{\alpha}{-} x$, since $a, x \in S_\delta(u) \cap S_\gamma(b)$. Now, we consider the color of the edge from

x to y . If $x \xrightarrow{\alpha} y$, we get a contradiction with Proposition 17, since then we would have $x \xrightarrow{\alpha} a, d, y$ and $x \xrightarrow{\beta} c$ and $x \xrightarrow{\gamma} b$. If $x \xrightarrow{\beta} y$, we get a contradiction with Proposition 18, by consideration of $x \xrightarrow{\beta} y \xrightarrow{\beta} d \xrightarrow{\beta} c \xrightarrow{\beta} x$ and $x \xrightarrow{\alpha} d$ and $y \xrightarrow{\alpha} c$. If $x \xrightarrow{\gamma} y$, we get a contradiction with Proposition 18, by consideration of $x \xrightarrow{\gamma} y \xrightarrow{\gamma} a \xrightarrow{\gamma} b \xrightarrow{\gamma} x$ and $x \xrightarrow{\alpha} a$ and $y \xrightarrow{\alpha} b$. Thus, we must have $x \xrightarrow{\delta} y$. But this gives a monochromatic triangle, since $x \xrightarrow{\delta} u \xrightarrow{\delta} y$. thus, we have shown that either $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2$ or $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(d)\| = 2$.

Without loss of generality, we may suppose that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2.$$

By Lemma 4, there exist $x_0, x_1, x_2 \in S_\delta(u) \cap S_\beta(c)$ and $y_0, y_1, y_2 \in S_\delta(v) \cap S_\beta(d)$ such that both $P_{\alpha, \gamma}(b, x_0, x_1, x_2, d)$ and $P_{\alpha, \gamma}(b, y_0, y_1, y_2, d)$ hold.

Now, we show that $a \xrightarrow{\alpha} x_0, x_2, y_0, y_2$. Suppose not. Without loss of generality, we may suppose that either $a \xrightarrow{\alpha} x_0$ fails or else $a \xrightarrow{\alpha} x_2$ fails.

Suppose that $a \xrightarrow{\alpha} x_0$ fails. Since $a \xrightarrow{\gamma} b$ and $a \xrightarrow{\beta} d$, we must have $S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a) \subseteq \{x_1, x_2\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a)\| = 2$, so that $a \xrightarrow{\alpha} x_1, x_2$. But this gives a monochromatic triangle, since $x_1 \xrightarrow{\alpha} x_2$.

Suppose that $a \xrightarrow{\alpha} x_2$ fails. Since $a \xrightarrow{\gamma} b$ and $a \xrightarrow{\beta} d$, we must have $S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a) \subseteq \{x_0, x_1\}$. By Lemma 6(3), we see that $\|S_\delta(u) \cap S_\beta(c) \cap S_\alpha(a)\| = 2$, so that $a \xrightarrow{\alpha} x_0, x_1$. But this gives a monochromatic triangle, since $x_0 \xrightarrow{\alpha} x_1$.

Thus, we have

$$a \xrightarrow{\alpha} x_0, x_2, y_0, y_2,$$

as desired.

Now, we note the following:

$$\text{Either } x_0 \xrightarrow{\gamma} y_2 \text{ or } x_0 \xrightarrow{\delta} y_2 \text{ since } x_0, y_2 \in S_\beta(c) \cap S_\alpha(a).$$

$$\text{Either } x_2 \xrightarrow{\gamma} y_0 \text{ or } x_2 \xrightarrow{\delta} y_0 \text{ since } x_2, y_0 \in S_\beta(c) \cap S_\alpha(a).$$

Note further that $b, x_0, x_1, x_2, d, y_2, y_1, y_0 \in S_\beta(c)$. Note also that the following hold:

$$\{x_0, x_1, x_2\} \cap \{y_2, y_1, y_0\} = \emptyset$$

$$P_{\alpha, \gamma}(b, x_0, x_1, x_2, d)$$

$$P_{\alpha, \gamma}(d, y_2, y_1, y_0, b)$$

Thus, we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, x_0, x_1, x_2, d, y_2, y_1, y_0)$$

to see that $\|S_\beta(c)\| \leq 14$.

But this is easily seen to be impossible. By Proposition 17, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(c)\| \in \{1, 3\}$ for some $\eta \in \{\alpha, \gamma\}$, since $\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(c)\| = 2$. Thus, we have $\|S_\eta(c)\| \leq 14$ for some $\eta \in \{\alpha, \gamma\}$, which is impossible, by Proposition 4, since $\|S_\beta(c)\| \leq 14$.

The proof is complete. \square

Proposition 20. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 6$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 2, 1]. \end{aligned}$$

Proof. Suppose not. Then, by Proposition 17, we may suppose, without loss of generality, that there exists some $x \in S_\delta(u) \cap S_\delta(v)$ such that

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| = 3,$$

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| = 2,$$

and

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\| = 0.$$

The induced coloring on the complete graph with vertex set $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)$ has three edges, each of which is colored with either β or γ . Thus, there exists some $y \in S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)$ such that both edges from y are the same color.

By Lemma 5, there exist $x_0, \dots, x_{15} \in S_\delta(u)$ such that one of the following cases hold:

- (1) $C_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_1, x_5\}$.
- (2) $C_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_2, x_4\}$.
- (3) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_1, x_5\}$.
- (4) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_3, x_2, x_4\}$.
- (5) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_2$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_2, x_4, x_5\}$.
- (6) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_2$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_2, x_1, x_3\}$.
- (7) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_1$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_1, x_3, x_4\}$.
- (8) $C_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ where $x = x_0$ and $y = x_1$ and $S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x) = \{x_1, x_2, x_5\}$.

Note that in all cases, we have

$$S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \subseteq \{x_6, x_7, x_8, x_9, x_{10}\}.$$

For case (1), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_1 \xrightarrow{\gamma} x_5$ and $x_3 \xrightarrow{\gamma} x_8$ and $x_1 \xrightarrow{\beta} x_8 \xrightarrow{\beta} x_5 \xrightarrow{\beta} x_3 \xrightarrow{\beta} x_1$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\beta} x_7$ and $x_1 \xrightarrow{\beta} x_3$ and $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_7 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$. Also, $x_9 \notin S_\delta(u) \cap S_\delta(v) \cap$

$S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\alpha} x_5 \xrightarrow{\alpha} x_9 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$ and $x_0 \xrightarrow{\beta} x_9$ and $x_5 \xrightarrow{\beta} x_3$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \xrightarrow{\beta} x_1, x_5, x_6, x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (1) lead to contradiction.

For case (2), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_8 \xrightarrow{\alpha} x_4 \xrightarrow{\alpha} x_0$ and $x_0 \xrightarrow{\beta} x_8$ and $x_2 \xrightarrow{\beta} x_4$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Lemma 8, by consideration of $x_0 \xrightarrow{\alpha} x_3$ and $x_6 \xrightarrow{\alpha} x_{10}$ and $x_0 \xrightarrow{\beta} x_{10} \xrightarrow{\beta} x_3 \xrightarrow{\beta} x_6 \xrightarrow{\beta} x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Lemma 7, by consideration of $x_4 \xrightarrow{\beta} x_2, x_6, x_7$ and $x_4 \xrightarrow{\alpha} x_0$ and $x_4 \xrightarrow{\gamma} x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Lemma 7, by consideration of $x_2 \xrightarrow{\beta} x_4, x_9, x_{10}$ and $x_2 \xrightarrow{\alpha} x_0$ and $x_2 \xrightarrow{\gamma} x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Lemma 7, by consideration of $x_6 \xrightarrow{\beta} x_0, x_3, x_4$ and $x_6 \xrightarrow{\alpha} x_2$ and $x_6 \xrightarrow{\gamma} x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_{10}\}$, by Lemma 7, by consideration of $x_{10} \xrightarrow{\beta} x_0, x_2, x_3$ and $x_{10} \xrightarrow{\alpha} x_4$ and $x_{10} \xrightarrow{\gamma} x_7$. Thus, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{x_7, x_9\}$ for case (2). We handle this situation below.

For case (3), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_1 \xrightarrow{\beta} x_8 \xrightarrow{\beta} x_5 \xrightarrow{\beta} x_3 \xrightarrow{\beta} x_1$ and $x_1 \xrightarrow{\gamma} x_5$ and $x_3 \xrightarrow{\gamma} x_8$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\beta} x_7$ and $x_1 \xrightarrow{\beta} x_3$ and $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_7 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$. Also, $x_9 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\alpha} x_5 \xrightarrow{\alpha} x_9 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$ and $x_0 \xrightarrow{\beta} x_9$ and $x_5 \xrightarrow{\beta} x_3$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \xrightarrow{\beta} x_1, x_5, x_6, x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (3) lead to contradiction.

For case (4), first note that $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_8 \xrightarrow{\alpha} x_4 \xrightarrow{\alpha} x_0$ and $x_0 \xrightarrow{\beta} x_8$ and $x_2 \xrightarrow{\beta} x_4$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Lemma 8, by consideration of $x_0 \xrightarrow{\alpha} x_3$ and $x_6 \xrightarrow{\alpha} x_{10}$ and $x_0 \xrightarrow{\beta} x_{10} \xrightarrow{\beta} x_3 \xrightarrow{\beta} x_6 \xrightarrow{\beta} x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Lemma 9, by consideration of $x_0 \xrightarrow{\alpha} x_2$ and $x_6 \xrightarrow{\alpha} x_7$ and $x_7 \xrightarrow{\gamma} x_2 \xrightarrow{\beta} x_6 \xrightarrow{\beta} x_0 \xrightarrow{\beta} x_7$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Lemma 9, by consideration of $x_9 \xrightarrow{\gamma} x_4 \xrightarrow{\beta} x_{10} \xrightarrow{\beta} x_0 \xrightarrow{\beta} x_9$ and $x_0 \xrightarrow{\alpha} x_4$ and $x_{10} \xrightarrow{\alpha} x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Lemma 7, by consideration of $x_6 \xrightarrow{\beta} x_0, x_3, x_4$ and $x_6 \xrightarrow{\alpha} x_2$ and $x_6 \xrightarrow{\gamma} x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_{10}\}$, by Lemma 7, by consideration of $x_{10} \xrightarrow{\alpha} x_4$ and $x_{10} \xrightarrow{\gamma} x_7$ and $x_{10} \xrightarrow{\beta} x_0, x_2, x_3$. Thus, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) = \{x_7, x_9\}$, for case (4). We handle this situation below.

For case (5), first note that $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_2 \xrightarrow{\beta} x_5 \xrightarrow{\beta} x_7 \xrightarrow{\beta} x_4 \xrightarrow{\beta} x_2$ and $x_2 \xrightarrow{\gamma} x_7$ and $x_4 \xrightarrow{\gamma} x_5$. Also, $x_8 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\beta} x_8$

and $x_2 \xrightarrow{\beta} x_4$ and $x_0 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_8 \xrightarrow{\alpha} x_4 \xrightarrow{\alpha} x_0$. Also, $x_{10} \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_{10} \xrightarrow{\gamma} x_5 \xrightarrow{\alpha} x_0 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_{10}$ and $x_0 \xrightarrow{\beta} x_{10}$ and $x_2 \xrightarrow{\beta} x_5$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_2 \xrightarrow{\beta} x_4, x_5, x_6, x_9$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (5) lead to contradiction.

For case (6), first note that $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_7 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$ and $x_0 \xrightarrow{\beta} x_7$ and $x_1 \xrightarrow{\beta} x_3$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_8\}$, by Proposition 19, by consideration of $x_2 \xrightarrow{\alpha} x_8 \xrightarrow{\gamma} x_6 \xrightarrow{\gamma} x_1 \xrightarrow{\gamma} x_2$ and $x_1 \xrightarrow{\beta} x_8$ and $x_2 \xrightarrow{\beta} x_6$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_9 \xrightarrow{\beta} x_0, x_1, x_2$ and $x_9 \xrightarrow{\alpha} x_3$ and $x_9 \xrightarrow{\gamma} x_6$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_{10}\}$, by Proposition 17, by consideration of $x_3 \xrightarrow{\beta} x_1, x_6, x_{10}$ and $x_3 \xrightarrow{\alpha} x_0$ and $x_3 \xrightarrow{\gamma} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 17, by consideration of $x_1 \xrightarrow{\beta} x_3, x_8, x_9$ and $x_1 \xrightarrow{\alpha} x_0$ and $x_1 \xrightarrow{\gamma} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_{10}\}$, by Proposition 19, by consideration of $x_1 \xrightarrow{\beta} x_8 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_{10} \xrightarrow{\alpha} x_1$ and $x_1 \xrightarrow{\gamma} x_2$ and $x_8 \xrightarrow{\gamma} x_{10}$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Proposition 19, by consideration of $x_2 \xrightarrow{\gamma} x_3 \xrightarrow{\alpha} x_9 \xrightarrow{\alpha} x_{10} \xrightarrow{\alpha} x_2$ and $x_2 \xrightarrow{\beta} x_9$ and $x_3 \xrightarrow{\beta} x_{10}$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (6) lead to contradiction.

For case (7), first note that $x_6 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_6 \xrightarrow{\alpha} x_4 \xrightarrow{\beta} x_1 \xrightarrow{\beta} x_3 \xrightarrow{\beta} x_6$ and $x_1 \xrightarrow{\gamma} x_6$ and $x_3 \xrightarrow{\gamma} x_4$. Also, $x_7 \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 18, by consideration of $x_0 \xrightarrow{\beta} x_7$ and $x_1 \xrightarrow{\beta} x_3$ and $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_7 \xrightarrow{\alpha} x_3 \xrightarrow{\alpha} x_0$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 17, by consideration of $x_1 \xrightarrow{\beta} x_3, x_4, x_8, x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_9, x_{10}\}$, by Proposition 19, by consideration of $x_0 \xrightarrow{\alpha} x_4$ and $x_9 \xrightarrow{\alpha} x_{10}$ and $x_4 \xrightarrow{\gamma} x_9 \xrightarrow{\beta} x_0 \xrightarrow{\beta} x_{10} \xrightarrow{\beta} x_4$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_{10}\}$, by Proposition 17, by consideration of $x_{10} \xrightarrow{\beta} x_0, x_3, x_4$ and $x_{10} \xrightarrow{\alpha} x_1$ and $x_{10} \xrightarrow{\gamma} x_8$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (7) lead to contradiction.

For case (8), first note that $x_{10} \notin S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$, by Proposition 19, by consideration of $x_5 \xrightarrow{\gamma} x_{10} \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_0 \xrightarrow{\alpha} x_5$ and $x_0 \xrightarrow{\beta} x_{10}$ and $x_5 \xrightarrow{\beta} x_2$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_7\}$, by Proposition 19, by consideration of $x_7 \xrightarrow{\gamma} x_2 \xrightarrow{\beta} x_6 \xrightarrow{\beta} x_0 \xrightarrow{\beta} x_7$ and $x_0 \xrightarrow{\alpha} x_2$ and $x_6 \xrightarrow{\alpha} x_7$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_8\}$, by Proposition 17, by consideration of $x_1 \xrightarrow{\gamma} x_2, x_5, x_6$ and $x_1 \xrightarrow{\alpha} x_0$ and $x_1 \xrightarrow{\beta} x_8$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_6, x_9\}$, by Proposition 17, by consideration of $x_1 \xrightarrow{\gamma} x_2, x_5, x_6$ and $x_1 \xrightarrow{\alpha} x_0$ and $x_1 \xrightarrow{\beta} x_9$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_7, x_8\}$, by Proposition 17, by consideration of $x_5 \xrightarrow{\alpha} x_0$ and $x_5 \xrightarrow{\gamma} x_1$ and $x_5 \xrightarrow{\beta} x_2, x_7, x_8$. Also, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq$

$\{x_7, x_9\}$, by Proposition 17, by consideration of $x_9 \xrightarrow{\beta} x_0, x_1, x_2$ and $x_9 \xrightarrow{\alpha} x_5$ and $x_9 \xrightarrow{\gamma} x_7$. Finally, $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x) \neq \{x_8, x_9\}$, by Proposition 18, by consideration of $x_0 \xrightarrow{\beta} x_8 \xrightarrow{\beta} x_1 \xrightarrow{\beta} x_9 \xrightarrow{\beta} x_0$ and $x_0 \xrightarrow{\alpha} x_1$ and $x_8 \xrightarrow{\alpha} x_9$. Thus, all possibilities for $S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)$ in case (8) lead to contradiction.

We have thus shown that for some $i \in \{0, 1\}$, there exist $x_0, \dots, x_{15} \in S_\delta(u)$ with $C_{\alpha, \beta, \gamma}^i(x_0, \dots, x_{15})$ such that $x = x_0$ and $y = x_3$ and $S_\delta(u) \cap S_\delta(v) = \{x_0, x_2, x_3, x_4, x_7, x_9\}$.

Switching the roles of u and v in this argument shows that for some $j \in \{0, 1\}$, there exist $y_0, \dots, y_{15} \in S_\delta(v)$ with $C_{\alpha, \beta, \gamma}^j(y_0, \dots, y_{15})$ such that $x = y_0$ and $y = y_3$ and $S_\delta(u) \cap S_\delta(v) = \{y_0, y_2, y_3, y_4, y_7, y_9\}$.

Since $x = x_0 = y_0$, there are two possibilities. The first is

$$x_0 = y_0$$

$$x_2 = y_2$$

$$x_3 = y_3$$

$$x_4 = y_4$$

$$x_7 = y_7$$

$$x_9 = y_9$$

The second is

$$x_0 = y_0$$

$$x_2 = y_4$$

$$x_3 = y_3$$

$$x_4 = y_2$$

$$x_7 = y_9$$

$$x_9 = y_7$$

Actually, these possibilities represent isomorphic situations. Thus, we may assume, without loss of generality, that the former possibility holds. As luck (and a clever choice of notation!) would have it, we may ignore the indices i and j . We will use only edges where the twistedness or untwistedness does not matter. Let $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e, f\}$, where the following hold:

$$a = x_0 = y_0$$

$$b = x_2 = y_2$$

$$c = x_3 = y_3$$

$$d = x_4 = y_4$$

$$e = x_7 = y_7$$

$$f = x_9 = y_9$$

Now, we note the following:

$$x_{11} \xrightarrow{\delta} y_{11} \text{ since } x_{11}, y_{11} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(e).$$

$$x_{12} \xrightarrow{\delta} y_{12} \text{ since } x_{12}, y_{12} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_{14} \xrightarrow{\delta} y_{14} \text{ since } x_{14}, y_{14} \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(d).$$

$$x_{15} \xrightarrow{\delta} y_{15} \text{ since } x_{15}, y_{15} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(f).$$

$$\text{Either } x_{11} \xrightarrow{\beta} y_{12} \text{ or } x_{11} \xrightarrow{\delta} y_{12} \text{ since } x_{11}, y_{12} \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_{12} \xrightarrow{\beta} y_{11} \text{ or } x_{12} \xrightarrow{\delta} y_{11} \text{ since } x_{12}, y_{11} \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_{14} \xrightarrow{\beta} y_{15} \text{ or } x_{14} \xrightarrow{\delta} y_{15} \text{ since } x_{14}, y_{15} \in S_\gamma(a) \cap S_\alpha(b).$$

$$\text{Either } x_{15} \xrightarrow{\beta} y_{14} \text{ or } x_{15} \xrightarrow{\delta} y_{14} \text{ since } x_{15}, y_{14} \in S_\gamma(a) \cap S_\alpha(b).$$

$$\text{Either } x_{11} \xrightarrow{\beta} y_{15} \text{ or } x_{11} \xrightarrow{\delta} y_{15} \text{ since } x_{11}, y_{15} \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_{15} \xrightarrow{\beta} y_{11} \text{ or } x_{15} \xrightarrow{\delta} y_{11} \text{ since } x_{15}, y_{11} \in S_\gamma(a) \cap S_\alpha(c).$$

Note further that $x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15} \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\} &= \emptyset \\ P_{\beta, \alpha}(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}) & \\ P_{\beta, \alpha}(y_{11}, y_{12}, y_{13}, y_{14}, y_{15}) & \end{aligned}$$

Since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(a)\| = 3$, we see, by Proposition 3, that $\|S_\alpha(a)\| \leq 14$, so that, by Proposition 1, we have

$$\|S_\gamma(a)\| \geq 15.$$

Thus, we may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{11}, y_{12}, y_{13}, y_{14}, y_{15})$$

to see that

$$\begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^0(x_{12}, x_{11}, x_{15}, x_{14}, x_{13}, y_{12}, y_{11}, y_{15}, y_{14}, y_{13}) \\ &\text{or } M_{\beta, \alpha, \delta}^0(x_{13}, x_{12}, x_{11}, x_{15}, x_{14}, y_{13}, y_{12}, y_{11}, y_{15}, y_{14}) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_{15}, x_{14}, x_{13}, x_{12}, x_{11}, y_{15}, y_{14}, y_{13}, y_{12}, y_{11}). \end{aligned}$$

Thus, we see that

$$x_{12} \xrightarrow{\alpha} y_{14} \text{ and } x_{14} \xrightarrow{\alpha} y_{12}.$$

Finally, note that $d, y_{14}, y_8, y_{12}, b, x_{12}, x_8, x_{14} \in S_\gamma(c)$. Note also that the following hold:

$$\begin{aligned} \{y_{14}, y_8, y_{12}\} \cap \{x_{12}, x_8, x_{14}\} &= \emptyset \\ P_{\beta, \alpha}(d, y_{14}, y_8, y_{12}, b) \\ P_{\beta, \alpha}(b, x_{12}, x_8, x_{14}, d) \end{aligned}$$

Since $\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(c)\| = 3$, we see, by Proposition 3, that $\|S_\alpha(c)\| \leq 14$, so that, by Proposition 1, we have

$$\|S_\gamma(c)\| \geq 15.$$

Thus, we may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (d, y_{14}, y_8, y_{12}, b, x_{12}, x_8, x_{14})$$

to produce the desired contradiction. The proof is complete. \square

Theorem 10. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 6$.*

Proof. For any $\eta \in \{\alpha, \beta, \gamma\}$, we define

$$U_\eta = \{x \in S_\delta(u) \cap S_\delta(v) \mid \|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 1\}.$$

By Proposition 20, we see that

$$S_\delta(u) \cap S_\delta(v) = U_\alpha \uplus U_\beta \uplus U_\gamma.$$

Thus, we have

$$6 = \|U_\alpha\| + \|U_\beta\| + \|U_\gamma\|.$$

Note that the number of η colored edges in the induced coloring on the complete graph with vertex set $S_\delta(u) \cap S_\delta(v)$ is

$$\frac{\|U_\eta\| + 2 \cdot (6 - \|U_\eta\|)}{2}.$$

Thus, $\|U_\eta\|$ is even for each $\eta \in \{\alpha, \beta, \gamma\}$. It follows that

$$(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|) \in [6, 0, 0] \cup [4, 2, 0] \cup [2, 2, 2].$$

Next, we show that $\|U_\eta\| \neq 2$ for all $\eta \in \{\alpha, \beta, \gamma\}$. Suppose not. Then $\|U_\eta\| = 2$ for some $\eta \in \{\alpha, \beta, \gamma\}$. We consider the graph whose edges are the η colored edges of $S_\delta(u) \cap S_\delta(v)$. This graph is a triangle free graph with 2 vertices of degree 1 and 4 vertices of degree 2. Such a graph has exactly 5 edges. There is only one such graph up to isomorphism. Thus, we may suppose that $S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3, x, y\}$ with $w_0 \overset{\eta}{-} w_1 \overset{\eta}{-} w_2 \overset{\eta}{-} w_3 \overset{\eta}{-} w_0 \overset{\eta}{-}$ and $x \overset{\eta}{-} y$. Lemma 8 then gives a contradiction. Thus, we have $\|U_\eta\| \neq 2$ for all $\eta \in \{\alpha, \beta, \gamma\}$.

It follows that

$$(\|U_\alpha\|, \|U_\beta\|, \|U_\gamma\|) \in [6, 0, 0],$$

so that, we may assume, without loss of generality, that

$$\|U_\alpha\| = 6,$$

and

$$\|U_\beta\| = \|U_\gamma\| = 0.$$

Such a coloring on a complete graph with 6 vertices is clearly impossible. The proof is complete. \square

3.11. Attaching Sets of Cardinality 5.

Proposition 21. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 5$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [4, 0, 0] \cup [2, 2, 0]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) &= \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ &\quad \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 5 &= \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ &\quad + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [4, 0, 0] \cup [3, 1, 1] \cup [2, 2, 0] \cup [2, 1, 1]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Proposition 22. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 5$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 2, 0]. \end{aligned}$$

Proof. Suppose not. Then by Proposition 21, there exist some $x \in S_\delta(u) \cap S_\delta(v)$ such that $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| = 4$ for some $\eta \in \{\alpha, \beta, \gamma\}$. We may suppose that $S_\delta(u) \cap S_\delta(v) = \{x, w_0, w_1, w_2, w_3\}$. Clearly, we cannot have $w_0 \overset{\eta}{\sim} w_i$ for any $i \in \{1, 2, 3\}$. Thus, we have $\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(w_0)\| = 1$, which contradicts Proposition 21.

The proof is complete. □

Proposition 23. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α, β, γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, $d = x_{15} = y_{14}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto v \mapsto u \\ a &\mapsto a \\ b &\mapsto e \mapsto b \\ c &\mapsto d \mapsto c \\ x_i &\mapsto y_i \mapsto x_i \quad \text{for all } i \in \{0, \dots, 15\} \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \alpha \\ \beta &\mapsto \beta \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

First, we note the following:

$$x_3 \overset{\delta}{\sim} y_1 \text{ since } x_3, y_1 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(b).$$

$$x_{10} \overset{\delta}{\sim} y_8 \text{ since } x_{10}, y_8 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(a).$$

$$x_3 \overset{\delta}{\sim} y_7 \text{ since } x_3, y_7 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_9 \overset{\delta}{\sim} y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_9 \overset{\delta}{\sim} y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$\text{Either } x_3 \overset{\beta}{\sim} y_8 \text{ or } x_3 \overset{\delta}{\sim} y_8 \text{ since } x_3, y_8 \in S_\gamma(c) \cap S_\alpha(e).$$

$$\text{Either } x_9 \overset{\alpha}{\sim} y_8 \text{ or } x_9 \overset{\delta}{\sim} y_8 \text{ since } x_9, y_8 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_9 \overset{\alpha}{\sim} y_7 \text{ or } x_9 \overset{\delta}{\sim} y_7 \text{ since } x_9, y_7 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_3, x_1, x_9, x_0, x_{10}, y_1, y_4, y_7, y_0, y_8 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{aligned} \{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_1, y_4, y_7, y_0, y_8\} &= \emptyset \\ P_{\beta, \alpha}(x_3, x_1, x_9, x_0, x_{10}) & \\ P_{\beta, \alpha}(y_1, y_4, y_7, y_0, y_8) & \end{aligned}$$

Now, we show that if $\|S_\gamma(c)\| \geq 15$, then $x_9 \overset{\delta}{-} y_1, y_4, y_7, y_0, y_8$. To that end, suppose that $\|S_\gamma(c)\| \geq 15$. Applying Lemma 14 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_4, y_7, y_0, y_8)$$

to see that either $x_9 \overset{\delta}{-} y_1, y_4, y_7, y_0, y_8$ or else there are exactly two δ -colored edges from x_9 to $\{y_1, y_4, y_7, y_0, y_8\}$. as the former is what we seek to show, we assume the latter. We already have two such edges above, namely $x_9 \overset{\delta}{-} y_1$ and $x_9 \overset{\delta}{-} y_4$. Thus, none of the edges from x_9 to $\{y_7, y_0, y_8\}$ can be δ -colored. But, by the above, we have either $x_9 \overset{\alpha}{-} y_8$ or $x_9 \overset{\delta}{-} y_8$. Also by the above, we have $x_9 \overset{\alpha}{-} y_7$ or $x_9 \overset{\delta}{-} y_7$. Thus, we must have $x_9 \overset{\alpha}{-} y_8, y_7$. But this is impossible, since otherwise the fact that $y_8 \overset{\alpha}{-} y_7$ would give a monochromatic triangle, thus giving a contradiction. Thus, we have shown that

$$(1) \quad \text{if } \|S_\gamma(c)\| \geq 15, \text{ then } x_9 \overset{\delta}{-} y_1, y_4, y_7, y_0, y_8,$$

as desired.

Now, we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_4, y_7, y_0, y_8)$$

to see, with the help of (1), that $\|S_\gamma(c)\| \leq 14$ so that, by Proposition 1, we have

$$(2) \quad \|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15.$$

Next, we note the following:

$$\begin{aligned} x_4 \overset{\delta}{-} y_9 \text{ since } x_4, y_9 \in S_\beta(c) \cap S_\alpha(a) \cap S_\gamma(b). \\ x_4 \overset{\delta}{-} y_5 \text{ since } x_4, y_5 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(b). \end{aligned}$$

Note further that $d, y_5, y_9, y_{10}, b, x_8, x_4, x_6 \in S_\beta(c)$. Note also that the following hold:

$$\begin{aligned} \{y_5, y_9, y_{10}\} \cap \{x_8, x_4, x_6\} &= \emptyset \\ P_{\alpha, \gamma}(d, y_5, y_9, y_{10}, b) & \\ P_{\alpha, \gamma}(b, x_8, x_4, x_6, d) & \end{aligned}$$

Since $\|S_\beta(c)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (d, y_5, y_9, y_{10}, b, x_8, x_4, x_6)$$

to see that

$$(3) \quad N_{\delta, \gamma, \alpha}^2(d, x_6, x_4, x_8, b, y_{10}, y_9, y_5).$$

Now, we note the following:

$$x_5 \xrightarrow{\delta} y_6 \text{ since } x_5, y_6 \in S_\alpha(c) \cap S_\beta(d) \cap S_\gamma(a).$$

$$x_7 \xrightarrow{\delta} y_6 \text{ since } x_7, y_6 \in S_\alpha(c) \cap S_\beta(b) \cap S_\gamma(a).$$

Note further that $a, y_2, y_6, y_3, e, x_7, x_5, x_2 \in S_\alpha(c)$. Note also that the following hold:

$$\begin{aligned} \{y_2, y_6, y_3\} \cap \{x_7, x_5, x_2\} &= \emptyset \\ P_{\beta, \gamma}(a, y_2, y_6, y_3, e) \\ P_{\beta, \gamma}(e, x_7, x_5, x_2, a) \end{aligned}$$

Since $\|S_\alpha(c)\| \geq 15$ by (2), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_2, y_6, y_3, e, x_7, x_5, x_2)$$

to see that

$$(4) \quad N_{\delta, \gamma, \beta}^2(a, x_2, x_5, x_7, e, y_3, y_6, y_2).$$

Now, we note the following:

$$x_6 \xrightarrow{\alpha} y_9 \text{ by (3).}$$

$$x_7 \xrightarrow{\delta} y_3 \text{ since } x_7, y_3 \in S_\beta(e) \cap S_\alpha(c) \cap S_\gamma(a).$$

$$x_1 \xrightarrow{\delta} y_3 \text{ since } x_1, y_3 \in S_\beta(e) \cap S_\alpha(b) \cap S_\gamma(d).$$

Note further that $a, y_9, y_3, y_7, d, x_6, x_7, x_1 \in S_\beta(e)$. Note also that the following hold:

$$\begin{aligned} \{y_9, y_3, y_7\} \cap \{x_6, x_7, x_1\} &= \emptyset \\ P_{\alpha, \gamma}(a, y_9, y_3, y_7, d) \\ P_{\alpha, \gamma}(d, x_6, x_7, x_1, a) \end{aligned}$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_9, y_3, y_7, d, x_6, x_7, x_1)$$

to see that $\|S_\beta(a)\| \leq 14$ so that, by Proposition 1, we have

$$(5) \quad \|S_\gamma(e)\| \geq 15.$$

From (3), we see that

$$(6) \quad x_6 \xrightarrow{\alpha} y_9.$$

An application of the symmetry Θ to (6) gives

$$(7) \quad x_9 \overset{\alpha}{\text{---}} y_6.$$

Finally, we note the following:

$$x_5 \overset{\delta}{\text{---}} y_4 \text{ since } x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(d).$$

$$x_8 \overset{\delta}{\text{---}} y_{10} \text{ by (3).}$$

$$x_2 \overset{\delta}{\text{---}} y_2 \text{ by (4).}$$

$$x_5 \overset{\delta}{\text{---}} y_2 \text{ by (4).}$$

$$x_2 \overset{\beta}{\text{---}} y_6 \text{ by (4).}$$

$$x_9 \overset{\alpha}{\text{---}} y_6 \text{ by (7).}$$

Note further that $x_0, x_8, x_5, x_2, x_9, y_0, y_{10}, y_4, y_2, y_6 \in S_\gamma(e)$. Note also that the following hold:

$$\{x_0, x_8, x_5, x_2, x_9\} \cap \{y_0, y_{10}, y_4, y_2, y_6\} = \emptyset$$

$$P_{\beta, \alpha}(x_0, x_8, x_5, x_2, x_9)$$

$$P_{\beta, \alpha}(y_0, y_{10}, y_4, y_2, y_6)$$

Since $\|S_\gamma(e)\| \geq 15$ by (5), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_8, x_5, x_2, x_9)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_{10}, y_4, y_2, y_6)$$

to produce the desired contradiction. The proof is complete. \square

Proposition 24. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. If the graph has at least 62 vertices, then for any $w \in \{a, b, c, d, e\}$ we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted. (Note that both $\|S_\alpha(w)\|, \|S_\beta(w)\| \geq 15$ in this case, so that it makes sense to ask whether they are twisted in this last sentence.)*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ a &\longmapsto a \\ b &\longmapsto c \longmapsto e \longmapsto d \longmapsto b \\ x_0 &\longmapsto x_0 \end{aligned}$$

$$\begin{aligned}
& x_3 \mapsto x_8 \mapsto x_3 \\
& x_1 \mapsto x_7 \mapsto x_5 \mapsto x_9 \mapsto x_1 \\
& x_2 \mapsto x_{10} \mapsto x_4 \mapsto x_6 \mapsto x_2 \\
& y_0 \mapsto y_0 \\
& y_3 \mapsto y_8 \mapsto y_3 \\
& y_1 \mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\
& y_2 \mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2
\end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
\alpha & \mapsto \beta \mapsto \alpha \\
\gamma & \mapsto \gamma \\
\delta & \mapsto \delta
\end{aligned}$$

We assume that the graph has at least 62 vertices. First, we note the following:

$$x_2 \overset{\delta}{-} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e).$$

$$x_6 \overset{\delta}{-} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_{10} \overset{\delta}{-} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_4 \overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$\text{Either } x_{10} \overset{\alpha}{-} y_2 \text{ or } x_{10} \overset{\delta}{-} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(a) \cap S_\beta(e).$$

$$\text{Either } x_6 \overset{\alpha}{-} y_4 \text{ or } x_6 \overset{\delta}{-} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

$$\text{Either } x_2 \overset{\alpha}{-} y_{10} \text{ or } x_2 \overset{\delta}{-} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(a) \cap S_\beta(e).$$

$$\text{Either } x_4 \overset{\alpha}{-} y_6 \text{ or } x_4 \overset{\delta}{-} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\beta(b).$$

$$\text{Either } x_2 \overset{\beta}{-} y_6 \text{ or } x_2 \overset{\delta}{-} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_4 \overset{\beta}{-} y_{10} \text{ or } x_4 \overset{\delta}{-} y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \overset{\beta}{-} y_2 \text{ or } x_6 \overset{\delta}{-} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_{10} \overset{\beta}{-} y_4 \text{ or } x_{10} \overset{\delta}{-} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\alpha(d).$$

Note further that $x_2, x_6, x_0, x_{10}, x_4, y_2, y_6, y_0, y_{10}, y_4 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned}
& \{x_2, x_6, x_0, x_{10}, x_4\} \cap \{y_2, y_6, y_0, y_{10}, y_4\} = \emptyset \\
& P_{\beta, \alpha}(x_2, x_6, x_0, x_{10}, x_4) \\
& P_{\beta, \alpha}(y_2, y_6, y_0, y_{10}, y_4)
\end{aligned}$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_6, x_0, x_{10}, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_6, y_0, y_{10}, y_4)$$

to see that

$$(8) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M_{\beta, \alpha, \delta}^0(x_2, x_6, x_0, x_{10}, x_4, y_2, y_6, y_0, y_{10}, y_4).$$

Next, we note the following:

$$x_1 \xrightarrow{\delta} y_1 \text{ since } x_1, y_1 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$

$$x_9 \xrightarrow{\delta} y_9 \text{ since } x_9, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(a).$$

$$x_{10} \xrightarrow{\delta} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_3 \xrightarrow{\delta} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a).$$

$$\text{Either } x_9 \xrightarrow{\alpha} y_{10} \text{ or } x_9 \xrightarrow{\delta} y_{10} \text{ since } x_9, y_{10} \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_9 \text{ or } x_{10} \xrightarrow{\delta} y_9 \text{ since } x_{10}, y_9 \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_9 \xrightarrow{\alpha} y_3 \text{ or } x_9 \xrightarrow{\delta} y_3 \text{ since } x_9, y_3 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_3 \xrightarrow{\alpha} y_9 \text{ or } x_3 \xrightarrow{\delta} y_9 \text{ since } x_3, y_9 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_1 \xrightarrow{\beta} y_9 \text{ or } x_1 \xrightarrow{\delta} y_9 \text{ since } x_1, y_9 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_9 \xrightarrow{\beta} y_1 \text{ or } x_9 \xrightarrow{\delta} y_1 \text{ since } x_9, y_1 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_{10} \xrightarrow{\beta} y_3 \text{ or } x_{10} \xrightarrow{\delta} y_3 \text{ since } x_{10}, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_3 \xrightarrow{\beta} y_{10} \text{ or } x_3 \xrightarrow{\delta} y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_1, x_9, x_0, x_{10}, x_3, y_1, y_9, y_0, y_{10}, y_3 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_1, x_9, x_0, x_{10}, x_3\} \cap \{y_1, y_9, y_0, y_{10}, y_3\} = \emptyset$$

$$P_{\beta, \alpha}(x_1, x_9, x_0, x_{10}, x_3)$$

$$P_{\beta, \alpha}(y_1, y_9, y_0, y_{10}, y_3)$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_1, x_9, x_0, x_{10}, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_9, y_0, y_{10}, y_3)$$

to see that

$$(9) \quad \text{if } \|S_\gamma(b)\| \geq 15, \text{ then } M_{\beta, \alpha, \gamma}^0(x_0, x_{10}, x_3, x_1, x_9, y_0, y_{10}, y_3, y_1, y_9).$$

Next, we note that $x_4 \xrightarrow{\delta} y_4$ since $x_4, y_4 \in S_\beta(b) \cap S_\alpha(d) \cap S_\gamma(a)$. Note further that $c, y_6, y_4, y_8, a, x_8, x_4, x_6 \in S_\beta(b)$. Note also that the following hold:

$$\{y_6, y_4, y_8\} \cap \{x_8, x_4, x_6\} = \emptyset$$

$$P_{\alpha,\gamma}(c, y_6, y_4, y_8, a)$$

$$P_{\alpha,\gamma}(a, x_8, x_4, x_6, c)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_6, y_4, y_8, a, x_8, x_4, x_6)$$

to see that

$$(10) \quad \text{if } \|S_\beta(b)\| \geq 15, \text{ then } S_\beta(b) \text{ is twisted}$$

$$\text{and either } N_{\delta,\gamma,\alpha}^2(c, y_6, y_4, y_8, a, x_8, x_4, x_6)$$

$$\text{or } N_{\delta,\gamma,\alpha}^2(c, x_6, x_4, x_8, a, y_8, y_4, y_6).$$

Next, we note that $x_5 \stackrel{\delta}{-} y_5$ since $x_5, y_5 \in S_\alpha(b) \cap S_\beta(c) \cap S_\gamma(d)$. Note further that $e, y_2, y_5, y_7, d, x_7, x_5, x_2 \in S_\alpha(b)$. Note also that the following hold:

$$\{y_2, y_5, y_7\} \cap \{x_7, x_5, x_2\} = \emptyset$$

$$P_{\beta,\gamma}(e, y_2, y_5, y_7, d)$$

$$P_{\beta,\gamma}(d, x_7, x_5, x_2, e)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (e, y_2, y_5, y_7, d, x_7, x_5, x_2)$$

to see that

$$(11) \quad \text{if } \|S_\alpha(b)\| \geq 15, \text{ then } S_\alpha(b) \text{ is twisted}$$

$$\text{and either } N_{\delta,\gamma,\beta}^2(e, y_2, y_5, y_7, d, x_7, x_5, x_2)$$

$$\text{or } N_{\delta,\gamma,\beta}^2(e, x_2, x_5, x_7, d, y_7, y_5, y_2).$$

Next, we note that $x_3 \stackrel{\delta}{-} y_3$ since $x_3, y_3 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(b)$. Note further that $b, y_7, y_3, y_9, e, x_9, x_3, x_7 \in S_\beta(a)$. Note also that the following hold:

$$\{y_7, y_3, y_9\} \cap \{x_9, x_3, x_7\} = \emptyset$$

$$P_{\alpha,\gamma}(b, y_7, y_3, y_9, e)$$

$$P_{\alpha,\gamma}(e, x_9, x_3, x_7, b)$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_7, y_3, y_9, e, x_9, x_3, x_7)$$

to see that

$$(12) \quad \text{if } \|S_\beta(a)\| \geq 15, \text{ then } S_\beta(a) \text{ is twisted}$$

$$\text{and either } N_{\delta,\gamma,\alpha}^2(b, y_7, y_3, y_9, e, x_9, x_3, x_7)$$

$$\text{or } N_{\delta,\gamma,\alpha}^2(b, x_7, x_3, x_9, e, y_9, y_3, y_7).$$

From (8), we see that

$$(13) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } x_0 \stackrel{\delta}{-} y_{10} \text{ and } x_6 \stackrel{\alpha}{-} y_4 \text{ and } x_4 \stackrel{\alpha}{-} y_6.$$

From (9), we see that

$$(14) \quad \text{if } \|S_\gamma(b)\| \geq 15, \text{ then } x_0 \xrightarrow{\beta} y_{10} \text{ and } x_9 \xrightarrow{\delta} y_3 \text{ and } x_3 \xrightarrow{\delta} y_9.$$

From (10), we see that

$$(15) \quad \text{if } \|S_\beta(b)\| \geq 15, \text{ then } S_\beta(b) \text{ is twisted} \\ \text{and either } x_6 \xrightarrow{\delta} y_4 \text{ or } x_4 \xrightarrow{\delta} y_6.$$

From (12), we see that

$$(16) \quad \text{if } \|S_\beta(a)\| \geq 15, \text{ then } S_\beta(a) \text{ is twisted} \\ \text{and either } x_9 \xrightarrow{\alpha} y_3 \text{ or } x_3 \xrightarrow{\alpha} y_9.$$

From (13) and (14), we see that

$$(17) \quad \text{either } \|S_\gamma(a)\| \leq 14 \text{ or } \|S_\gamma(b)\| \leq 14.$$

From (13) and (15), we see that

$$(18) \quad \text{either } \|S_\gamma(a)\| \leq 14 \text{ or } \|S_\beta(b)\| \leq 14.$$

From (14) and (16), we see that

$$(19) \quad \text{either } \|S_\gamma(b)\| \leq 14 \text{ or } \|S_\beta(a)\| \leq 14.$$

From (17) and (18), by Proposition 1, we see that

$$(20) \quad \|S_\gamma(a)\| \leq 14.$$

From (19) and (20), by Proposition 1, we see that

$$(21) \quad \|S_\gamma(b)\| \leq 14.$$

From (20) and (16), by Proposition 1, we see that

$$(22) \quad \|S_\gamma(a)\| \leq 14 \text{ and } S_\beta(a) \text{ is twisted.}$$

By an application of the symmetry Θ to (22), we see that

$$(23) \quad \|S_\gamma(a)\| \leq 14 \text{ and } S_\alpha(a) \text{ is twisted.}$$

From (22) and (23), we see that

$$(24) \quad \|S_\gamma(a)\| \leq 14 \text{ and both } S_\beta(a) \text{ and } S_\alpha(a) \text{ are twisted.}$$

From (21), (11), and (15), by Proposition 1, we see that

$$(25) \quad \|S_\gamma(b)\| \leq 14 \text{ and both } S_\beta(b) \text{ and } S_\alpha(b) \text{ are twisted.}$$

By an application of the symmetry Θ to (25), we see that

$$(26) \quad \|S_\gamma(c)\| \leq 14 \text{ and both } S_\alpha(c) \text{ and } S_\beta(c) \text{ are twisted.}$$

By an application of the symmetry Θ to (26), we see that

$$(27) \quad \|S_\gamma(e)\| \leq 14 \text{ and both } S_\beta(e) \text{ and } S_\alpha(e) \text{ are twisted.}$$

By an application of the symmetry Θ to (27), we see that

$$(28) \quad \|S_\gamma(d)\| \leq 14 \text{ and both } S_\alpha(d) \text{ and } S_\beta(d) \text{ are twisted.}$$

By (24), (25), (26),(27), and (28), we see that the proof is complete. \square

Proposition 25. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto u \\ v &\mapsto v \\ a &\mapsto a \\ b &\mapsto c \mapsto e \mapsto d \mapsto b \\ x_0 &\mapsto x_0 \\ x_3 &\mapsto x_8 \mapsto x_3 \\ x_1 &\mapsto x_9 \mapsto x_5 \mapsto x_7 \mapsto x_1 \\ x_2 &\mapsto x_6 \mapsto x_4 \mapsto x_{10} \mapsto x_2 \\ y_0 &\mapsto y_0 \\ y_3 &\mapsto y_8 \mapsto y_3 \\ y_1 &\mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\ y_2 &\mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \beta \mapsto \alpha \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

First, we note the following:

$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e).$$

$$x_{10} \xrightarrow{\delta} y_6 \text{ since } x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_6 \xrightarrow{\delta} y_{10} \text{ since } x_6, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_4 \xrightarrow{\delta} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$\text{Either } x_6 \xrightarrow{\alpha} y_2 \text{ or } x_6 \xrightarrow{\delta} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\beta(e).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_4 \text{ or } x_{10} \xrightarrow{\delta} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Either $x_2 \xrightarrow{\alpha} y_{10}$ or $x_2 \xrightarrow{\delta} y_{10}$ since $x_2, y_{10} \in S_\gamma(a) \cap S_\beta(e)$.

Either $x_4 \xrightarrow{\alpha} y_6$ or $x_4 \xrightarrow{\delta} y_6$ since $x_4, y_6 \in S_\gamma(a) \cap S_\beta(b)$.

Either $x_2 \xrightarrow{\beta} y_6$ or $x_2 \xrightarrow{\delta} y_6$ since $x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_4 \xrightarrow{\beta} y_{10}$ or $x_4 \xrightarrow{\delta} y_{10}$ since $x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_6 \xrightarrow{\beta} y_4$ or $x_6 \xrightarrow{\delta} y_4$ since $x_6, y_4 \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_{10} \xrightarrow{\beta} y_2$ or $x_{10} \xrightarrow{\delta} y_2$ since $x_{10}, y_2 \in S_\gamma(a) \cap S_\alpha(c)$.

Note further that $x_2, x_{10}, x_0, x_6, x_4, y_2, y_6, y_0, y_{10}, y_4 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_2, x_{10}, x_0, x_6, x_4\} \cap \{y_2, y_6, y_0, y_{10}, y_4\} &= \emptyset \\ P_{\beta, \alpha}(x_2, x_{10}, x_0, x_6, x_4) & \\ P_{\beta, \alpha}(y_2, y_6, y_0, y_{10}, y_4) & \end{aligned}$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_{10}, x_0, x_6, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_6, y_0, y_{10}, y_4)$$

to see that

$$(29) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M_{\beta, \alpha, \delta}^0(x_2, x_{10}, x_0, x_6, x_4, y_2, y_6, y_0, y_{10}, y_4).$$

Next, we note the following:

$x_3 \xrightarrow{\delta} y_3$ since $x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a)$.

$x_7 \xrightarrow{\delta} y_9$ since $x_7, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(a)$.

$x_6 \xrightarrow{\delta} y_{10}$ since $x_6, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c)$.

$x_5 \xrightarrow{\delta} y_9$ since $x_5, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c)$.

$x_7 \xrightarrow{\delta} y_1$ since $x_7, y_1 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d)$.

Either $x_5 \xrightarrow{\alpha} y_{10}$ or $x_5 \xrightarrow{\delta} y_{10}$ since $x_5, y_{10} \in S_\gamma(b) \cap S_\beta(c)$.

Either $x_7 \xrightarrow{\alpha} y_3$ or $x_7 \xrightarrow{\delta} y_3$ since $x_7, y_3 \in S_\gamma(b) \cap S_\beta(a)$.

Either $x_6 \xrightarrow{\alpha} y_9$ or $x_6 \xrightarrow{\delta} y_9$ since $x_6, y_9 \in S_\gamma(b) \cap S_\beta(c)$.

Either $x_3 \xrightarrow{\alpha} y_9$ or $x_3 \xrightarrow{\delta} y_9$ since $x_3, y_9 \in S_\gamma(b) \cap S_\beta(a)$.

Either $x_5 \xrightarrow{\beta} y_1$ or $x_5 \xrightarrow{\delta} y_1$ since $x_5, y_1 \in S_\gamma(b) \cap S_\alpha(a)$.

Either $x_6 \xrightarrow{\beta} y_3$ or $x_6 \xrightarrow{\delta} y_3$ since $x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d)$.

Either $x_3 \xrightarrow{\beta} y_{10}$ or $x_3 \xrightarrow{\delta} y_{10}$ since $x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d)$.

Note further that $x_3, x_5, x_7, x_0, x_6, y_3, y_1, y_9, y_0, y_{10} \in S_\gamma(b)$. Note also that the following hold:

$$\begin{aligned} \{x_3, x_5, x_7, x_0, x_6\} \cap \{y_3, y_1, y_9, y_0, y_{10}\} &= \emptyset \\ P_{\beta, \alpha}(x_3, x_5, x_7, x_0, x_6) \\ P_{\beta, \alpha}(y_3, y_1, y_9, y_0, y_{10}) \end{aligned}$$

We may now apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_5, x_7, x_0, x_6)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_3, y_1, y_9, y_0, y_{10})$$

to see that

$$(30) \quad \text{if } \|S_\gamma(b)\| \geq 15, \text{ then } M_{\beta, \alpha, \gamma}^2(x_0, x_6, x_3, x_5, x_7, y_3, y_{10}, y_0, y_9, y_1).$$

By an application of the symmetry Θ to (30), we see that

$$(31) \quad \text{if } \|S_\gamma(c)\| \geq 15, \text{ then } M_{\alpha, \beta, \delta}^2(x_0, x_4, x_8, x_7, x_1 y_8, y_4, y_0, y_1, y_7).$$

By an application of the symmetry Θ to (31), we see that

$$(32) \quad \text{if } \|S_\gamma(e)\| \geq 15, \text{ then } M_{\beta, \alpha, \delta}^2(x_0, x_{10}, x_3, x_1, x_9, y_3, y_6, y_0, y_7, y_5).$$

By an application of the symmetry Θ to (32), we see that

$$(33) \quad \text{if } \|S_\gamma(d)\| \geq 15, \text{ then } M_{\alpha, \beta, \delta}^2(x_0, x_2, x_8, x_9, x_5, y_8, y_2, y_0, y_5, y_9).$$

Now, note that $c, y_6, y_4, y_8, a, x_8, x_4, x_{10} \in S_\beta(b)$. Note also that the following hold:

$$\begin{aligned} \{y_6, y_4, y_8\} \cap \{x_8, x_4, x_{10}\} &= \emptyset \\ P_{\alpha, \gamma}(c, y_6, y_4, y_8, a) \\ P_{\alpha, \gamma}(a, x_8, x_4, x_{10}, c) \end{aligned}$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_6, y_4, y_8, a, x_8, x_4, x_{10})$$

to see that

$$(34) \quad \text{if } x_{10} \xrightarrow{\alpha} y_4 \text{ and } x_4 \xrightarrow{\alpha} y_{10}, \text{ then } \|S_\beta(b)\| \leq 14.$$

Next, we note the following:

$$x_5 \xrightarrow{\delta} y_5 \text{ since } x_5, y_5 \in S_\beta(c) \cap S_\alpha(a) \cap S_\gamma(b).$$

$$x_5 \xrightarrow{\delta} y_9 \text{ since } x_5, y_9 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(b).$$

Note further that $b, y_5, y_9, y_{10}, d, x_6, x_5, x_9 \in S_\beta(c)$. Note also that the following hold:

$$\begin{aligned} \{y_5, y_9, y_{10}\} \cap \{x_6, x_5, x_9\} &= \emptyset \\ P_{\alpha, \gamma}(b, y_5, y_9, y_{10}, d) \\ P_{\alpha, \gamma}(d, x_6, x_5, x_9, b) \end{aligned}$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_5, y_9, y_{10}, d, x_6, x_5, x_9)$$

to see that

$$(35) \quad \text{if } \|S_\beta(c)\| \geq 15, \text{ then } N_{\delta, \gamma, \alpha}^2(b, x_9, x_5, x_6, d, y_{10}, y_9, y_5).$$

Next, we note the following:

$$\begin{aligned} x_1 \overset{\delta}{-} y_1 & \text{ since } x_1, y_1 \in S_\beta(d) \cap S_\alpha(a) \cap S_\gamma(c). \\ x_1 \overset{\delta}{-} y_7 & \text{ since } x_1, y_7 \in S_\beta(d) \cap S_\alpha(b) \cap S_\gamma(c). \end{aligned}$$

Note further that $e, y_1, y_7, y_6, c, x_{10}, x_1, x_7 \in S_\beta(d)$. Note also that the following hold:

$$\begin{aligned} \{y_1, y_7, y_6\} \cap \{x_{10}, x_1, x_7\} &= \emptyset \\ P_{\alpha, \gamma}(e, y_1, y_7, y_6, c) & \\ P_{\alpha, \gamma}(c, x_{10}, x_1, x_7, e) & \end{aligned}$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (e, y_1, y_7, y_6, c, x_{10}, x_1, x_7)$$

to see that

$$(36) \quad \text{if } \|S_\beta(d)\| \geq 15, \text{ then } N_{\delta, \gamma, \alpha}^2(e, x_7, x_1, x_{10}, c, y_6, y_7, y_1).$$

Finally, note that $b, y_7, y_3, y_9, e, x_7, x_3, x_9 \in S_\beta(a)$. note also that the following hold:

$$\begin{aligned} \{y_7, y_3, y_9\} \cap \{x_7, x_3, x_9\} &= \emptyset \\ P_{\alpha, \gamma}(b, y_7, y_3, y_9, e) & \\ P_{\alpha, \gamma}(e, x_7, x_3, x_9, b) & \end{aligned}$$

We may now apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_7, y_3, y_9, e, x_7, x_3, x_9)$$

to see that

$$(37) \quad \text{if } x_9 \overset{\alpha}{-} y_9 \text{ and } x_7 \overset{\alpha}{-} y_7, \text{ then } \|S_\beta(a)\| \leq 14.$$

Now, we show that $\|S_\gamma(a)\| \leq 14$. Suppose not. Then $\|S_\gamma(a)\| \geq 15$, so that $x_0 \overset{\delta}{-} y_0$ by (29). But then $\|S_\gamma(b)\| \leq 14$, since otherwise, we would have $x_0 \overset{\alpha}{-} y_0$ by (30). Since $\|S_\gamma(b)\| \leq 14$, we must have $\|S_\beta(b)\| \geq 15$, by Proposition 1. But (29) implies that $x_{10} \overset{\alpha}{-} y_4$ and $x_4 \overset{\alpha}{-} y_{10}$, which gives a contradiction by (34). Thus, we have

$$(38) \quad \|S_\gamma(a)\| \leq 14,$$

as desired.

Next, we show that either $\|S_\gamma(c)\| \geq 15$ or $\|S_\gamma(d)\| \geq 15$ (or both). Suppose not. Then $\|S_\gamma(c)\|, \|S_\gamma(d)\| \leq 14$. But this, in turn, implies that $\|S_\beta(c)\|, \|S_\beta(d)\| \geq 15$,

by Proposition 1. Thus, we now have $x_9 \stackrel{\alpha}{\sim} y_9$ by (35) and $x_7 \stackrel{\alpha}{\sim} y_7$ by (36). But then (37) implies that $\|S_\beta(a)\| \leq 14$, which contradicts (38). Thus, we have

$$(39) \quad \text{either } \|S_\gamma(c)\| \geq 15 \text{ or } \|S_\gamma(d)\| \geq 15,$$

as desired.

By (39),(31), and (33), we have

$$(40) \quad x_0 \stackrel{\alpha}{\sim} y_0.$$

By an application of the symmetry Θ to (40) we have

$$(41) \quad x_0 \stackrel{\beta}{\sim} y_0.$$

(40) and (41) together produce the desired contradiction. The proof is complete. \square

Proposition 26. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha,\beta,\gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha,\beta,\gamma}^0(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, $e = x_{12} = y_{12}$. If the graph has fewer than 62 vertices, then for any $w \in \{a, b, c, d, e\}$ we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted. (Note that both $\|S_\alpha(w)\|, \|S_\beta(w)\| \geq 15$ in this case, so that it makes sense to ask whether they are twisted in this last sentence.)*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ a &\longmapsto a \\ b &\longmapsto c \longmapsto e \longmapsto d \longmapsto b \\ x_0 &\longmapsto x_0 \\ x_3 &\longmapsto x_8 \longmapsto x_3 \\ x_1 &\longmapsto x_9 \longmapsto x_5 \longmapsto x_7 \longmapsto x_1 \\ x_2 &\longmapsto x_6 \longmapsto x_4 \longmapsto x_{10} \longmapsto x_2 \\ y_0 &\longmapsto y_0 \\ y_3 &\longmapsto y_8 \longmapsto y_3 \\ y_1 &\longmapsto y_9 \longmapsto y_5 \longmapsto y_7 \longmapsto y_1 \\ y_2 &\longmapsto y_6 \longmapsto y_4 \longmapsto y_{10} \longmapsto y_2 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\longmapsto \beta \longmapsto \alpha \\ \gamma &\longmapsto \gamma \end{aligned}$$

$$\delta \mapsto \delta$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ

$$\begin{aligned} u &\mapsto u \\ v &\mapsto v \\ a &\mapsto b \mapsto c \mapsto d \mapsto e \mapsto a \\ x_0 &\mapsto x_0 \\ x_1 &\mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_5 \mapsto x_0 \\ x_6 &\mapsto x_7 \mapsto x_8 \mapsto x_9 \mapsto x_{10} \mapsto x_6 \\ y_0 &\mapsto y_0 \\ y_1 &\mapsto y_2 \mapsto y_3 \mapsto y_4 \mapsto y_5 \mapsto y_0 \\ y_6 &\mapsto y_7 \mapsto y_8 \mapsto y_9 \mapsto y_{10} \mapsto y_6 \end{aligned}$$

Note that the symmetry Ψ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \alpha \\ \beta &\mapsto \beta \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

We assume that the graph has at least 62 vertices.

First, we note the following:

$$\begin{aligned} x_2 &\overset{\delta}{-} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(e). \\ x_{10} &\overset{\delta}{-} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\ x_6 &\overset{\delta}{-} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c). \\ x_4 &\overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \\ \text{Either } x_6 &\overset{\alpha}{-} y_2 \text{ or } x_6 \overset{\delta}{-} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\beta(e). \\ \text{Either } x_2 &\overset{\alpha}{-} y_6 \text{ or } x_2 \overset{\delta}{-} y_6 \text{ since } x_2, y_6 \in S_\gamma(a) \cap S_\beta(e). \\ \text{Either } x_{10} &\overset{\alpha}{-} y_4 \text{ or } x_{10} \overset{\delta}{-} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b). \\ \text{Either } x_4 &\overset{\alpha}{-} y_{10} \text{ or } x_4 \overset{\delta}{-} y_{10} \text{ since } x_4, y_{10} \in S_\gamma(a) \cap S_\beta(b). \\ \text{Either } x_2 &\overset{\beta}{-} y_{10} \text{ or } x_2 \overset{\delta}{-} y_{10} \text{ since } x_2, y_{10} \in S_\gamma(a) \cap S_\alpha(c). \\ \text{Either } x_{10} &\overset{\beta}{-} y_2 \text{ or } x_{10} \overset{\delta}{-} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(a) \cap S_\alpha(c). \\ \text{Either } x_6 &\overset{\beta}{-} y_4 \text{ or } x_6 \overset{\delta}{-} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\alpha(d). \\ \text{Either } x_4 &\overset{\beta}{-} y_6 \text{ or } x_4 \overset{\delta}{-} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d). \end{aligned}$$

Note further that $x_2, x_{10}, x_0, x_6, x_4, y_2, y_{10}, y_0, y_6, y_4 \in S_\gamma(a)$. note also that the following hold:

$$\begin{aligned} \{x_2, x_{10}, x_0, x_6, x_4\} \cap \{y_2, y_{10}, y_0, y_6, y_4\} &= \emptyset \\ P_{\beta, \alpha}(x_2, x_{10}, x_0, x_6, x_4) & \\ P_{\beta, \alpha}(y_2, y_{10}, y_0, y_6, y_4) & \end{aligned}$$

We may now apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_2, x_{10}, x_0, x_6, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_2, y_{10}, y_0, y_6, y_4)$$

to see that

$$(42) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } M_{\beta, \alpha, \delta}^2(x_2, x_{10}, x_0, x_6, x_4, y_2, y_{10}, y_0, y_6, y_4).$$

Now, note that $x_6 \xrightarrow{\delta} y_6$ since $x_6, y_6 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(a)$. Note further that $a, y_3, y_6, y_4, b, x_4, x_6, x_3 \in S_\alpha(d)$. Note also that the following hold:

$$\begin{aligned} \{y_3, y_6, y_4\} \cap \{x_4, x_6, x_3\} &= \emptyset \\ P_{\beta, \gamma}(a, y_3, y_6, y_4, b) & \\ P_{\beta, \gamma}(b, x_4, x_6, x_3, a) & \end{aligned}$$

We may now apply Lemma 9 with

$$(u_1, \dots, u_8) = (a, y_3, y_6, y_4, b, x_4, x_6, x_3)$$

to see that

$$(43) \quad \begin{aligned} \text{if } \|S_\alpha(d)\| \geq 15, \text{ then } S_\alpha(d) \text{ is twisted} \\ \text{and either } N_{\delta, \gamma, \beta}^2(a, y_3, y_6, y_4, b, x_4, x_6, x_3) \\ \text{or } N_{\delta, \gamma, \beta}^2(a, x_3, x_6, x_4, b, y_4, y_6, y_3). \end{aligned}$$

By (43), we see that

$$(44) \quad \begin{aligned} \text{if } \|S_\alpha(d)\| \geq 15, \text{ then } S_\alpha(d) \text{ is twisted} \\ \text{and either } x_6 \xrightarrow{\delta} y_4 \text{ or } x_4 \xrightarrow{\delta} y_6. \end{aligned}$$

Suppose that $\|S_\gamma(a)\| \geq 15$. Then, by (42), we have $x_6 \xrightarrow{\beta} y_4$ and $x_4 \xrightarrow{\beta} y_6$. By (44), this implies that $\|S_\alpha(d)\| \leq 14$. Thus, we have shown that

$$(45) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\alpha(d)\| \leq 14.$$

From (45), by Proposition 1, we see that

$$(46) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\gamma(d)\| \geq 15.$$

By an application of the symmetry Θ to (45), we see that

$$(47) \quad \text{if } \|S_\gamma(a)\| \geq 15, \text{ then } \|S_\beta(e)\| \leq 14.$$

By an application of the symmetry Ψ to (45), we see that

$$(48) \quad \text{if } \|S_\gamma(b)\| \geq 15, \text{ then } \|S_\alpha(e)\| \leq 14.$$

Repeated applications of the symmetry Ψ to (46) shows that either $\|S_\gamma(w)\| \geq 15$ for all $w \in \{a, b, c, d, e\}$ or else $\|S_\gamma(w)\| \leq 14$ for all $w \in \{a, b, c, d, e\}$. Suppose that $\|S_\gamma(w)\| \geq 15$ for all $w \in \{a, b, c, d, e\}$. Then by (48), we have $\|S_\alpha(e)\| \leq 14$. But (47) implies that $\|S_\beta(e)\| \leq 14$, which is a contradiction. Thus, we have

$$(49) \quad \|S_\gamma(w)\| \geq 15 \text{ for all } w \in \{a, b, c, d, e\}.$$

By (49) and (43), we see that

$$(50) \quad S_\alpha(d) \text{ is twisted.}$$

By repeated applications of the symmetry Ψ to (50), we see that

$$(51) \quad S_\alpha(w) \text{ is twisted for all } w \in \{a, b, c, d, e\}.$$

by an application of the symmetry θ to (51), we see that

$$(52) \quad S_\beta(w) \text{ is twisted for all } w \in \{a, b, c, d, e\}.$$

By (49),(51),and (52), we see that the proof is complete. \square

Theorem 11. *Let V be the vertex set of a complete graph with a good edge coloring with four colors. Suppose that $\|V\| = 62$ and let $u, v \in V$ be such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ and $\|S_\delta(u) \cap S_\delta(v)\| = 5$, for some color δ . Then there exist $x_0, \dots, x_{15} \in S_\delta(u)$ and $y_0, \dots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 15\}$ and there exists some $j \in \{0, 1\}$ and colors α, β , and γ , such that $B_{\alpha, \beta, \gamma}^j(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^j(y_0, \dots, y_{15})$ with*

$$S_\delta(u) \cap S_\delta(v) = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} = \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\}.$$

Furthermore, for each $w \in S_\delta(u) \cap S_\delta(v)$, we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted.

Proof. Since $\|S_\delta(u) \cap S_\delta(v)\| = 5$, we may suppose that

$$S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3, w_4\}.$$

By Proposition 22, we may suppose, without loss of generality, that

$$(53) \quad w_0 \xrightarrow{\alpha} w_1, w_4$$

and

$$(54) \quad w_0 \xrightarrow{\beta} w_2, w_3.$$

By (53), we cannot have $w_1 \xrightarrow{\alpha} w_4$, since otherwise we would have a monochromatic triangle. Thus, since $w_1 \xrightarrow{\alpha} w_4$ by (53), we see, by Proposition 22, that either $w_1 \xrightarrow{\alpha} w_2$ or $w_1 \xrightarrow{\alpha} w_3$. Without loss of generality, we may suppose that

$$(55) \quad w_1 \xrightarrow{\alpha} w_2.$$

By (54), we cannot have $w_2 \overset{\beta}{-} w_3$, since otherwise we would have a monochromatic triangle. Thus, since $w_2 \overset{\alpha}{-} w_1$ by (55) and $w_2 \overset{\beta}{-} w_0$ by (54), we see, by Proposition 22, that

$$(56) \quad w_2 \overset{\alpha}{-} w_3$$

and

$$(57) \quad w_2 \overset{\beta}{-} w_4.$$

By (53), we cannot have $w_4 \overset{\alpha}{-} w_1$, since otherwise we would have a monochromatic triangle. Thus, since $w_4 \overset{\alpha}{-} w_0$ by (53) and $w_4 \overset{\beta}{-} w_2$ by (57), we see, by Proposition 22, that

$$(58) \quad w_4 \overset{\beta}{-} w_1$$

and

$$(59) \quad w_4 \overset{\alpha}{-} w_3.$$

Since $w_1 \overset{\alpha}{-} w_0, w_2$ by (53) and (55), and $w_1 \overset{\beta}{-} w_4$ by (58), we see, by Proposition 22, that

$$(60) \quad w_1 \overset{\beta}{-} w_3.$$

Thus, we see, by (53), (54), (55), (56), (57), (58), (59), and (60), that

$$P_{\alpha, \beta}(w_0, w_1, w_2, w_3, w_4).$$

The conclusion now follows by Proposition 23, Proposition 24, Proposition 25, and Proposition 26.

The proof is complete. □

3.12. Attaching Sets of Cardinality 4.

Proposition 27. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 4$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [3, 0, 0] \cup [2, 1, 0]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x))$$

$$\uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)),$$

so that

$$4 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|.$$

Thus, we have

$$(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [3, 0, 0] \cup [2, 1, 0] \cup [1, 1, 1].$$

The proposition now follows by an application of Proposition 4. \square

Proposition 28. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 4$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$(\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 1, 0].$$

Proof. Suppose not. Then, by Proposition 27, we may suppose, without loss of generality, that $S_\delta(u) \cap S_\delta(v) = \{x, w_0, w_1, w_2\}$ with $x \xrightarrow{\alpha} w_0, w_1, w_2$. We cannot have $w_0 \xrightarrow{\alpha} w_i$ for any $i \in \{1, 2\}$, since otherwise we would have a monochromatic triangle. Since $w_0 \xrightarrow{\alpha} x$, we may, by Proposition 27, suppose, without loss of generality, that $w_0 \xrightarrow{\beta} w_1, w_2$. Now, we must have $w_1 \xrightarrow{\gamma} w_2$, since $w_1, w_2 \in S_\alpha(x) \cap S_\beta(w_0) \cap S_\delta(u)$. Thus, we have $w_1 \xrightarrow{\alpha} x$ and $w_1 \xrightarrow{\beta} w_0$ and $w_1 \xrightarrow{\gamma} w_2$, which contradicts Proposition 27.

The proof is complete. \square

Proposition 29. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_0 = y_0$, $b = x_1 = y_1$, $c = x_{12} = y_{12}$, and $d = x_4 = y_4$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ a &\longmapsto b \longmapsto c \longmapsto d \longmapsto a \\ x_2 &\longmapsto x_{13} \longmapsto x_9 \longmapsto x_6 \longmapsto x_2 \\ x_3 &\longmapsto x_7 \longmapsto x_{14} \longmapsto x_{11} \longmapsto x_3 \\ x_5 &\longmapsto x_{10} \longmapsto x_{15} \longmapsto x_8 \longmapsto x_5 \end{aligned}$$

$$\begin{aligned}
y_2 &\mapsto y_{13} \mapsto y_9 \mapsto y_6 \mapsto y_2 \\
y_3 &\mapsto y_7 \mapsto y_{14} \mapsto y_{11} \mapsto y_3 \\
y_5 &\mapsto y_{10} \mapsto y_{15} \mapsto y_8 \mapsto y_5
\end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
\alpha &\mapsto \alpha \\
\beta &\mapsto \gamma \mapsto \beta \\
\delta &\mapsto \delta
\end{aligned}$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ :

$$\begin{aligned}
u &\mapsto v \mapsto u \\
a &\mapsto a \\
b &\mapsto b \\
c &\mapsto c \\
d &\mapsto d \\
x_i &\mapsto y_i \mapsto x_i \quad \text{for all } i \in \{0, \dots, 15\}
\end{aligned}$$

Note that the symmetry Ψ acts on the colors as follows:

$$\begin{aligned}
\alpha &\mapsto \alpha \\
\beta &\mapsto \beta \\
\gamma &\mapsto \gamma \\
\delta &\mapsto \delta
\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\gamma(a) = \{x_{11}, c, x_{13}, x_{14}, x_{15}\} \subseteq S_\gamma(a)$ and $S_\delta(v) \cap S_\gamma(a) = \{y_{11}, c, y_{13}, y_{14}, y_{15}\} \subseteq S_\gamma(a)$. Note further that both $P_{\beta,\alpha}(x_{11}, c, x_{13}, x_{14}, x_{15})$ and $P_{\beta,\alpha}(y_{11}, c, y_{13}, y_{14}, y_{15})$ hold and that

$$\|\{x_{11}, c, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, c, y_{13}, y_{14}, y_{15}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\gamma(a)\| \leq 14$ so that

$$(61) \quad \|S_\alpha(a)\|, \|S_\beta(a)\| \geq 15.$$

Repeated applications of the symmetry Θ to (61) give

$$(62) \quad \|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15,$$

$$(63) \quad \|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15,$$

and

$$(64) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.$$

Now, we note the following:

$$x_6 \overset{\delta}{-} y_6 \text{ since } x_6, y_6 \in S_\beta(a) \cap S_\alpha(d) \cap S_\gamma(b).$$

$$x_9 \overset{\delta}{-} y_9 \text{ since } x_9, y_9 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(d).$$

$$x_7 \overset{\delta}{-} y_7 \text{ since } x_7, y_7 \in S_\beta(a) \cap S_\alpha(b) \cap S_\gamma(c).$$

$$\text{Either } x_{10} \overset{\gamma}{-} y_{10} \text{ or } x_{10} \overset{\delta}{-} y_{10} \text{ since } x_{10}, y_{10} \in S_\beta(a) \cap S_\alpha(b).$$

$$\text{Either } x_8 \overset{\gamma}{-} y_8 \text{ or } x_8 \overset{\delta}{-} y_8 \text{ since } x_8, y_8 \in S_\beta(a) \cap S_\alpha(d).$$

$$\text{Either } x_7 \overset{\gamma}{-} y_{10} \text{ or } x_7 \overset{\delta}{-} y_{10} \text{ since } x_7, y_{10} \in S_\beta(a) \cap S_\alpha(b).$$

$$\text{Either } x_{10} \overset{\gamma}{-} y_7 \text{ or } x_{10} \overset{\delta}{-} y_7 \text{ since } x_{10}, y_7 \in S_\beta(a) \cap S_\alpha(b).$$

$$\text{Either } x_6 \overset{\gamma}{-} y_8 \text{ or } x_6 \overset{\delta}{-} y_8 \text{ since } x_6, y_8 \in S_\beta(a) \cap S_\alpha(d).$$

$$\text{Either } x_8 \overset{\gamma}{-} y_6 \text{ or } x_8 \overset{\delta}{-} y_6 \text{ since } x_8, y_6 \in S_\beta(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \overset{\alpha}{-} y_7 \text{ or } x_6 \overset{\delta}{-} y_7 \text{ since } x_6, y_7 \in S_\beta(a) \cap S_\gamma(c).$$

$$\text{Either } x_7 \overset{\alpha}{-} y_6 \text{ or } x_7 \overset{\delta}{-} y_6 \text{ since } x_7, y_6 \in S_\beta(a) \cap S_\gamma(c).$$

Note further that $x_8, x_6, x_9, x_7, x_{10}, y_8, y_6, y_9, y_7, y_{10} \in S_\beta(a)$. Note also that the following hold:

$$\begin{aligned} \{x_8, x_6, x_9, x_7, x_{10}\} \cap \{y_8, y_6, y_9, y_7, y_{10}\} &= \emptyset \\ P_{\gamma, \alpha}(x_8, x_6, x_9, x_7, x_{10}) & \\ P_{\gamma, \alpha}(y_8, y_6, y_9, y_7, y_{10}) & \end{aligned}$$

Since $\|S_\beta(a)\| \geq 15$ by (61), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_8, x_6, x_9, x_7, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_8, y_6, y_9, y_7, y_{10})$$

to see that one of the following six possibilities must hold:

$$(Ai) \quad M_{\gamma, \alpha, \delta}^0(x_9, x_7, x_{10}, x_8, x_6, y_9, y_7, y_{10}, y_8, y_6)$$

$$(Aii) \quad M_{\gamma, \alpha, \delta}^0(x_7, x_{10}, x_8, x_6, x_9, y_7, y_{10}, y_8, y_6, y_9)$$

$$(Aiii) \quad M_{\gamma, \alpha, \delta}^2(x_{10}, x_7, x_9, x_6, x_8, y_{10}, y_7, y_9, y_6, y_8)$$

$$(Aiv) \quad M_{\gamma, \alpha, \delta}^2(x_8, x_6, x_9, x_7, x_{10}, y_8, y_6, y_9, y_7, y_{10})$$

$$(Av) \quad M_{\gamma, \alpha, \delta}^2(x_9, x_6, x_8, x_{10}, x_7, y_6, y_8, y_{10}, y_7, y_9)$$

$$(Avi) \quad M_{\gamma, \alpha, \delta}^2(x_9, x_7, x_{10}, x_8, x_6, y_7, y_{10}, y_8, y_6, y_9)$$

test

Now, we show that (Ai) must fail. Suppose not. Then (Ai) holds, so that $x_8 \overset{\gamma}{-} y_6$ and $x_6 \overset{\gamma}{-} y_8$. Note that $a, y_{11}, y_8, y_6, c, x_6, x_8, x_{11} \in S_\alpha(d)$. Note also

that the following hold:

$$\begin{aligned} \{y_{11}, y_8, y_6\} \cap \{x_6, x_8, x_{11}\} &= \emptyset \\ P_{\gamma, \beta}(a, y_{11}, y_8, y_6, c) \\ P_{\gamma, \beta}(c, x_6, x_8, x_{11}, a) \end{aligned}$$

Since $\|S_\alpha(d)\| \geq 15$ by (64), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_{11}, y_8, y_6, c, x_6, x_8, x_{11})$$

to get a contradiction. Thus, (Ai) must fail.

Now, we show that (Aii) must fail. Suppose not. Then (Aii) holds, so that $x_7 \xrightarrow{\gamma} y_{10}$ and $x_{10} \xrightarrow{\gamma} y_7$. Note that $a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13} \in S_\alpha(b)$. Note also that the following hold:

$$\begin{aligned} \{y_{13}, y_{10}, y_7\} \cap \{x_7, x_{10}, x_{13}\} &= \emptyset \\ P_{\gamma, \beta}(a, y_{13}, y_{10}, y_7, c) \\ P_{\gamma, \beta}(c, x_7, x_{10}, x_{13}, a) \end{aligned}$$

Since $\|S_\alpha(b)\| \geq 15$ by (62), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13})$$

to get a contradiction. Thus, (Aii) must fail.

Since both (Ai) and (Aii) fail, one of (Aiii), (Aiv), (Av), or (Avi) must hold. An application of the symmetry Θ to this says that one of (Biii), (Biv), (Bv), or (Bvi) must hold.

$$\begin{aligned} \text{(Biii)} \quad & M_{\beta, \alpha, \delta}^2(x_{15}, x_{14}, x_6, x_2, x_5, y_{15}, y_{14}, y_6, y_2, y_5) \\ \text{(Biv)} \quad & M_{\beta, \alpha, \delta}^2(x_5, x_2, x_6, x_{14}, x_{15}, y_5, y_2, y_6, y_{14}, y_{15}) \\ \text{(Bv)} \quad & M_{\beta, \alpha, \delta}^2(x_6, x_2, x_5, x_{15}, x_{14}, y_2, y_5, y_{15}, y_{14}, y_6) \\ \text{(Bvi)} \quad & M_{\beta, \alpha, \delta}^2(x_6, x_{14}, x_{15}, x_5, x_2, y_{14}, y_{15}, y_5, y_2, y_6) \end{aligned}$$

An application of the symmetry Θ to this says that one of (Ciii), (Civ), (Cv), or (Cvi) must hold.

$$\begin{aligned} \text{(Ciii)} \quad & M_{\gamma, \alpha, \delta}^2(x_8, x_{11}, x_2, x_{13}, x_{10}, y_8, y_{11}, y_2, y_{13}, y_{10}) \\ \text{(Civ)} \quad & M_{\gamma, \alpha, \delta}^2(x_{10}, x_{13}, x_2, x_{11}, x_8, y_{10}, y_{13}, y_2, y_{11}, y_8) \\ \text{(Cv)} \quad & M_{\gamma, \alpha, \delta}^2(x_2, x_{13}, x_{10}, x_8, x_{11}, y_{13}, y_{10}, y_8, y_{11}, y_2) \\ \text{(Cvi)} \quad & M_{\gamma, \alpha, \delta}^2(x_2, x_{11}, x_8, x_{10}, x_{13}, y_{11}, y_8, y_{10}, y_{13}, y_2) \end{aligned}$$

An application of the symmetry Θ to this says that one of (Diii), (Div), (Dv), or (Dvi) must hold.

$$\begin{aligned} \text{(Diii)} \quad & M_{\beta, \alpha, \delta}^2(x_5, x_3, x_{13}, x_9, x_{15}, y_5, y_3, y_{13}, y_9, y_{15}) \\ \text{(Div)} \quad & M_{\beta, \alpha, \delta}^2(x_{15}, x_9, x_{13}, x_3, x_5, y_5, y_3, y_{13}, y_9, y_{15}) \\ \text{(Dv)} \quad & M_{\beta, \alpha, \delta}^2(x_{13}, x_9, x_{15}, x_5, x_3, y_9, y_{15}, y_5, y_3, y_{13}) \\ \text{(Dvi)} \quad & M_{\beta, \alpha, \delta}^2(x_{13}, x_3, x_5, x_{15}, x_9, y_3, y_5, y_{15}, y_9, y_{13}) \end{aligned}$$

Now, we show that (Biii) must fail. Suppose not. Then (Biii) holds. If (Ciii) holds, then $x_{11} \xrightarrow{\alpha} y_2$. But $x_{14} \xrightarrow{\alpha} y_2$ by (Biii), which gives a monochromatic triangle since $x_{11} \xrightarrow{\alpha} x_{14}$. Thus, (Ciii) fails. If (Civ) holds, then $x_2 \xrightarrow{\alpha} y_{11}$. But $x_2 \xrightarrow{\alpha} y_{14}$ by (Biii), which gives a monochromatic triangle since $y_{11} \xrightarrow{\alpha} y_{14}$. Thus, (Civ) fails. If (Cv) holds, then $x_{10} \xrightarrow{\alpha} y_2$. But $x_6 \xrightarrow{\alpha} y_2$ by (Biii), which gives a monochromatic triangle since $x_6 \xrightarrow{\alpha} x_{10}$. Thus, (Cv) fails. Therefore, we conclude that (Cvi) must hold. If (d.iii) holds, then $x_3 \xrightarrow{\alpha} y_{13}$. But $x_{11} \xrightarrow{\alpha} y_{13}$ by (Cvi), which gives a monochromatic triangle since $x_{11} \xrightarrow{\alpha} y_3$. Thus, (Diii) fails. If (Div) holds, then $x_9 \xrightarrow{\alpha} y_{13}$. But $x_8 \xrightarrow{\alpha} y_{13}$ by (Cvi), which gives a monochromatic triangle since $x_9 \xrightarrow{\alpha} x_8$. Thus (Div) fails. Therefore, we conclude that either (Dv) or (Dvi) must hold. But then $x_5 \xrightarrow{\beta} y_5$, which is a contradiction, since $x_5 \xrightarrow{\delta} y_5$ by (Biii). Thus, we have shown that (Biii) fails, as desired.

Note that $M_{\beta, \alpha, \gamma}^2(y_{15}, y_{14}, y_6, y_2, y_5, x_{15}, x_{14}, x_6, x_2, x_5)$ must also fail, by an application of the symmetry Ψ . It is trivial to see, directly from Definition 2(4), that $M_{\beta, \alpha, \gamma}^2(y_{15}, y_{14}, y_6, y_2, y_5, x_{15}, x_{14}, x_6, x_2, x_5)$ is equivalent to (Biv). Thus, we have shown that (Biv) fails, also. That is, both (Biii) and (Biv) must fail. Thus, we have

$$(65) \quad \text{either (Bv) holds or (Bvi) holds.}$$

Repeated applications of the symmetry Θ give

$$(66) \quad \text{either (Cv) holds or (Cvi) holds,}$$

$$(67) \quad \text{either (Dv) holds or (Dvi) holds,}$$

and

$$(68) \quad \text{either (Av) holds or (Avi) holds.}$$

Now, suppose that (Cv) holds. Then we must have $x_{10} \xrightarrow{\gamma} y_{10}$, by (Cv). Also, $x_{13} \xrightarrow{\gamma} y_2$ by (Cv), and $y_2 \xrightarrow{\gamma} y_7$ so that the edge from x_{13} and y_7 cannot be of color γ , since otherwise we would have a monochromatic triangle. Note that $a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13} \in S_\alpha(b)$. Note also that the following hold:

$$\begin{aligned} \{y_{13}, y_{10}, y_7\} \cap \{x_7, x_{10}, x_{13}\} &= \emptyset \\ P_{\gamma, \beta}(a, y_{13}, y_{10}, y_7, c) & \\ P_{\gamma, \beta}(c, x_7, x_{10}, x_{13}, a) & \end{aligned}$$

Since $\|S_\alpha(b)\| \geq 15$ by (62), We may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_{13}, y_{10}, y_7, c, x_7, x_{10}, x_{13})$$

to get

$$N_{\delta, \beta, \gamma}^1(a, x_{13}, x_{10}, x_7, c, y_7, y_{10}, y_{13}).$$

But then $x_{13} \xrightarrow{\beta} y_7$. If (Dvi) holds, then $x_{13} \xrightarrow{\beta} y_5$, which gives a monochromatic triangle, since $y_5 \xrightarrow{\beta} y_7$. Thus, (Dvi) fails, so that by (67), (Dv) must hold. Thus,

we have shown that

$$(69) \quad \text{if (Cv) holds, then (Dv) holds.}$$

By an application of the symmetry Θ to (69), we see that

$$(70) \quad \text{if (Dv) holds, then (Av) holds.}$$

Now, we show that (Cv) must fail. Suppose not. Then (Cv) holds. By (69) and (70), (Av) also holds. But $x_5 \stackrel{\delta}{\sim} y_{15}$ by (Cv), whereas $x_5 \stackrel{\alpha}{\sim} y_{15}$ by (Av), which is a contradiction. Thus, we have shown that

$$(71) \quad \text{(Cv) fails,}$$

as desired.

By (66) and (71), we see that

$$(72) \quad \text{(Cvi) holds.}$$

The fact that $M_{\gamma, \alpha, \delta}^2(y_2, y_{13}, y_{10}, y_8, y_{11}, x_{13}, x_{10}, x_8, x_{11}, x_2)$ is equivalent to (Cvi) follows trivially from Definition 2(4). Thus, we have

$$(73) \quad M_{\gamma, \alpha, \delta}^2(y_2, y_{13}, y_{10}, y_8, y_{11}, x_{13}, x_{10}, x_8, x_{11}, x_2)$$

by (72). By an application of the symmetry Ψ to (73) we see that

$$(74) \quad \text{(Cv) holds.}$$

Thus, we have the desired contradiction, by (71) and (74). The proof is complete. \square

Proposition 30. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_0 = y_0$, $b = x_1 = y_4$, $c = x_{12} = y_{12}$, and $d = x_4 = y_1$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ a &\longmapsto b \longmapsto c \longmapsto d \longmapsto a \\ x_2 &\longmapsto x_{13} \longmapsto x_9 \longmapsto x_6 \longmapsto x_2 \\ x_3 &\longmapsto x_7 \longmapsto x_{14} \longmapsto x_{11} \longmapsto x_3 \\ x_5 &\longmapsto x_{10} \longmapsto x_{15} \longmapsto x_8 \longmapsto x_5 \\ y_2 &\longmapsto y_6 \longmapsto y_9 \longmapsto y_{13} \longmapsto y_2 \\ y_3 &\longmapsto y_{11} \longmapsto y_{14} \longmapsto y_7 \longmapsto y_3 \\ y_5 &\longmapsto y_8 \longmapsto y_{15} \longmapsto y_{10} \longmapsto y_5 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}\alpha &\mapsto \alpha \\ \beta &\mapsto \gamma \mapsto \beta \\ \delta &\mapsto \delta\end{aligned}$$

Next, we note that the hypotheses of the proposition are preserved by the following symmetry Ψ :

$$\begin{aligned}u &\mapsto v \mapsto u \\ a &\mapsto a \\ b &\mapsto d \mapsto b \\ c &\mapsto c \\ x_i &\mapsto y_i \mapsto x_i \quad \text{for all } i \in \{0, \dots, 15\}\end{aligned}$$

Note that the symmetry Ψ acts on the colors as follows:

$$\begin{aligned}\alpha &\mapsto \alpha \\ \beta &\mapsto \beta \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\gamma(a) = \{x_{11}, c, x_{13}, x_{14}, x_{15}\} \subseteq S_\gamma(a)$ and $S_\delta(v) \cap S_\gamma(a) = \{y_{11}, c, y_{13}, y_{14}, y_{15}\} \subseteq S_\gamma(a)$. Note further that both $P_{\beta,\alpha}(x_{11}, c, x_{13}, x_{14}, x_{15})$ and $P_{\beta,\alpha}(y_{11}, c, y_{13}, y_{14}, y_{15})$ hold and that

$$\|\{x_{11}, c, x_{13}, x_{14}, x_{15}\} \cap \{y_{11}, c, y_{13}, y_{14}, y_{15}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\gamma(a)\| \leq 14$ so that

$$(75) \quad \|S_\alpha(a)\|, \|S_\beta(a)\| \geq 15.$$

Repeated applications of the symmetry Θ to (75) give

$$(76) \quad \|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15,$$

$$(77) \quad \|S_\alpha(c)\|, \|S_\beta(c)\| \geq 15,$$

and

$$(78) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.$$

Note that either $x_2 \xrightarrow{\gamma} y_2$ or $x_2 \xrightarrow{\delta} y_2$ since $x_2, y_2 \in S_\alpha(a) \cap S_\beta(c)$. Note further that $d, y_3, y_5, y_2, b, x_3, x_5, x_2 \in S_\alpha(a)$. Note also that the following hold:

$$\begin{aligned}\{y_3, y_5, y_2\} \cap \{x_3, x_5, x_2\} &= \emptyset \\ P_{\beta,\gamma}(d, y_3, y_5, y_2, b) \\ P_{\beta,\gamma}(b, x_3, x_5, x_2, d)\end{aligned}$$

Since $\|S_\alpha(a)\| \geq 15$ by (75), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (d, y_3, y_5, y_2, b, x_3, x_5, x_2)$$

to see that

$$(79) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \beta}^1(d, x_2, x_5, x_3, b, y_2, y_5, y_3) \\ &\text{or } N_{\delta, \gamma, \beta}^2(d, x_2, x_5, x_3, b, y_2, y_5, y_3). \end{aligned}$$

By repeated applications of the symmetry Θ , we see that

$$(80) \quad \begin{aligned} &\text{either } N_{\delta, \beta, \gamma}^1(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}) \\ &\text{or } N_{\delta, \beta, \gamma}^2(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}), \end{aligned}$$

$$(81) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \beta}^1(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}) \\ &\text{or } N_{\delta, \gamma, \beta}^2(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}), \end{aligned}$$

and

$$(82) \quad \begin{aligned} &\text{either } N_{\delta, \beta, \gamma}^1(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7) \\ &\text{or } N_{\delta, \beta, \gamma}^2(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7). \end{aligned}$$

Now, we note the following:

$$x_2 \overset{\delta}{-} y_3 \text{ by (79).}$$

$$x_{14} \overset{\delta}{-} y_{15} \text{ by (81).}$$

$$x_2 \overset{\delta}{-} y_3 \text{ since } x_2, y_3 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$

$$x_{14} \overset{\delta}{-} y_9 \text{ since } x_{14}, y_9 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(d).$$

$$\text{Either } x_2 \overset{\alpha}{-} y_9 \text{ or } x_2 \overset{\delta}{-} y_9 \text{ since } x_2, y_9 \in S_\gamma(b) \cap S_\beta(d).$$

$$\text{Either } x_2 \overset{\alpha}{-} y_{13} \text{ or } x_2 \overset{\delta}{-} y_{13} \text{ since } x_2, y_{13} \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_{14} \overset{\alpha}{-} y_3 \text{ or } x_{14} \overset{\delta}{-} y_3 \text{ since } x_{14}, y_3 \in S_\gamma(b) \cap S_\beta(d).$$

$$\text{Either } x_6 \overset{\alpha}{-} y_9 \text{ or } x_6 \overset{\delta}{-} y_9 \text{ since } x_6, y_9 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_2 \overset{\beta}{-} y_5 \text{ or } x_2 \overset{\delta}{-} y_5 \text{ since } x_2, y_5 \in S_\gamma(b) \cap S_\alpha(a).$$

$$\text{Either } x_5 \overset{\beta}{-} y_5 \text{ or } x_5 \overset{\delta}{-} y_5 \text{ since } x_5, y_5 \in S_\gamma(b) \cap S_\alpha(a).$$

$$\text{Either } x_6 \overset{\beta}{-} y_{13} \text{ or } x_6 \overset{\delta}{-} y_{13} \text{ since } x_6, y_{13} \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_{15} \overset{\beta}{-} y_9 \text{ or } x_{15} \overset{\delta}{-} y_9 \text{ since } x_{15}, y_9 \in S_\gamma(b) \cap S_\alpha(c).$$

Note further that $x_{15}, x_5, x_2, x_6, x_{14}, y_{15}, y_5, y_3, y_{13}, y_9 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_{15}, x_5, x_2, x_6, x_{14}\} \cap \{y_{15}, y_5, y_3, y_{13}, y_9\} = \emptyset$$

$$P_{\beta,\alpha}(x_{15}, x_5, x_2, x_6, x_{14})$$

$$P_{\beta,\alpha}(y_{15}, y_5, y_3, y_{13}, y_9)$$

Since $\|S_\gamma(b)\| \geq 15$ by (76), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{15}, x_5, x_2, x_6, x_{14})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{15}, y_5, y_3, y_{13}, y_9)$$

to see that if $x_{15} \xrightarrow{\delta} y_{15}$, then we get a contradiction. Thus, we cannot have $x_{15} \xrightarrow{\delta} y_{15}$, so that, since $x_{15}, y_{15} \in S_\gamma(b) \cap S_\alpha(c)$, we must have $x_{15} \xrightarrow{\beta} y_{15}$. But this, combined with (81), implies that

$$(83) \quad N_{\delta,\gamma,\beta}^1(b, x_9, x_{15}, x_{14}, d, y_9, y_{15}, y_{14}).$$

By repeated applications of the symmetry Θ , we see that

$$(84) \quad N_{\delta,\beta,\gamma}^1(c, x_6, x_8, x_{11}, a, y_{13}, y_{10}, y_7),$$

$$(85) \quad N_{\delta,\gamma,\beta}^1(d, x_2, x_5, x_3, b, y_2, y_5, y_3),$$

and

$$(86) \quad N_{\delta,\beta,\gamma}^1(a, x_{13}, x_{10}, x_7, c, y_6, y_8, y_{11}).$$

Finally, we note the following:

$$x_{15} \xrightarrow{\delta} y_9 \text{ by (83).}$$

$$x_2 \xrightarrow{\delta} y_3 \text{ by (85).}$$

$$x_{14} \xrightarrow{\delta} y_{15} \text{ by (83).}$$

$$x_5 \xrightarrow{\beta} y_5 \text{ by (85).}$$

$$x_6 \xrightarrow{\beta} y_{13} \text{ by (84).}$$

$$x_{15} \xrightarrow{\beta} y_{15} \text{ by (83).}$$

Note further that $x_{15}, x_5, x_2, x_6, x_{14}, y_9, y_{13}, y_3, y_5, y_{15} \in S_\gamma(b)$. Note also that the following hold:

$$\{x_{15}, x_5, x_2, x_6, x_{14}\} \cap \{y_9, y_{13}, y_3, y_5, y_{15}\} = \emptyset$$

$$P_{\beta,\alpha}(x_{15}, x_5, x_2, x_6, x_{14})$$

$$P_{\beta,\alpha}(y_9, y_{13}, y_3, y_5, y_{15})$$

Since $\|S_\gamma(b)\| \geq 15$ by (76), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{15}, x_5, x_2, x_6, x_{14})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_9, y_{13}, y_3, y_5, y_{15})$$

to get the desired contradiction. The proof is complete. \square

Proposition 31. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, and $d = x_{15} = y_{14}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_8, x_2, x_{10}, x_{11}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{13}, y_8, y_2, y_{10}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha, \gamma}(b, x_8, x_2, x_{10}, x_{11})$ and $P_{\alpha, \gamma}(b, y_{13}, y_8, y_2, y_{10})$ hold and that

$$\|\{b, x_8, x_2, x_{10}, x_{11}\} \cap \{b, y_{13}, y_8, y_2, y_{10}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(87) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{d, x_{11}, x_1, x_8, x_5\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{d, y_{13}, y_3, y_{10}, y_4\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta, \gamma}(d, x_{11}, x_1, x_8, x_5)$ and $P_{\beta, \gamma}(d, y_{13}, y_3, y_{10}, y_4)$ hold and that

$$\|\{d, x_{11}, x_1, x_8, x_5\} \cap \{d, y_{13}, y_3, y_{10}, y_4\}\| = \|\{d\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \leq 14$ so that

$$(88) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_2, x_5, x_7, x_{11}\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{13}, y_3, y_6, y_2\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta, \gamma}(a, x_2, x_5, x_7, x_{11})$ and $P_{\beta, \gamma}(a, y_{13}, y_3, y_6, y_2)$ hold and that

$$\|\{a, x_2, x_5, x_7, x_{11}\} \cap \{a, y_{13}, y_3, y_6, y_2\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$(89) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\beta(d) = \{c, x_5, x_9, x_{10}, x_{11}\} \subseteq S_\beta(d)$ and $S_\delta(v) \cap S_\beta(d) = \{c, y_{13}, y_8, y_4, y_6\} \subseteq S_\beta(d)$. Note further that both $P_{\alpha, \gamma}(c, x_5, x_9, x_{10}, x_{11})$ and $P_{\alpha, \gamma}(c, y_{13}, y_8, y_4, y_6)$ hold and that

$$\|\{c, x_5, x_9, x_{10}, x_{11}\} \cap \{c, y_{13}, y_8, y_4, y_6\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(d)\| \leq 14$ so that

$$(90) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.$$

Now, we note the following:

$$x_9 \xrightarrow{\delta} y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_9 \xrightarrow{\delta} y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$\text{Either } x_9 \xrightarrow{\alpha} y_8 \text{ or } x_9 \xrightarrow{\delta} y_8 \text{ since } x_9, y_8 \in S_\gamma(c) \cap S_\beta(d).$$

Either $x_9 \overset{\alpha}{-} y_7$ or $x_9 \overset{\delta}{-} y_7$ since $x_9, y_7 \in S_\gamma(c) \cap S_\beta(b)$.

Note further that $x_1, x_9, x_0, x_{10}, x_3, y_1, y_8, y_0, y_7, y_4 \in s_\gamma(c)$. We also note the following:

$$\begin{aligned} \{x_1, x_9, x_0, x_{10}, x_3\} \cap \{y_1, y_8, y_0, y_7, y_4\} &= \emptyset \\ P_{\beta, \alpha}(x_1, x_9, x_0, x_{10}, x_3) & \\ P_{\beta, \alpha}(y_1, y_8, y_0, y_7, y_4) & \end{aligned}$$

Since $\|S_\gamma(c)\| \geq 15$ by (89), we may apply Lemma 14 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_1, x_9, x_0, x_{10}, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_8, y_0, y_7, y_4)$$

to see that either $x_9 \overset{\delta}{-} y_1, y_8, y_0, y_7, y_4$ or else $x_9 \overset{\alpha}{-} y_8, y_7$. But we cannot have $x_9 \overset{\alpha}{-} y_8, y_7$ since $y_8 \overset{\alpha}{-} y_7$, so we must have

$$(91) \quad x_9 \overset{\delta}{-} y_1, y_8, y_0, y_7, y_4.$$

Next, we note the following:

$$x_9 \overset{\delta}{-} y_7 \text{ by (91).}$$

$$x_3 \overset{\delta}{-} y_7 \text{ since } S_\beta(b) \cap S_\alpha(d) \cap S_\gamma(a).$$

Note further that $a, y_1, y_7, y_6, c, x_7, x_3, x_9 \in S_\beta(b)$. Note also that the following hold:

$$\begin{aligned} \{y_1, y_7, y_6\} \cap \{x_7, x_3, x_9\} &= \emptyset \\ P_{\alpha, \gamma}(a, y_1, y_7, y_6, c) & \\ P_{\alpha, \gamma}(c, x_7, x_3, x_9, a) & \end{aligned}$$

Since $\|S_\beta(b)\| \geq 15$ by (88), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_1, y_7, y_6, c, x_7, x_3, x_9)$$

to get

$$(92) \quad N_{\delta, \gamma, \alpha}^2(a, y_1, y_7, y_6, c, x_7, x_3, x_9).$$

Now, we note the following:

$$x_6 \overset{\delta}{-} y_5 \text{ since } x_6, y_5 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_7 \overset{\delta}{-} y_6 \text{ by (92).}$$

$$x_3 \overset{\delta}{-} y_7 \text{ by (92).}$$

$$x_3 \overset{\delta}{-} y_6 \text{ by (92).}$$

$$x_5 \overset{\delta}{-} y_6 \text{ since } x_5, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_7 \overset{\alpha}{-} y_7 \text{ by (92).}$$

Either $x_3 \xrightarrow{\beta} y_5$ or $x_3 \xrightarrow{\delta} y_5$ since $x_3, y_5 \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_6 \xrightarrow{\beta} y_7$ or $x_6 \xrightarrow{\delta} y_7$ since $x_6, y_7 \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_7 \xrightarrow{\beta} y_3$ or $x_7 \xrightarrow{\delta} y_3$ since $x_7, y_3 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_5 \xrightarrow{\beta} y_3$ or $x_5 \xrightarrow{\delta} y_3$ since $x_5, y_3 \in S_\gamma(a) \cap S_\alpha(c)$.

Note further that $x_6, x_0, x_7, x_5, x_3, y_5, y_3, y_6, y_0, y_7 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_6, x_0, x_7, x_5, x_3\} \cap \{y_5, y_3, y_6, y_0, y_7\} &= \emptyset \\ P_{\beta, \alpha}(x_6, x_0, x_7, x_5, x_3) \\ P_{\beta, \alpha}(y_5, y_3, y_6, y_0, y_7) \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (87), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_0, x_7, x_5, x_3)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_5, y_3, y_6, y_0, y_7)$$

to see that

$$(93) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_5, x_7, x_0, x_6, x_3, y_6, y_0, y_7, y_5, y_3) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_0, x_7, x_5, x_3, x_6, y_6, y_3, y_5, y_7, y_0). \end{aligned}$$

Next, we note the following:

$$x_6 \xrightarrow{\delta} y_5 \text{ by (93).}$$

$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(b) \cap S_\alpha(c) \cap S_\beta(a).$$

$$x_4 \xrightarrow{\delta} y_9 \text{ since } x_4, y_9 \in S_\gamma(b) \cap S_\alpha(a) \cap S_\beta(c).$$

$$x_0 \xrightarrow{\alpha} y_5 \text{ by (93).}$$

$$\text{Either } x_6 \xrightarrow{\alpha} y_0 \text{ or } x_6 \xrightarrow{\delta} y_0 \text{ by (93).}$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_2 \text{ or } x_{10} \xrightarrow{\delta} y_2 \text{ since } x_{10}, y_2 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_8 \text{ or } x_{10} \xrightarrow{\delta} y_8 \text{ since } x_{10}, y_8 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_4 \xrightarrow{\alpha} y_5 \text{ or } x_4 \xrightarrow{\delta} y_5 \text{ since } x_4, y_5 \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_2 \xrightarrow{\alpha} y_8 \text{ or } x_2 \xrightarrow{\delta} y_8 \text{ since } x_2, y_8 \in S_\gamma(b) \cap S_\beta(a).$$

$$\text{Either } x_6 \xrightarrow{\alpha} y_9 \text{ or } x_6 \xrightarrow{\delta} y_9 \text{ since } x_6, y_9 \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_6 \xrightarrow{\beta} y_2 \text{ or } x_6 \xrightarrow{\delta} y_2 \text{ since } x_6, y_2 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_2 \xrightarrow{\beta} y_5 \text{ or } x_2 \xrightarrow{\delta} y_5 \text{ since } x_2, y_5 \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_8, y_5, y_2, y_9, y_0 \in S_\gamma(b)$. Note also that the following hold:

$$\begin{aligned} \{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_8, y_5, y_2, y_9, y_0\} &= \emptyset \\ P_{\beta, \alpha}(x_0, x_6, x_2, x_4, x_{10}) & \\ P_{\beta, \alpha}(y_8, y_5, y_2, y_9, y_0) & \end{aligned}$$

Since $\|S_\gamma(b)\| \geq 15$ by (88), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_8, y_5, y_2, y_9, y_0)$$

to see that

$$(94) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^1(x_6, x_0, x_{10}, x_4, x_2, y_5, y_8, y_0, y_9, y_2) \\ &\text{or } M_{\beta, \alpha, \delta}^0(x_{10}, x_0, x_6, x_2, x_4, y_8, y_0, y_9, y_2, y_5) \\ &\text{or } M_{\beta, \alpha, \delta}^1(x_6, x_0, x_{10}, x_4, x_2, y_5, y_2, y_9, y_0, y_8). \end{aligned}$$

Note that $x_2 \overset{\beta}{-} y_5$ by (94). We cannot have $x_5 \overset{\beta}{-} y_5$, since if we did, the fact that $x_2 \overset{\beta}{-} x_5$ would result in a monochromatic triangle. Since we cannot have $x_5 \overset{\beta}{-} y_5$, we see from (93) that

$$(95) \quad M_{\beta, \alpha, \delta}^2(x_0, x_7, x_5, x_3, x_6, y_6, y_3, y_5, y_7, y_0).$$

Now, note that $x_6 \overset{\delta}{-} y_0$ by (95), so that we see from (94) that

$$(96) \quad M_{\beta, \alpha, \delta}^0(x_{10}, x_0, x_6, x_2, x_4, y_8, y_0, y_9, y_2, y_5).$$

By (95), we have $x_3 \overset{\beta}{-} y_0$. By (96), we have $x_{10} \overset{\beta}{-} y_0$. But this gives a monochromatic triangle since $x_3 \overset{\beta}{-} x_{10}$, thus producing the desired contradiction. The proof is complete. \square

Proposition 32. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, e\}$ where $a = x_{12} = y_{12}$, $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(a) = \{c, x_{15}, x_9, x_1, x_4\} \subseteq S_\alpha(a)$ and $S_\delta(v) \cap S_\alpha(a) = \{c, y_9, y_1, y_4, y_{14}\} \subseteq S_\alpha(a)$. Note further that both $P_{\beta, \gamma}(c, x_{15}, x_9, x_1, x_4)$ and $P_{\beta, \gamma}(c, y_9, y_1, y_4, y_{14})$ hold and that

$$\|\{c, x_{15}, x_9, x_1, x_4\} \cap \{c, y_9, y_1, y_4, y_{14}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(a)\| \leq 14$ so that

$$(97) \quad \|S_\beta(a)\|, \|S_\gamma(a)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{e, x_{15}, x_5, x_8, x_1\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{e, y_3, y_{10}, y_4, y_{14}\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta,\gamma}(e, x_{15}, x_5, x_8, x_1)$ and $P_{\beta,\gamma}(e, y_3, y_{10}, y_4, y_{14})$ hold and that

$$\|\{e, x_{15}, x_5, x_8, x_1\} \cap \{e, y_3, y_{10}, y_4, y_{14}\}\| = \|\{e\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \leq 14$ so that

$$(98) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\beta(c) = \{b, x_8, x_4, x_6, x_{15}\} \subseteq S_\beta(c)$ and $S_\delta(v) \cap S_\beta(c) = \{b, y_{10}, y_9, y_5, y_{14}\} \subseteq S_\beta(c)$. Note further that both $P_{\alpha,\gamma}(b, x_8, x_4, x_6, x_{15})$ and $P_{\alpha,\gamma}(b, y_{10}, y_9, y_5, y_{14})$ hold and that

$$\|\{b, x_8, x_4, x_6, x_{15}\} \cap \{b, y_{10}, y_9, y_5, y_{14}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(c)\| \leq 14$ so that

$$(99) \quad \|S_\alpha(c)\|, \|S_\gamma(c)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\beta(e) = \{a, x_1, x_7, x_6, x_{15}\} \subseteq S_\beta(e)$ and $S_\delta(v) \cap S_\beta(e) = \{a, y_9, y_3, y_7, y_{14}\} \subseteq S_\beta(e)$. Note further that both $P_{\alpha,\gamma}(a, x_1, x_7, x_6, x_{15})$ and $P_{\alpha,\gamma}(a, y_9, y_3, y_7, y_{14})$ hold and that

$$\|\{a, x_1, x_7, x_6, x_{15}\} \cap \{a, y_9, y_3, y_7, y_{14}\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(e)\| \leq 14$ so that

$$(100) \quad \|S_\alpha(e)\|, \|S_\gamma(e)\| \geq 15.$$

Note that $x_2 \stackrel{\delta}{-} y_2$ since $x_2, y_2 \in S_\beta(a) \cap S_\alpha(c) \cap S_\gamma(e)$. Note further that $e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10} \in S_\beta(a)$. Note also that the following hold:

$$\begin{aligned} \{y_8, y_2, y_{10}\} \cap \{x_8, x_2, x_{10}\} &= \emptyset \\ P_{\alpha,\gamma}(e, y_8, y_2, y_{10}, b) \\ P_{\alpha,\gamma}(b, x_8, x_2, x_{10}, e) \end{aligned}$$

Since $\|S_\beta(a)\| \geq 15$ by (97), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10})$$

to see that

$$(101) \quad \begin{aligned} &\text{either } N_{\delta,\gamma,\alpha}^2(e, y_8, y_2, y_{10}, b, x_8, x_2, x_{10}) \\ &\text{or } N_{\delta,\gamma,\alpha}^2(e, x_{10}, x_2, x_8, b, y_{10}, y_2, y_8). \end{aligned}$$

Next, we note the following:

$$x_8 \stackrel{\delta}{-} y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$

$$x_2 \stackrel{\delta}{-} y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$

$$\text{Either } x_5 \stackrel{\beta}{-} y_2 \text{ or } x_5 \stackrel{\delta}{-} y_2 \text{ since } x_5, y_2 \in S_\gamma(e) \cap S_\alpha(c).$$

Either $x_5 \overset{\beta}{-} y_4$ or $x_5 \overset{\delta}{-} y_4$ since $x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b)$.
 Either $x_2 \overset{\beta}{-} y_6$ or $x_2 \overset{\delta}{-} y_6$ since $x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c)$.
 Either $x_5 \overset{\beta}{-} y_{10}$ or $x_5 \overset{\delta}{-} y_{10}$ since $x_5, y_{10} \in S_\gamma(e) \cap S_\alpha(b)$.
 Either $x_8 \overset{\beta}{-} y_4$ or $x_8 \overset{\delta}{-} y_4$ since $x_8, y_4 \in S_\gamma(e) \cap S_\alpha(b)$.
 Either $x_5 \overset{\beta}{-} y_6$ or $x_5 \overset{\delta}{-} y_6$ since $x_5, y_6 \in S_\gamma(e) \cap S_\alpha(b)$.
 Either $x_9 \overset{\beta}{-} y_4$ or $x_9 \overset{\delta}{-} y_4$ since $x_9, y_4 \in S_\gamma(e) \cap S_\alpha(a)$.
 Either $x_8 \overset{\alpha}{-} y_2$ or $x_8 \overset{\delta}{-} y_2$ since $x_8, y_2 \in S_\gamma(e) \cap S_\beta(a)$.
 Either $x_2 \overset{\alpha}{-} y_{10}$ or $x_2 \overset{\delta}{-} y_{10}$ since $x_2, y_{10} \in S_\gamma(e) \cap S_\beta(a)$.

Note further that $x_9, x_0, x_8, x_5, x_2, y_6, y_0, y_{10}, y_4, y_2 \in S_\gamma(e)$. Note also that the following hold:

$$\begin{aligned}
 \{x_9, x_0, x_8, x_5, x_2\} \cap \{y_6, y_0, y_{10}, y_4, y_2\} &= \emptyset \\
 P_{\beta, \alpha}(x_9, x_0, x_8, x_5, x_2) & \\
 P_{\beta, \alpha}(y_6, y_0, y_{10}, y_4, y_2) &
 \end{aligned}$$

Since $\|S_\gamma(e)\| \geq 15$ by (100), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_8, x_5, x_2)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_6, y_0, y_{10}, y_4, y_2)$$

to see that

$$(102) \quad \text{if } x_9 \overset{\delta}{-} y_6, \text{ then } M_{\beta, \alpha, \gamma}^0(x_8, x_5, x_2, x_9, x_0, y_{10}, y_4, y_2, y_6, y_0).$$

Now, if $x_9 \overset{\delta}{-} y_6$, then we would have both $x_8 \overset{\delta}{-} y_2$ and $x_2 \overset{\delta}{-} y_{10}$ by (102), which would then contradict (101). Thus, we cannot have $x_9 \overset{\delta}{-} y_6$, so that, since $x_9, y_6 \in S_\gamma(e) \cap S_\beta(b)$, we must have

$$(103) \quad x_9 \overset{\alpha}{-} y_6.$$

Note that $x_9, x_0, x_8, x_5, x_2, y_6, y_0, y_{10}, y_4, y_2 \in S_\gamma(e)$. Note also that the following hold:

$$\begin{aligned}
 \{x_9, x_0, x_8, x_5, x_2\} \cap \{y_6, y_0, y_{10}, y_4, y_2\} &= \emptyset \\
 P_{\beta, \alpha}(x_9, x_0, x_8, x_5, x_2) & \\
 P_{\beta, \alpha}(y_6, y_0, y_{10}, y_4, y_2) &
 \end{aligned}$$

Noting also that $\|S_\gamma(e)\| \geq 15$ by (100), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_8, x_5, x_2)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_6, y_0, y_{10}, y_4, y_2).$$

We now show that $x_5 \overset{\delta}{-} y_4$. Suppose not. Then $x_5 \overset{\beta}{-} y_4$ since $x_5, y_4 \in S_\gamma(e) \cap S_\alpha(b)$. We also note the following:

$$x_9 \overset{\alpha}{-} y_6 \text{ by (103).}$$

$$x_8 \overset{\delta}{-} y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$

$$x_2 \overset{\delta}{-} y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$

$$x_5 \overset{\delta}{-} y_2 \text{ since } x_5, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(y_4).$$

$$x_5 \overset{\delta}{-} y_{10} \text{ since } x_5, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(y_4).$$

$x_5 \overset{\beta}{-} y_6$ by Lemma 14, since $x_5 \overset{\delta}{-} y_2, y_{10}$ and $x_5 \overset{\beta}{-} y_4$ and $x_5, y_6 \in S_\alpha(x_9)$.

$$x_5 \overset{\alpha}{-} y_6 \text{ by Lemma 14, since } x_5 \overset{\delta}{-} y_2, y_{10} \text{ and } x_5 \overset{\beta}{-} y_4, y_6.$$

$$x_2 \overset{\delta}{-} y_6 \text{ since } x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(x_5).$$

$$x_8 \overset{\delta}{-} y_4 \text{ since } x_8, y_4 \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(x_5).$$

$$x_8 \overset{\delta}{-} y_6 \text{ since } x_8, y_6 \in S_\gamma(e) \cap S_\alpha(x_9) \cap S_\beta(x_5).$$

$$x_8 \overset{\delta}{-} y_2, y_0 \text{ by Lemma 14, since } x_8 \overset{\delta}{-} y_6, y_{10}, y_4.$$

Now, by Lemma 14, since $x_9 \overset{\alpha}{-} x_8 \overset{\delta}{-} y_6, y_0, y_{10}, y_4, y_2$, we must have two α -colored edges from x_9 to $\{y_6, y_0, y_{10}, y_4, y_2\}$. But $x_9 \overset{\delta}{-} y_2$ and $x_9, y_4, y_{10} \in S_\alpha(y_6)$ and $x_9, x_0 \in S_\alpha(y_5)$, so that this is impossible, thus giving a contradiction. Therefore, we have

$$(104) \quad x_5 \overset{\delta}{-} y_4,$$

as desired.

Finally, we note the following:

$$x_8 \overset{\delta}{-} y_{10} \text{ since } x_8, y_{10} \in S_\gamma(e) \cap S_\alpha(b) \cap S_\beta(a).$$

$$x_5 \overset{\delta}{-} y_4 \text{ by (104).}$$

$$x_2 \overset{\delta}{-} y_2 \text{ since } x_2, y_2 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(a).$$

$$x_9 \overset{\alpha}{-} y_6 \text{ by (103).}$$

Either $x_2 \overset{\beta}{-} y_6$ or $x_2 \overset{\delta}{-} y_6$ since $x_2, y_6 \in S_\gamma(e) \cap S_\alpha(c)$.

Note further that $x_0, x_8, x_5, x_2, x_9, y_0, y_{10}, y_4, y_2, y_6 \in S_\gamma(e)$. Note also that the following hold:

$$\{x_0, x_8, x_5, x_2, x_9\} \cap \{y_0, y_{10}, y_4, y_2, y_6\} = \emptyset$$

$$P_{\beta, \alpha}(x_0, x_8, x_5, x_2, x_9)$$

$$P_{\beta, \alpha}(y_0, y_{10}, y_4, y_2, y_6)$$

Since $\|S_\gamma(e)\| \geq 15$ by (100), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_8, x_5, x_2, x_9)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_{10}, y_4, y_2, y_6)$$

to see that

$$(105) \quad M_{\beta, \alpha, \delta}^1(x_2, x_9, x_0, x_8, x_5, y_2, y_4, y_{10}, y_0, y_6).$$

Finally, we note the following:

$$x_5 \xrightarrow{\beta} y_2 \text{ by (105).}$$

$$x_7 \xrightarrow{\delta} y_6 \text{ since } x_7, y_6 \in S_\gamma(e) \cap S_\alpha(c) \cap S_\beta(b).$$

Note further that $a, y_2, y_6, y_3, e, x_7, x_5, x_2 \in S_\alpha(c)$. Note also that the following hold:

$$\begin{aligned} \{y_2, y_6, y_3\} \cap \{x_7, x_5, x_2\} &= \emptyset \\ P_{\beta, \gamma}(a, y_2, y_6, y_3, e) \\ P_{\beta, \gamma}(e, x_7, x_5, x_2, a) \end{aligned}$$

Since $\|S_\alpha(c)\| \geq 15$ by (99), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_2, y_6, y_3, e, x_7, x_5, x_2)$$

to produce the desired contradiction. The theorem is proved. \square

Proposition 33. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{13} = y_{11}$, $c = x_{14} = y_{15}$, $d = x_{15} = y_{14}$, and $e = x_{11} = y_{13}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto v \mapsto u \\ b &\mapsto e \mapsto b \\ c &\mapsto d \mapsto c \\ x_i &\mapsto y_i \mapsto x_i \quad \text{for all } i \in \{0, \dots, 15\} \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \alpha \\ \beta &\mapsto \beta \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(c) = \{e, x_{12}, x_2, x_5, x_7\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{e, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(e, x_{12}, x_2, x_5, x_7)$ and $P_{\beta,\gamma}(e, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{e, x_{12}, x_2, x_5, x_7\} \cap \{e, y_{12}, y_2, y_6, y_3\}\| = \|\{e\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$(106) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

An application of the symmetry Θ to (106) gives

$$(107) \quad \|S_\beta(d)\|, \|S_\gamma(d)\| \geq 15.$$

First, note that $c, y_6, y_4, y_8, e, x_{10}, x_9, x_5 \in S_\beta(d)$. Note also that the following hold:

$$\begin{aligned} \{y_6, y_4, y_8\} \cap \{x_{10}, x_9, x_5\} &= \emptyset \\ P_{\alpha,\gamma}(c, y_6, y_4, y_8, e) \\ P_{\alpha,\gamma}(e, x_{10}, x_9, x_5, c) \end{aligned}$$

Since $\|S_\beta(d)\| \geq 15$ by (107), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_6, y_4, y_8, e, x_{10}, x_9, x_5)$$

to see that

$$(108) \quad \text{if } x_9 \xrightarrow{\alpha} y_8, \text{ then } N_{\delta,\gamma,\alpha}^2(c, y_6, y_4, y_8, e, x_{10}, x_9, x_5).$$

Next, we show that $x_9 \xrightarrow{\delta} y_8$. Suppose not. Then $x_9 \xrightarrow{\alpha} y_8$, since $x_9, y_8 \in S_\beta(d) \cap S_\gamma(c)$. But then, we have $x_5 \xrightarrow{\alpha} y_4$ by (108), which is impossible since $x_5, y_4 \in S_\alpha(b)$. Thus,

$$(109) \quad x_9 \xrightarrow{\delta} y_8,$$

as desired.

Note that $x_3, x_1, x_9, x_0, x_{10}, y_7, y_0, y_8, y_1, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{aligned} \{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_7, y_0, y_8, y_1, y_4\} &= \emptyset \\ P_{\beta,\alpha}(x_3, x_1, x_9, x_0, x_{10}) \\ P_{\beta,\alpha}(y_7, y_0, y_8, y_1, y_4) \end{aligned}$$

Noting also that $\|S_\gamma(c)\| \geq 15$ by (106), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_0, y_8, y_1, y_4).$$

We now show that $x_{10} \overset{\delta}{-} y_4$. Suppose not. Then $x_{10} \overset{\delta}{-} y_4$, since $x_{10}, y_4 \in S_\gamma(c) \cap S_\beta(d)$. We also note the following:

$$x_9 \overset{\delta}{-} y_4 \text{ since } x_9, y_4 \in S_\gamma(c) \cap S_\alpha(x_{10}) \cap S_\beta(d).$$

$$\text{Either } x_9 \overset{\alpha}{-} y_1 \text{ or } x_9 \overset{\delta}{-} y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_9 \overset{\alpha}{-} y_7 \text{ or } x_9 \overset{\delta}{-} y_7 \text{ since } x_9, y_7 \in S_\gamma(c) \cap S_\beta(b).$$

Also, $x_9 \overset{\delta}{-} y_1, y_0, y_7$ since otherwise we would have, by Lemma 14, in view of the fact that $x_9 \overset{\delta}{-} y_8, y_4$, that $x_9 \overset{\alpha}{-} y_1, y_7$, which is impossible since $y_1 \overset{\alpha}{-} y_7$. But now, since $x_3 \overset{\delta}{-} y_1, y_7$ and $x_3 \overset{\alpha}{-} y_9$, we must have two α -colored edges from x_3 to $\{y_8, y_0, y_4\}$, by Lemma 14. But $x_3, y_8 \in S_\alpha(e)$, so that we must have $x_3 \overset{\alpha}{-} y_0, y_4$. but this gives a contradiction since $y_0 \overset{\alpha}{-} y_4$. Thus, we have

$$(110) \quad x_{10} \overset{\delta}{-} y_4,$$

as desired.

Now, note that the following hold:

$$x_3 \overset{\delta}{-} y_7 \text{ since } x_3, y_7 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_9 \overset{\delta}{-} y_8 \text{ by (109).}$$

$$x_{10} \overset{\delta}{-} y_4 \text{ by (110).}$$

$$x_3 \overset{\delta}{-} y_1 \text{ since } x_3, y_1 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(b).$$

$$x_{10} \overset{\delta}{-} y_8 \text{ since } x_{10}, y_8 \in S_\gamma(c) \cap S_\alpha(e) \cap S_\beta(d).$$

$$\text{Either } x_1 \overset{\alpha}{-} y_7 \text{ or } x_1 \overset{\delta}{-} y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(e).$$

$$\text{Either } x_3 \overset{\beta}{-} y_8 \text{ or } x_3 \overset{\delta}{-} y_8 \text{ since } x_3, y_8 \in S_\gamma(c) \cap S_\alpha(e).$$

$$\text{Either } x_9 \overset{\alpha}{-} y_1 \text{ or } x_9 \overset{\delta}{-} y_1 \text{ since } x_9, y_1 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_3, x_1, x_9, x_0, x_{10}, y_7, y_0, y_8, y_1, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\{x_3, x_1, x_9, x_0, x_{10}\} \cap \{y_7, y_0, y_8, y_1, y_4\} = \emptyset$$

$$P_{\beta, \alpha}(x_3, x_1, x_9, x_0, x_{10})$$

$$P_{\beta, \alpha}(y_7, y_0, y_8, y_1, y_4)$$

Since $\|S_\gamma(e)\| \geq 15$ by (111), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_3, x_1, x_9, x_0, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_0, y_8, y_1, y_4)$$

to get the desired contradiction. The proof is complete. \square

Proposition 34. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto u \\ v &\mapsto v \\ b &\mapsto c \mapsto e \mapsto d \mapsto b \\ x_0 &\mapsto x_0 \\ x_{13} &\mapsto x_{13} \\ x_3 &\mapsto x_8 \mapsto x_3 \\ x_1 &\mapsto x_7 \mapsto x_5 \mapsto x_9 \mapsto x_1 \\ x_2 &\mapsto x_{10} \mapsto x_4 \mapsto x_6 \mapsto x_2 \\ y_0 &\mapsto y_0 \\ y_{13} &\mapsto y_{13} \\ y_3 &\mapsto y_8 \mapsto y_3 \\ y_1 &\mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\ y_2 &\mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \beta \mapsto \alpha \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_6, x_4, x_8, x_{13}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_6, y_4, y_8, y_{13}\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha, \gamma}(c, x_6, x_4, x_8, x_{13})$ and $P_{\alpha, \gamma}(c, y_6, y_4, y_8, y_{13})$ hold and that

$$\|\{c, x_6, x_4, x_8, x_{13}\} \cap \{c, y_6, y_4, y_8, y_{13}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$(111) \quad \|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15.$$

An application of the symmetry Θ to (111) gives

$$(112) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Now, note that $x_9 \overset{\delta}{\sim} y_9$. Note further that $d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10} \in S_\beta(c)$. Note also that the following hold:

$$\begin{aligned} \{y_{10}, y_9, y_5\} \cap \{x_5, x_9, x_{10}\} &= \emptyset \\ P_{\alpha, \gamma}(d, y_{10}, y_9, y_5, b) \\ P_{\alpha, \gamma}(b, x_5, x_9, x_{10}, d) \end{aligned}$$

Since $\|S_\beta(c)\| \geq 15$ by (112), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10})$$

to see that

$$(113) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \alpha}^2(d, y_{10}, y_9, y_5, b, x_5, x_9, x_{10}) \\ &\text{or } N_{\delta, \gamma, \alpha}^2(d, x_{10}, x_9, x_5, b, y_5, y_9, y_{10}). \end{aligned}$$

Next, note that either $x_{10} \overset{\delta}{\sim} y_9$ or $x_9 \overset{\delta}{\sim} y_{10}$ by (113). Note also that the following hold:

$$\begin{aligned} &x_9 \overset{\delta}{\sim} y_9 \text{ since } x_9, y_9 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c). \\ &x_{10} \overset{\delta}{\sim} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c). \\ &x_1 \overset{\delta}{\sim} y_1 \text{ since } x_1, y_1 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d). \\ &\text{Either } x_3 \overset{\beta}{\sim} y_3 \text{ or } x_3 \overset{\delta}{\sim} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(c). \\ &\text{Either } x_3 \overset{\beta}{\sim} y_{10} \text{ or } x_3 \overset{\delta}{\sim} y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d). \\ &\text{Either } x_{10} \overset{\beta}{\sim} y_3 \text{ or } x_{10} \overset{\delta}{\sim} y_3 \text{ since } x_{10}, y_3 \in S_\gamma(b) \cap S_\alpha(d). \\ &\text{Either } x_1 \overset{\beta}{\sim} y_9 \text{ or } x_1 \overset{\delta}{\sim} y_9 \text{ since } x_1, y_9 \in S_\gamma(b) \cap S_\alpha(e). \\ &\text{Either } x_9 \overset{\beta}{\sim} y_1 \text{ or } x_9 \overset{\delta}{\sim} y_1 \text{ since } x_9, y_1 \in S_\gamma(b) \cap S_\alpha(e). \\ &\text{Either } x_9 \overset{\alpha}{\sim} y_{10} \text{ or } x_9 \overset{\delta}{\sim} y_{10} \text{ since } x_9, y_{10} \in S_\gamma(b) \cap S_\beta(c). \\ &\text{Either } x_{10} \overset{\alpha}{\sim} y_9 \text{ or } x_{10} \overset{\delta}{\sim} y_9 \text{ since } x_{10}, y_9 \in S_\gamma(b) \cap S_\beta(c). \end{aligned}$$

Note further that $x_9, x_0, x_{10}, x_3, x_1, y_9, y_0, y_{10}, y_3, y_1 \in S_\gamma(b)$. Note also that the following hold:

$$\begin{aligned} \{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_9, y_0, y_{10}, y_3, y_1\} &= \emptyset \\ P_{\beta, \alpha}(x_9, x_0, x_{10}, x_3, x_1) \\ P_{\beta, \alpha}(y_9, y_0, y_{10}, y_3, y_1) \end{aligned}$$

Since $\|S_\gamma(b)\| \geq 15$ by (111), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_9, y_0, y_{10}, y_3, y_1)$$

to see that

$$(114) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_3, x_{10}, x_0, x_9, x_1, y_{10}, y_3, y_1, y_9, y_0) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_0, x_9, x_1, x_3, x_{10}, y_1, y_9, y_0, y_{10}, y_3). \end{aligned}$$

By (114), we have

$$(115) \quad x_0 \xrightarrow{\alpha} y_0.$$

Applying the symmetry Θ to (115) gives

$$(116) \quad x_0 \xrightarrow{\beta} y_0.$$

(115) and (116) together produce the desired contradiction. The proof is complete. \square

Proposition 35. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_{11}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_7, x_3, x_9, x_{12}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_7, y_3, y_9, y_{12}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha, \gamma}(b, x_7, x_3, x_9, x_{12})$ and $P_{\alpha, \gamma}(b, y_7, y_3, y_9, y_{12})$ hold and that

$$\|\{b, x_7, x_3, x_9, x_{12}\} \cap \{b, y_7, y_3, y_9, y_{12}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(117) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_{12}, x_2, x_6, x_3\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta, \gamma}(a, x_{12}, x_2, x_6, x_3)$ and $P_{\beta, \gamma}(a, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{a, x_{12}, x_2, x_6, x_3\} \cap \{a, y_{12}, y_2, y_6, y_3\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$(118) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Now, note that $x_8 \xrightarrow{\delta} y_8$ since $x_8, y_8 \in S_\alpha(a) \cap S_\beta(b) \cap S_\gamma(c)$. Note further that $d, y_1, y_8, y_5, c, x_5, x_8, x_1 \in S_\alpha(a)$. Note also that the following hold:

$$\begin{aligned} &\{y_1, y_8, y_5\} \cap \{x_5, x_8, x_1\} = \emptyset \\ &P_{\beta, \alpha}(d, y_1, y_8, y_5, c) \\ &P_{\beta, \alpha}(c, x_5, x_8, x_1, d) \end{aligned}$$

Since $\|S_\alpha(a)\| \geq 15$ by (117), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (d, y_1, y_8, y_5, c, x_5, x_8, x_1)$$

to see that

$$(119) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \beta}^2(d, y_1, y_8, y_5, c, x_5, x_8, x_1) \\ &\text{or } N_{\delta, \gamma, \beta}^2(d, x_1, x_8, x_5, c, y_5, y_8, y_1). \end{aligned}$$

Thus, by (119), we see that

$$(120) \quad \text{either } x_8 \overset{\beta}{-} y_1 \text{ or } x_1 \overset{\beta}{-} y_8.$$

Also, by (119), we see that

$$(121) \quad \text{either } x_1 \overset{\delta}{-} y_8 \text{ or } x_8 \overset{\delta}{-} y_1.$$

Next, we note the following:

$$x_1 \overset{\delta}{-} y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$x_8 \overset{\delta}{-} y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_7 \overset{\delta}{-} y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(a).$$

$$x_4 \overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$

$$\text{Either } x_1 \overset{\alpha}{-} y_7 \text{ or } x_1 \overset{\delta}{-} y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_8 \overset{\alpha}{-} y_4 \text{ or } x_8 \overset{\delta}{-} y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_7 \overset{\alpha}{-} y_1 \text{ or } x_7 \overset{\delta}{-} y_1 \text{ since } x_7, y_1 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_4 \overset{\alpha}{-} y_8 \text{ or } x_4 \overset{\delta}{-} y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).$$

Note further that $x_1, x_8, x_0, x_7, x_4, y_1, y_8, y_0, y_7, y_4 \in S_\gamma(c)$. Note also that the following hold:

$$\{x_1, x_8, x_0, x_7, x_4\} \cap \{y_1, y_8, y_0, y_7, y_4\} = \emptyset$$

$$P_{\beta, \alpha}(x_1, x_8, x_0, x_7, x_4)$$

$$P_{\beta, \alpha}(y_1, y_8, y_0, y_7, y_4)$$

Since $\|S_\gamma(c)\| \geq 15$ by (118), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_1, x_8, x_0, x_7, x_4)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_1, y_8, y_0, y_7, y_4)$$

to see, with the help of (120) and (121), that

$$(122) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \gamma}^2(x_0, x_7, x_4, x_1, x_8, y_0, y_7, y_4, y_1, y_8) \\ &\text{or } M_{\beta, \alpha, \gamma}^2(x_8, x_1, x_4, x_7, x_0, y_8, y_1, y_4, y_7, y_0). \end{aligned}$$

Thus, we have

$$(123) \quad \text{either } x_0 \overset{\alpha}{-} y_4 \text{ or } x_4 \overset{\alpha}{-} y_0,$$

by (122).

Finally, we note the following:

$$\begin{aligned} x_{10} &\overset{\delta}{-} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c). \\ x_4 &\overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \\ x_6 &\overset{\delta}{-} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\ &x_0 \overset{\delta}{-} y_0 \text{ by (122)}. \end{aligned}$$

Either $x_{10} \overset{\beta}{-} y_4$ or $x_{10} \overset{\delta}{-} y_4$ since $x_{10}, y_4 \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_4 \overset{\beta}{-} y_{10}$ or $x_4 \overset{\delta}{-} y_{10}$ since $x_4, y_{10} \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_2 \overset{\beta}{-} y_6$ or $x_2 \overset{\delta}{-} y_6$ since $x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_6 \overset{\beta}{-} y_2$ or $x_6 \overset{\delta}{-} y_2$ since $x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_6 \overset{\alpha}{-} y_4$ or $x_6 \overset{\delta}{-} y_4$ since $x_6, y_4 \in S_\gamma(a) \cap S_\beta(b)$.

Either $x_4 \overset{\alpha}{-} y_6$ or $x_4 \overset{\delta}{-} y_6$ since $x_4, y_6 \in S_\gamma(a) \cap S_\beta(b)$.

Note further that $x_{10}, x_4, x_2, x_6, x_0, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_{10}, x_4, x_2, x_6, x_0\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} &= \emptyset \\ P_{\beta, \alpha}(x_{10}, x_4, x_2, x_6, x_0) & \\ P_{\beta, \alpha}(y_{10}, y_4, y_2, y_6, y_0) & \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (117), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_{10}, x_4, x_2, x_6, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to get, with the help of (123), the desired contradiction. The proof is complete. \square

Proposition 36. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ b &\longmapsto c \longmapsto e \longmapsto d \longmapsto b \end{aligned}$$

$$\begin{aligned}
x_0 &\mapsto x_0 \\
x_{13} &\mapsto x_{13} \\
x_3 &\mapsto x_8 \mapsto x_3 \\
x_1 &\mapsto x_9 \mapsto x_5 \mapsto x_7 \mapsto x_1 \\
x_2 &\mapsto x_6 \mapsto x_4 \mapsto x_{10} \mapsto x_2 \\
y_0 &\mapsto y_0 \\
y_{13} &\mapsto y_{13} \\
y_3 &\mapsto y_8 \mapsto y_3 \\
y_1 &\mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\
y_2 &\mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2
\end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
\alpha &\mapsto \beta \mapsto \alpha \\
\gamma &\mapsto \gamma \\
\delta &\mapsto \delta
\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_{13}, x_8, x_4, x_{10}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_{13}, y_8, y_4, y_6\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha,\gamma}(c, x_{13}, x_8, x_4, x_{10})$ and $P_{\alpha,\gamma}(c, y_{13}, y_8, y_4, y_6)$ hold and that

$$\|\{c, x_{13}, x_8, x_4, x_{10}\} \cap \{c, y_{13}, y_8, y_4, y_6\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$(124) \quad \|S_\alpha(b)\|, \|S_\gamma(b)\| \geq 15.$$

An application of the symmetry Θ to (124) gives

$$(125) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Now, we note that $x_5 \stackrel{\delta}{\sim} y_9$ since $x_5, y_9 \in S_\beta(c) \cap S_\alpha(e) \cap S_\gamma(d)$. Note further that $b, y_5, y_9, y_{10}, d, x_6, x_5, x_9 \in S_\beta(c)$. note also that the following hold:

$$\begin{aligned}
\{y_5, y_9, y_{10}\} \cap \{x_6, x_5, x_9\} &= \emptyset \\
P_{\alpha,\gamma}(b, y_5, y_9, y_{10}, d) \\
P_{\alpha,\gamma}(d, x_6, x_5, x_9, b)
\end{aligned}$$

Since $\|S_\beta(c)\| \geq 15$ by (125), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (b, y_5, y_9, y_{10}, d, x_6, x_5, x_9)$$

to see that

$$(126) \quad \begin{aligned} &\text{either } N_{\delta,\gamma,\alpha}^2(b, y_5, y_9, y_{10}, d, x_6, x_5, x_9) \\ &\text{or } N_{\delta,\gamma,\alpha}^2(b, x_9, x_5, x_6, d, y_{10}, y_9, y_5). \end{aligned}$$

By an application of the symmetry Θ to (126), we see that

$$(127) \quad \begin{aligned} & \text{either } N_{\delta, \gamma, \beta}^2(c, y_9, y_1, y_4, b, x_4, x_7, x_5) \\ & \text{or } N_{\delta, \gamma, \beta}^2(c, x_5, x_7, x_4, b, y_4, y_1, y_9). \end{aligned}$$

By (126), we see that

$$(128) \quad \text{either } x_5 \overset{\delta}{-} y_{10} \text{ and } x_6 \overset{\alpha}{-} y_9 \text{ or } x_5 \overset{\alpha}{-} y_{10} \text{ and } x_6 \overset{\delta}{-} y_9.$$

By (127), we see that

$$(129) \quad \text{either } x_7 \overset{\beta}{-} y_9 \text{ and } x_5 \overset{\delta}{-} y_1 \text{ or } x_7 \overset{\delta}{-} y_9 \text{ and } x_5 \overset{\beta}{-} y_1.$$

Next, we note the following:

$$x_5 \overset{\delta}{-} y_9 \text{ by (127).}$$

$$x_6 \overset{\delta}{-} y_{10} \text{ by (126).}$$

$$x_7 \overset{\delta}{-} y_1 \text{ by (127).}$$

$$\text{Either } x_3 \overset{\beta}{-} y_{10} \text{ or } x_3 \overset{\delta}{-} y_{10} \text{ since } x_3, y_{10} \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_3 \overset{\beta}{-} y_3 \text{ or } x_3 \overset{\delta}{-} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_6 \overset{\beta}{-} y_3 \text{ or } x_6 \overset{\delta}{-} y_3 \text{ since } x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

Note further that $x_5, x_3, x_6, x_0, x_7, y_9, y_0, y_{10}, y_3, y_1 \in S_\gamma(b)$. Note also that the following hold:

$$\{x_5, x_3, x_6, x_0, x_7\} \cap \{y_9, y_0, y_{10}, y_3, y_1\} = \emptyset$$

$$P_{\beta, \alpha}(x_5, x_3, x_6, x_0, x_7)$$

$$P_{\beta, \alpha}(y_9, y_0, y_{10}, y_3, y_1)$$

Since $\|S_\gamma(b)\| \geq 15$ by (124), we may apply Lemma 9 and Lemma 10 to see that

$$(130) \quad \begin{aligned} & \text{either } M_{\beta, \alpha, \delta}^1(x_6, x_0, x_7, x_5, x_3, y_{10}, y_3, y_1, y_9, y_0) \\ & \text{or } M_{\beta, \alpha, \delta}^1(x_0, x_6, x_3, x_5, x_7, y_9, y_1, y_3, y_{10}, y_0) \\ & \text{or } M_{\beta, \alpha, \delta}^2(x_0, x_6, x_3, x_5, x_7, y_{10}, y_3, y_1, y_9, y_0) \\ & \text{or } M_{\beta, \alpha, \delta}^2(x_3, x_6, x_0, x_7, x_5, y_{10}, y_0, y_9, y_1, y_3) \\ & \text{or } M_{\beta, \alpha, \delta}^2(x_3, x_5, x_7, x_0, x_6, y_1, y_9, y_0, y_{10}, y_3) \\ & \text{or } M_{\beta, \alpha, \delta}^2(x_0, x_7, x_5, x_3, x_6, y_9, y_1, y_3, y_{10}, y_0). \end{aligned}$$

By (130), we see that

$$(131) \quad \begin{aligned} & \text{if } x_0 \overset{\delta}{-} y_0, \text{ then either } x_5 \overset{\beta}{-} y_0 \text{ and } x_0 \overset{\beta}{-} y_9 \\ & \text{or } x_5 \overset{\alpha}{-} y_0 \text{ and } x_0 \overset{\delta}{-} y_9. \end{aligned}$$

Again by (130), we see that

$$(132) \quad \text{if } x_0 \overset{\delta}{-} y_0, \text{ then either } x_7 \overset{\delta}{-} y_0 \text{ and } x_0 \overset{\alpha}{-} y_1$$

$$\text{or } x_7 \overset{\beta}{\text{---}} y_0 \text{ and } x_0 \overset{\beta}{\text{---}} y_1.$$

Again by (130), we see that

$$(133) \quad \text{if } x_0 \overset{\beta}{\text{---}} y_0, \text{ then either } x_5 \overset{\delta}{\text{---}} y_0 \text{ and } x_0 \overset{\delta}{\text{---}} y_9 \\ \text{or } x_5 \overset{\alpha}{\text{---}} y_0 \text{ and } x_0 \overset{\alpha}{\text{---}} y_9.$$

Again by (130), we see that

$$(134) \quad \text{if } x_0 \overset{\alpha}{\text{---}} y_0, \text{ then } x_7 \overset{\alpha}{\text{---}} y_0 \text{ and } x_0 \overset{\beta}{\text{---}} y_1.$$

By an application of the symmetry Θ to (131), we see that

$$(135) \quad \text{if } x_0 \overset{\delta}{\text{---}} y_0, \text{ then either } x_7 \overset{\alpha}{\text{---}} y_0 \text{ and } x_0 \overset{\alpha}{\text{---}} y_1 \\ \text{or } x_7 \overset{\beta}{\text{---}} y_0 \text{ and } x_0 \overset{\delta}{\text{---}} y_1.$$

From (132) and (135) and the fact that $x_0, y_0 \in S_\gamma(a)$, we conclude that

$$(136) \quad \text{either } x_0 \overset{\alpha}{\text{---}} y_0 \text{ or } x_0 \overset{\beta}{\text{---}} y_0.$$

By an application of the symmetry Θ to (133), we see that

$$(137) \quad \text{if } x_0 \overset{\alpha}{\text{---}} y_0, \text{ then either } x_7 \overset{\delta}{\text{---}} y_0 \text{ and } x_0 \overset{\delta}{\text{---}} y_1 \\ \text{or } x_7 \overset{\beta}{\text{---}} y_0 \text{ and } x_0 \overset{\beta}{\text{---}} y_1.$$

From (134), (137), and (136), we conclude that

$$(138) \quad x_0 \overset{\beta}{\text{---}} y_0.$$

An application of the symmetry Θ to (138) gives

$$(139) \quad x_0 \overset{\alpha}{\text{---}} y_0.$$

Now, (138) and (139) together produce the desired contradiction. The proof is complete. \square

Proposition 37. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha,\beta,\gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha,\beta,\gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_{11}$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\alpha(c) = \{a, x_3, x_{10}, x_2, x_{12}\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_3, y_6, y_2, y_{12}\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_3, x_{10}, x_2, x_{12})$ and $P_{\beta,\gamma}(a, y_3, y_6, y_2, y_{12})$ hold and that

$$\|\{a, x_3, x_{10}, x_2, x_{12}\} \cap \{a, y_3, y_6, y_2, y_{12}\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$(140) \quad \|S_\gamma(c)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\beta(a) = \{b, x_9, x_3, x_7, x_{12}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_7, y_3, y_9, y_{12}\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_9, x_3, x_7, x_{12})$ and $P_{\alpha,\gamma}(b, y_7, y_3, y_9, y_{12})$ hold and that

$$\|\{b, x_9, x_3, x_7, x_{12}\} \cap \{b, y_7, y_3, y_9, y_{12}\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(141) \quad \|S_\gamma(a)\| \geq 15.$$

Now, we note the following:

$$x_4 \overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_1 \overset{\delta}{-} y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$x_8 \overset{\delta}{-} y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_1 \overset{\delta}{-} y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(d).$$

$$\text{Either } x_7 \overset{\alpha}{-} y_7 \text{ or } x_7 \overset{\delta}{-} y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_7 \overset{\alpha}{-} y_1 \text{ or } x_7 \overset{\delta}{-} y_1 \text{ since } x_7, y_1 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_4 \overset{\alpha}{-} y_8 \text{ or } x_4 \overset{\delta}{-} y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_8 \overset{\alpha}{-} y_4 \text{ or } x_8 \overset{\delta}{-} y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_1 \overset{\beta}{-} y_8 \text{ or } x_1 \overset{\delta}{-} y_8 \text{ since } x_1, y_8 \in S_\gamma(c) \cap S_\alpha(a).$$

$$\text{Either } x_8 \overset{\beta}{-} y_1 \text{ or } x_8 \overset{\delta}{-} y_1 \text{ since } x_8, y_1 \in S_\gamma(c) \cap S_\alpha(a).$$

Note further that $x_7, x_4, x_1, x_8, x_0, y_7, y_4, y_1, y_8, y_0 \in S_\gamma(c)$. Note also that the following hold:

$$\{x_7, x_4, x_1, x_8, x_0\} \cap \{y_7, y_4, y_1, y_8, y_0\} = \emptyset$$

$$P_{\beta,\alpha}(x_7, x_4, x_1, x_8, x_0)$$

$$P_{\beta,\alpha}(y_7, y_4, y_1, y_8, y_0)$$

Since $\|S_\gamma(d)\| \geq 15$ by (140), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_7, x_4, x_1, x_8, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_7, y_4, y_1, y_8, y_0)$$

to see that

$$(142) \quad \text{either } M_{\beta,\alpha,\delta}^1(x_4, x_7, x_0, x_8, x_1, y_4, y_1, y_8, y_0, y_7) \\ \text{or } M_{\beta,\alpha,\delta}^1(x_7, x_0, x_8, x_1, x_4, y_1, y_8, y_0, y_7, y_4).$$

Finally, we note the following:

$$x_4 \overset{\delta}{\text{---}} y_4 \text{ since } x_4, y_4 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_{10} \overset{\delta}{\text{---}} y_6 \text{ since } x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_6 \overset{\delta}{\text{---}} y_{10} \text{ since } x_6, y_{10} \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_0 \overset{\beta}{\text{---}} y_0 \text{ by (142).}$$

$$\text{Either } x_2 \overset{\beta}{\text{---}} y_2 \text{ or } x_2 \overset{\delta}{\text{---}} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b).$$

Note further that $x_4, x_2, x_{10}, x_0, x_6, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_4, x_2, x_{10}, x_0, x_6\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$

$$P_{\beta, \alpha}(x_4, x_2, x_{10}, x_0, x_6)$$

$$P_{\beta, \alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (141), we may apply Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_4, x_2, x_{10}, x_0, x_6)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to produce the desired contradiction. The proof is complete. \square

Proposition 38. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^0(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{b, c, d, e\}$ where $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$u \mapsto u$$

$$v \mapsto v$$

$$b \mapsto c \mapsto e \mapsto d \mapsto b$$

$$x_0 \mapsto x_0$$

$$x_{13} \mapsto x_{13}$$

$$x_3 \mapsto x_8 \mapsto x_3$$

$$x_1 \mapsto x_9 \mapsto x_5 \mapsto x_7 \mapsto x_1$$

$$x_2 \mapsto x_6 \mapsto x_4 \mapsto x_{10} \mapsto x_2$$

$$y_0 \mapsto y_0$$

$$y_{13} \mapsto y_{13}$$

$$\begin{aligned}
y_3 &\mapsto y_8 \mapsto y_3 \\
y_1 &\mapsto y_9 \mapsto y_5 \mapsto y_7 \mapsto y_1 \\
y_2 &\mapsto y_6 \mapsto y_4 \mapsto y_{10} \mapsto y_2
\end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
\alpha &\mapsto \beta \mapsto \alpha \\
\gamma &\mapsto \gamma \\
\delta &\mapsto \delta
\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(b) = \{c, x_{10}, x_4, x_8, x_{13}\} \subseteq S_\beta(b)$ and $S_\delta(v) \cap S_\beta(b) = \{c, y_{10}, y_4, y_8, y_{13}\} \subseteq S_\beta(b)$. Note further that both $P_{\alpha,\gamma}(c, x_{10}, x_4, x_8, x_{13})$ and $P_{\alpha,\gamma}(c, y_{10}, y_4, y_8, y_{13})$ hold and that

$$\|\{c, x_{10}, x_4, x_8, x_{13}\} \cap \{c, y_{10}, y_4, y_8, y_{13}\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(b)\| \leq 14$ so that

$$(143) \quad \|S_\gamma(b)\| \geq 15.$$

Now, we note the following:

$$x_5 \xrightarrow{\delta} y_5 \text{ since } x_5, y_5 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(c).$$

$$x_6 \xrightarrow{\delta} y_6 \text{ since } x_6, y_6 \in S_\gamma(b) \cap S_\alpha(d) \cap S_\beta(c).$$

$$x_7 \xrightarrow{\delta} y_7 \text{ since } x_7, y_7 \in S_\gamma(b) \cap S_\alpha(e) \cap S_\beta(d).$$

$$\text{Either } x_5 \xrightarrow{\beta} y_7 \text{ or } x_5 \xrightarrow{\delta} y_7 \text{ since } x_5, y_7 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_7 \xrightarrow{\beta} y_5 \text{ or } x_7 \xrightarrow{\delta} y_5 \text{ since } x_7, y_5 \in S_\gamma(b) \cap S_\alpha(e).$$

$$\text{Either } x_3 \xrightarrow{\beta} y_3 \text{ or } x_3 \xrightarrow{\delta} y_3 \text{ since } x_3, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_3 \xrightarrow{\beta} y_6 \text{ or } x_3 \xrightarrow{\delta} y_6 \text{ since } x_3, y_6 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_6 \xrightarrow{\beta} y_3 \text{ or } x_6 \xrightarrow{\delta} y_3 \text{ since } x_6, y_3 \in S_\gamma(b) \cap S_\alpha(d).$$

$$\text{Either } x_5 \xrightarrow{\alpha} y_6 \text{ or } x_5 \xrightarrow{\delta} y_6 \text{ since } x_5, y_6 \in S_\gamma(b) \cap S_\beta(c).$$

$$\text{Either } x_6 \xrightarrow{\alpha} y_5 \text{ or } x_6 \xrightarrow{\delta} y_5 \text{ since } x_6, y_5 \in S_\gamma(b) \cap S_\beta(c).$$

Note further that $x_5, x_3, x_6, x_0, x_7, y_5, y_3, y_6, y_0, y_7 \in S_\gamma(b)$. Note also the following:

$$\{x_5, x_3, x_6, x_0, x_7\} \cap \{y_5, y_3, y_6, y_0, y_7\} = \emptyset$$

$$P_{\beta,\alpha}(x_5, x_3, x_6, x_0, x_7)$$

$$P_{\beta,\alpha}(y_5, y_3, y_6, y_0, y_7)$$

Since $\|S_\gamma(b)\| \geq 15$ by (143), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_5, x_3, x_6, x_0, x_7)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_5, y_3, y_6, y_0, y_7)$$

to see that

$$(144) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^0(x_0, x_7, x_5, x_3, x_6, y_0, y_7, y_5, y_3, y_6) \\ &\text{or } M_{\beta, \alpha, \delta}^0(x_3, x_6, x_0, x_7, x_5, y_3, y_6, y_0, y_7, y_5) \\ &\text{or } M_{\beta, \alpha, \delta}^1(x_6, x_0, x_7, x_5, x_3, y_6, y_0, y_7, y_5, y_3) \\ &\text{or } M_{\beta, \alpha, \delta}^1(x_3, x_5, x_7, x_0, x_6, y_3, y_5, y_7, y_0, y_6) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_3, x_6, x_0, x_7, x_5, y_6, y_3, y_5, y_7, y_0) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_0, x_7, x_5, x_3, x_6, y_5, y_7, y_0, y_6, y_3). \end{aligned}$$

By (144), we see that

$$(145) \quad x_0 \overset{\delta}{-} y_5 \text{ and } x_5 \overset{\delta}{-} y_0.$$

By an application of the symmetry Θ to (145), we see that

$$(146) \quad x_0 \overset{\delta}{-} y_7 \text{ and } x_7 \overset{\delta}{-} y_0.$$

Now, from (146) and (144), we see that

$$(147) \quad M_{\beta, \alpha, \delta}^0(x_3, x_6, x_0, x_7, x_5, y_3, y_6, y_0, y_7, y_5).$$

By (147), we see that

$$(148) \quad x_3 \overset{\alpha}{-} y_5, y_7.$$

An application of the symmetry Θ to (148) gives

$$(149) \quad x_8 \overset{\beta}{-} y_7, y_1.$$

An application of the symmetry Θ to (149) gives

$$(150) \quad x_3 \overset{\alpha}{-} y_1, y_9.$$

By (148) and (150) we get $x_3 \overset{\alpha}{-} y_7, y_1$, which gives a monochromatic triangle, since $y_1 \overset{\alpha}{-} y_7$, thus producing the desired contradiction. The proof is complete. \square

Proposition 39. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^0(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{14} = y_{13}$, $b = x_{15} = y_{14}$, $c = x_{11} = y_{15}$, and $d = x_{12} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$u \longmapsto u$$

$$v \longmapsto v$$

$$\begin{aligned}
a &\mapsto b \mapsto d \mapsto c \mapsto a \\
x_0 &\mapsto x_0 \\
x_{13} &\mapsto x_{13} \\
x_3 &\mapsto x_8 \mapsto x_3 \\
x_1 &\mapsto x_9 \mapsto x_5 \mapsto x_7 \mapsto x_1 \\
x_2 &\mapsto x_6 \mapsto x_4 \mapsto x_{10} \mapsto x_2 \\
y_0 &\mapsto y_{10} \mapsto y_1 \mapsto y_4 \mapsto y_0 \\
y_2 &\mapsto y_3 \mapsto y_{12} \mapsto y_7 \mapsto y_2 \\
y_6 &\mapsto y_9 \mapsto y_{11} \mapsto y_8 \mapsto y_6
\end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
\alpha &\mapsto \beta \mapsto \alpha \\
\gamma &\mapsto \gamma \\
\delta &\mapsto \delta
\end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{13}, x_8, x_4, x_{10}\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{13}, x_8, x_4, x_{10})$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{13}, x_8, x_4, x_{10}\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(151) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Repeated applications of the symmetry Θ to (151) give

$$(152) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15,$$

$$(153) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15,$$

and

$$(154) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

First, we show that $x_5 \xrightarrow{\delta} y_0$. Suppose not. Then $x_5 \xrightarrow{\beta} y_0$ since $x_5, y_0 \in S_\alpha(d) \cap S_\gamma(a)$. Note also that $x_7 \xrightarrow{\delta} y_9$ since $x_7, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b)$. Note further that $a, y_9, y_0, y_6, b, x_5, x_7, x_4 \in S_\alpha(d)$. Note also that the following hold:

$$\begin{aligned}
\{y_9, y_0, y_6\} \cap \{x_5, x_7, x_4\} &= \emptyset \\
P_{\beta,\gamma}(a, y_9, y_0, y_6, b) \\
P_{\beta,\gamma}(b, x_5, x_7, x_4, a)
\end{aligned}$$

Since $\|S_\alpha(d)\| \geq 15$ by (153), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_9, y_0, y_6, b, x_5, x_7, x_4)$$

to get a contradiction. Thus, we have shown that

$$(155) \quad x_5 \overset{\delta}{-} y_0,$$

as desired.

By an application of the symmetry Θ to (155), we see that

$$(156) \quad x_7 \overset{\delta}{-} y_{10}.$$

Next, we note the following:

$$x_7 \overset{\delta}{-} y_{10} \text{ by (156).}$$

$$x_6 \overset{\delta}{-} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_5 \overset{\delta}{-} y_{10} \text{ by (155).}$$

$$x_6 \overset{\delta}{-} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_5 \overset{\delta}{-} y_6 \text{ since } x_5, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$\text{Either } x_7 \overset{\beta}{-} y_6 \text{ or } x_7 \overset{\delta}{-} y_6 \text{ since } x_7, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_7 \overset{\beta}{-} y_0 \text{ or } x_7 \overset{\delta}{-} y_0 \text{ since } x_7, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_3 \overset{\beta}{-} y_2 \text{ or } x_3 \overset{\delta}{-} y_2 \text{ since } x_3, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_3 \overset{\beta}{-} y_6 \text{ or } x_3 \overset{\delta}{-} y_6 \text{ since } x_3, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$

$$\text{Either } x_6 \overset{\alpha}{-} y_4 \text{ or } x_6 \overset{\delta}{-} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

$$\text{Either } x_5 \overset{\alpha}{-} y_4 \text{ or } x_5 \overset{\delta}{-} y_4 \text{ since } x_5, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_7, x_0, x_6, x_3, x_5, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_7, x_0, x_6, x_3, x_5\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} &= \emptyset \\ P_{\beta, \alpha}(x_7, x_0, x_6, x_3, x_5) & \\ P_{\beta, \alpha}(y_{10}, y_4, y_2, y_6, y_0) & \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (151), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_7, x_0, x_6, x_3, x_5)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to see that

$$(157) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_3, x_5, x_7, x_0, x_6, y_6, y_0, y_{10}, y_4, y_2) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_3, x_6, x_0, x_7, x_5, y_6, y_2, y_4, y_{10}, y_0). \end{aligned}$$

By (157), we have

$$(158) \quad x_0 \overset{\beta}{-} y_2.$$

Applying the symmetry Θ to (158), we get

$$(159) \quad x_0 \xrightarrow{\alpha} y_3.$$

Applying the symmetry Θ to (159), we get

$$(160) \quad x_0 \xrightarrow{\beta} y_{12}.$$

But (158) and (160) give a monochromatic triangle, since $y_2 \xrightarrow{\beta} y_{12}$, thus producing the desired contradiction. The proof is complete. \square

Proposition 40. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{14} = y_{13}$, $b = x_{15} = y_{14}$, $c = x_{11} = y_{15}$, and $d = x_{12} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto u \\ v &\mapsto v \\ a &\mapsto b \mapsto d \mapsto c \mapsto a \\ x_0 &\mapsto x_0 \\ x_{13} &\mapsto x_{13} \\ x_3 &\mapsto x_8 \mapsto x_3 \\ x_1 &\mapsto x_7 \mapsto x_5 \mapsto x_9 \mapsto x_1 \\ x_2 &\mapsto x_{10} \mapsto x_4 \mapsto x_6 \mapsto x_2 \\ y_0 &\mapsto y_{10} \mapsto y_1 \mapsto y_4 \mapsto y_0 \\ y_2 &\mapsto y_3 \mapsto y_{12} \mapsto y_7 \mapsto y_2 \\ y_6 &\mapsto y_9 \mapsto y_{11} \mapsto y_8 \mapsto y_6 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \beta \mapsto \alpha \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{13}, x_8, x_4, x_6\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha, \gamma}(b, x_{13}, x_8, x_4, x_6)$ and $P_{\alpha, \gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{13}, x_8, x_4, x_6\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(161) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Repeated applications of the symmetry Θ to (161) give

$$(162) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15,$$

$$(163) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15,$$

and

$$(164) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

First, we show that $x_9 \xrightarrow{\delta} y_0$. Suppose not. Then $x_9 \xrightarrow{\beta} y_0$ since $x_9, y_0 \in S_\alpha(d) \cap S_\gamma(c)$. Note also that $x_1 \xrightarrow{\delta} y_9$ since $x_1, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b)$. Note further that $a, y_9, y_0, y_6, b, x_9, x_1, x_4 \in S_\alpha(d)$. Note also that the following hold:

$$\begin{aligned} \{y_9, y_0, y_6\} \cap \{x_9, x_1, x_4\} &= \emptyset \\ P_{\beta, \gamma}(a, y_9, y_0, y_6, b) \\ P_{\beta, \gamma}(b, x_9, x_1, x_4, a) \end{aligned}$$

Since $\|S_\alpha(d)\| \geq 15$ by (163), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_9, y_0, y_6, b, x_9, x_1, x_4)$$

to get a contradiction. Thus, we have shown that

$$(165) \quad x_9 \xrightarrow{\delta} y_0,$$

as desired.

By an application of the symmetry Θ to (165), we see that

$$(166) \quad x_1 \xrightarrow{\delta} y_{10}.$$

Next, we note the following:

$$x_9 \xrightarrow{\delta} y_0 \text{ by (165).}$$

$$x_{10} \xrightarrow{\delta} y_2 \text{ since } S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_1 \xrightarrow{\delta} y_{10} \text{ by (166). } x_9 \xrightarrow{\delta} y_6 \text{ since } S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b).$$

$$x_{10} \xrightarrow{\delta} y_6 \text{ since } S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_3 \xrightarrow{\beta} y_6 \text{ or } x_3 \xrightarrow{\delta} y_6 \text{ since } x_3, y_6 \in S_\gamma(a) \cap S_\alpha(c).$$

$$x_3 \xrightarrow{\beta} y_2 \text{ or } x_3 \xrightarrow{\delta} y_2 \text{ since } x_3, y_2 \in S_\gamma(a) \cap S_\alpha(c).$$

$$x_1 \xrightarrow{\beta} y_0 \text{ or } x_1 \xrightarrow{\delta} y_0 \text{ since } x_1, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$x_1 \xrightarrow{\beta} y_6 \text{ or } x_1 \xrightarrow{\delta} y_6 \text{ since } x_1, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

$$x_9 \xrightarrow{\alpha} y_4 \text{ or } x_9 \xrightarrow{\delta} y_4 \text{ since } x_9, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

$$x_{10} \xrightarrow{\alpha} y_4 \text{ or } x_{10} \xrightarrow{\delta} y_4 \text{ since } x_{10}, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_9, x_0, x_{10}, x_3, x_1, y_0, y_6, y_2, y_4, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_0, y_6, y_2, y_4, y_{10}\} &= \emptyset \\ P_{\beta, \alpha}(x_9, x_0, x_{10}, x_3, x_1) & \\ P_{\beta, \alpha}(y_0, y_6, y_2, y_4, y_{10}) & \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (161), we may apply Lemma 9 and Lemma 10 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_6, y_2, y_4, y_{10})$$

to see that

$$(167) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_0, x_{10}, x_3, x_1, x_9, y_6, y_2, y_4, y_{10}, y_0) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_0, x_9, x_1, x_3, x_{10}, y_6, y_0, y_{10}, y_4, y_2). \end{aligned}$$

By (167), we see that

$$(168) \quad x_3 \xrightarrow{\beta} y_2.$$

Applying the symmetry Θ to (168) gives

$$(169) \quad x_8 \xrightarrow{\alpha} y_3.$$

Applying the symmetry Θ to (169) gives

$$(170) \quad x_3 \xrightarrow{\beta} y_{12}.$$

But (168) and (170) give a monochromatic triangle, since $y_2 \xrightarrow{\beta} y_{12}$, thus producing the desired contradiction. The proof is complete. \square

Proposition 41. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_5 = y_5$. Then the graph has fewer than 62 vertices.*

Proof. First, we note that the hypotheses of the proposition are preserved by the following symmetry Θ :

$$\begin{aligned} u &\mapsto u \\ v &\mapsto v \\ a &\mapsto b \mapsto d \mapsto c \mapsto a \\ x_0 &\mapsto x_{10} \mapsto x_1 \mapsto x_4 \mapsto x_0 \\ x_2 &\mapsto x_3 \mapsto x_{12} \mapsto x_7 \mapsto x_2 \\ x_6 &\mapsto x_9 \mapsto x_{11} \mapsto x_8 \mapsto x_6 \\ y_0 &\mapsto y_{10} \mapsto y_1 \mapsto y_4 \mapsto y_0 \end{aligned}$$

$$\begin{aligned} y_2 &\mapsto y_3 \mapsto y_{12} \mapsto y_7 \mapsto y_2 \\ y_6 &\mapsto y_9 \mapsto y_{11} \mapsto y_8 \mapsto y_6 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned} \alpha &\mapsto \beta \mapsto \alpha \\ \gamma &\mapsto \gamma \\ \delta &\mapsto \delta \end{aligned}$$

Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{12}, x_9, x_3, x_7\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha,\gamma}(b, x_{12}, x_9, x_3, x_7)$ and $P_{\alpha,\gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{12}, x_9, x_3, x_7\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(171) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Repeated applications of the symmetry Θ to (171) give

$$(172) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15,$$

$$(173) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15,$$

and

$$(174) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Now, we show that $x_4 \xrightarrow{\delta} y_4$. Suppose not. Then $x_4 \xrightarrow{\alpha} y_4$ since $x_4, y_4 \in S_\gamma(a) \cap S_\beta(b)$. We also note the following:

$$x_6 \xrightarrow{\delta} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\alpha(x_4) \cap S_\beta(b).$$

$$x_2 \xrightarrow{\delta} y_2 \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(d).$$

$$x_4 \xrightarrow{\delta} y_6 \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(y_4) \cap S_\beta(b).$$

$$x_6 \xrightarrow{\delta} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$\text{Either } x_0 \xrightarrow{\beta} y_6 \text{ or } x_0 \xrightarrow{\delta} y_6 \text{ since } x_0, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_0 \xrightarrow{\beta} y_0 \text{ or } x_0 \xrightarrow{\delta} y_0 \text{ since } x_0, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_6 \xrightarrow{\beta} y_0 \text{ or } x_6 \xrightarrow{\delta} y_0 \text{ since } x_6, y_0 \in S_\gamma(a) \cap S_\alpha(d).$$

$$\text{Either } x_{10} \xrightarrow{\alpha} y_{10} \text{ or } x_{10} \xrightarrow{\delta} y_{10} \text{ since } x_{10}, y_{10} \in S_\gamma(a) \cap S_\beta(c).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_{10}, y_4, y_2, y_6, y_0 \in S_\gamma(a)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_{10}, y_4, y_2, y_6, y_0\} = \emptyset$$

$$P_{\beta, \alpha}(x_0, x_6, x_2, x_4, x_{10})$$

$$P_{\beta, \alpha}(y_{10}, y_4, y_2, y_6, y_0)$$

Since $\|S_\gamma(a)\| \geq 15$ by (171), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_{10}, y_4, y_2, y_6, y_0)$$

to get a contradiction. Thus, we have

$$(175) \quad x_4 \overset{\delta}{\text{---}} y_4,$$

as desired.

By repeated applications of the symmetry Θ to (175), we see that

$$(176) \quad x_0 \overset{\delta}{\text{---}} y_0,$$

$$(177) \quad x_{10} \overset{\delta}{\text{---}} y_{10},$$

and

$$(178) \quad x_1 \overset{\delta}{\text{---}} y_1.$$

Now, note that $x_0 \overset{\delta}{\text{---}} y_0$ by (176). Note further that $a, y_9, y_0, y_6, b, x_6, x_0, x_9 \in S_\alpha(d)$. Note also the following:

$$\begin{aligned} \{y_9, y_0, y_6\} \cap \{x_6, x_0, x_9\} &= \emptyset \\ P_{\beta, \gamma}(a, y_9, y_0, y_6, b) \\ P_{\beta, \gamma}(b, x_6, x_0, x_9, a) \end{aligned}$$

Since $\|S_\alpha(d)\| \geq 15$ by (173), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_9, y_0, y_6, b, x_6, x_0, x_9)$$

to see that

$$(179) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \beta}^2(a, y_9, y_0, y_6, b, x_6, x_0, x_9) \\ &\text{or } N_{\delta, \gamma, \beta}^2(a, x_9, x_0, x_6, b, y_6, y_0, y_9). \end{aligned}$$

By (179), we have

$$(180) \quad \text{either } x_6 \overset{\beta}{\text{---}} y_0 \text{ or } x_0 \overset{\beta}{\text{---}} y_6.$$

Finally, we note the following:

$$x_0 \overset{\delta}{\text{---}} y_0 \text{ by (176).}$$

$$x_6 \overset{\delta}{\text{---}} y_6 \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b).$$

$$x_4 \overset{\delta}{\text{---}} y_4 \text{ by (175).}$$

$$x_{10} \overset{\delta}{\text{---}} y_{10} \text{ by (177).}$$

$$\text{Either } x_0 \overset{\beta}{\text{---}} y_6 \text{ or } x_0 \overset{\delta}{\text{---}} y_6 \text{ since } x_0, y_6 \in S_\gamma(a) \cap S_\alpha(d).$$

Either $x_6 \xrightarrow{\beta} y_0$ or $x_6 \xrightarrow{\delta} y_0$ since $x_6, y_0 \in S_\gamma(a) \cap S_\alpha(d)$.

Either $x_6 \xrightarrow{\beta} y_2$ or $x_6 \xrightarrow{\delta} y_2$ since $x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_2 \xrightarrow{\beta} y_6$ or $x_2 \xrightarrow{\delta} y_6$ since $x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c)$.

Either $x_6 \xrightarrow{\alpha} y_4$ or $x_6 \xrightarrow{\delta} y_4$ since $x_6, y_4 \in S_\gamma(a) \cap S_\beta(b)$.

Either $x_4 \xrightarrow{\alpha} y_6$ or $x_4 \xrightarrow{\delta} y_6$ since $x_4, y_6 \in S_\gamma(a) \cap S_\beta(b)$.

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_0, y_6, y_2, y_4, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_0, y_6, y_2, y_4, y_{10}\} &= \emptyset \\ P_{\beta, \alpha}(x_0, x_6, x_2, x_4, x_{10}) & \\ P_{\beta, \alpha}(y_0, y_6, y_2, y_4, y_{10}) & \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (171), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_6, y_2, y_4, y_{10})$$

to produce, with the help of (180), a contradiction. The proof is complete. \square

Proposition 42. *Let u and v be vertices with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ in a good edge coloring of a complete graph, colored with the four pairwise distinct colors α , β , γ , and δ . Suppose that $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, and $d = x_{11} = y_5$. Then the graph has fewer than 62 vertices.*

Proof. Working toward a contradiction, we assume that the graph has at least 62 vertices.

Note that $S_\delta(u) \cap S_\beta(a) = \{b, x_{12}, x_9, x_3, x_7\} \subseteq S_\beta(a)$ and $S_\delta(v) \cap S_\beta(a) = \{b, y_{12}, y_9, y_3, y_7\} \subseteq S_\beta(a)$. Note further that both $P_{\alpha, \gamma}(b, x_{12}, x_9, x_3, x_7)$ and $P_{\alpha, \gamma}(b, y_{12}, y_9, y_3, y_7)$ hold and that

$$\|\{b, x_{12}, x_9, x_3, x_7\} \cap \{b, y_{12}, y_9, y_3, y_7\}\| = \|\{b\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(a)\| \leq 14$ so that

$$(181) \quad \|S_\alpha(a)\|, \|S_\gamma(a)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(b) = \{d, x_7, x_5, x_2, x_{12}\} \subseteq S_\alpha(b)$ and $S_\delta(v) \cap S_\alpha(b) = \{d, y_2, y_{12}, y_{11}, y_7\} \subseteq S_\alpha(b)$. Note further that both $P_{\beta, \gamma}(d, x_7, x_5, x_2, x_{12})$ and $P_{\beta, \gamma}(d, y_2, y_{12}, y_{11}, y_7)$ hold and that

$$\|\{d, x_7, x_5, x_2, x_{12}\} \cap \{d, y_2, y_{12}, y_{11}, y_7\}\| = \|\{d\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(b)\| \leq 14$ so that

$$(182) \quad \|S_\beta(b)\|, \|S_\gamma(b)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\alpha(c) = \{a, x_{12}, x_2, x_6, x_3\} \subseteq S_\alpha(c)$ and $S_\delta(v) \cap S_\alpha(c) = \{a, y_{12}, y_2, y_6, y_3\} \subseteq S_\alpha(c)$. Note further that both $P_{\beta,\gamma}(a, x_{12}, x_2, x_6, x_3)$ and $P_{\beta,\gamma}(a, y_{12}, y_2, y_6, y_3)$ hold and that

$$\|\{a, x_{12}, x_2, x_6, x_3\} \cap \{a, y_{12}, y_2, y_6, y_3\}\| = \|\{a\}\| = 1.$$

By Lemma 15, we see that $\|S_\alpha(c)\| \leq 14$ so that

$$(183) \quad \|S_\beta(c)\|, \|S_\gamma(c)\| \geq 15.$$

Note also that $S_\delta(u) \cap S_\beta(d) = \{c, x_{12}, x_1, x_7, x_6\} \subseteq S_\beta(d)$ and $S_\delta(v) \cap S_\beta(d) = \{c, y_2, y_8, y_7, y_3\} \subseteq S_\beta(d)$. Note further that both $P_{\alpha,\gamma}(c, x_{12}, x_1, x_7, x_6)$ and $P_{\alpha,\gamma}(c, y_2, y_8, y_7, y_3)$ hold and that

$$\|\{c, x_{12}, x_1, x_7, x_6\} \cap \{c, y_2, y_8, y_7, y_3\}\| = \|\{c\}\| = 1.$$

By Lemma 15, we see that $\|S_\beta(d)\| \leq 14$ so that

$$(184) \quad \|S_\alpha(d)\|, \|S_\gamma(d)\| \geq 15.$$

Note that $x_6, x_2, x_4, x_{10}, x_0, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_6, x_2, x_4, x_{10}, x_0\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} &= \emptyset \\ P_{\beta,\alpha}(x_6, x_2, x_4, x_{10}, x_0) & \\ P_{\beta,\alpha}(y_4, y_2, y_6, y_0, y_{10}) & \end{aligned}$$

Noting also that $\|S_\gamma(a)\| \geq 15$ by (181), we will apply Lemma 14 several times in what follows with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_2, x_4, x_{10}, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10}).$$

We now show that $x_2 \xrightarrow{\delta} y_6$. Suppose not. Then $x_2 \xrightarrow{\beta} y_6$ since $x_2, y_6 \in S_\gamma(a) \cap S_\alpha(c)$. We also note the following:

$$\begin{aligned} x_6 \xrightarrow{\delta} y_6 & \text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\ x_4 \xrightarrow{\delta} y_6 & \text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \end{aligned}$$

Also, $x_{10} \xrightarrow{\beta} y_6$ since otherwise, by Lemma 14, in view of the fact that $x_{10}, y_6 \in S_\gamma(a) \cap S_\alpha(d)$, we would have $y_6 \xrightarrow{\delta} x_6, x_4, x_{10}$, so that $y_6 \xrightarrow{\delta} x_2, x_0$, also, which would contradict the fact that $x_2 \xrightarrow{\beta} y_6$. Now, note the following:

$$\begin{aligned} x_2 \xrightarrow{\delta} y_2 & \text{ since } x_2, y_2 \in S_\gamma(a) \cap S_\alpha(b) \cap S_\beta(y_6). \\ x_{10} \xrightarrow{\delta} y_0 & \text{ since } x_{10}, y_0 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(y_6). \\ x_6 \xrightarrow{\delta} y_2 & \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d). \end{aligned}$$

Either $x_4 \xrightarrow{\beta} y_0$ or $x_4 \xrightarrow{\delta} y_0$ since $x_4, y_0 \in S_\gamma(a) \cap S_\alpha(d)$.

Note further that $x_6, x_2, x_4, x_{10}, x_0, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\begin{aligned} \{x_6, x_2, x_4, x_{10}, x_0\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} &= \emptyset \\ P_{\beta, \alpha}(x_6, x_2, x_4, x_{10}, x_0) & \\ P_{\beta, \alpha}(y_4, y_2, y_6, y_0, y_{10}) & \end{aligned}$$

Since $\|S_\gamma(a)\| \geq 15$ by (181), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_6, x_2, x_4, x_{10}, x_0)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to get a contradiction. Thus, we have

$$(185) \quad x_2 \overset{\delta}{-} y_6,$$

as desired.

Now, we note the following:

$$\begin{aligned} x_2 \overset{\delta}{-} y_6 &\text{ by (185).} \\ x_6 \overset{\delta}{-} y_6 &\text{ since } x_6, y_6 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(b). \\ x_4 \overset{\delta}{-} y_6 &\text{ since } x_4, y_6 \in S_\gamma(a) \cap S_\alpha(d) \cap S_\beta(b). \end{aligned}$$

Thus, by Lemma 14, we have

$$(186) \quad x_6, x_2, x_4, x_{10}, x_0 \overset{\delta}{-} y_6.$$

Next, we note the following:

$$\begin{aligned} x_{10} \overset{\delta}{-} y_6 &\text{ by (186).} \\ x_{10} \overset{\delta}{-} y_9 &\text{ since } x_{10}, y_9 \in S_\alpha(d) \cap S_\beta(c) \cap S_\gamma(b). \end{aligned}$$

Note further that $a, y_9, y_0, y_6, b, x_4, x_{10}, x_3 \in S_\alpha(d)$. Note also that the following hold:

$$\begin{aligned} \{y_9, y_0, y_6\} \cap \{x_4, x_{10}, x_3\} &= \emptyset \\ P_{\beta, \gamma}(a, y_9, y_0, y_6, b) & \\ P_{\beta, \gamma}(b, x_4, x_{10}, x_3, a) & \end{aligned}$$

Since $\|S_\alpha(d)\| \geq 15$ by (184), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (a, y_9, y_0, y_6, b, x_4, x_{10}, x_3)$$

to see that

$$(187) \quad \begin{aligned} &\text{either } N_{\delta, \gamma, \beta}^0(a, y_9, y_0, y_6, b, x_4, x_{10}, x_3) \\ &\text{or } N_{\delta, \gamma, \beta}^1(a, y_9, y_0, y_6, b, x_4, x_{10}, x_3) \\ &\text{or } N_{\delta, \gamma, \beta}^1(a, x_3, x_{10}, x_4, b, y_6, y_0, y_9). \end{aligned}$$

Next, we note the following:

$$x_6 \overset{\delta}{-} y_2 \text{ since } x_6, y_2 \in S_\gamma(a) \cap S_\alpha(c) \cap S_\beta(d).$$

$$x_4 \overset{\delta}{-} y_0 \text{ by (187).}$$

$$x_0, x_6, x_2, x_4, x_{10} \overset{\delta}{-} y_6 \text{ by (186).}$$

$$x_{10} \overset{\beta}{-} y_0 \text{ by (187).}$$

$$\text{Either } x_6 \overset{\alpha}{-} y_4 \text{ or } x_6 \overset{\delta}{-} y_4 \text{ since } x_6, y_4 \in S_\gamma(a) \cap S_\beta(b).$$

Note further that $x_0, x_6, x_2, x_4, x_{10}, y_4, y_2, y_6, y_0, y_{10} \in S_\gamma(a)$. Note also that the following hold:

$$\{x_0, x_6, x_2, x_4, x_{10}\} \cap \{y_4, y_2, y_6, y_0, y_{10}\} = \emptyset$$

$$P_{\beta, \alpha}(x_0, x_6, x_2, x_4, x_{10})$$

$$P_{\beta, \alpha}(y_4, y_2, y_6, y_0, y_{10})$$

Since $\|S_\gamma(a)\| \geq 15$ by (181), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_0, x_6, x_2, x_4, x_{10})$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_4, y_2, y_6, y_0, y_{10})$$

to see that

$$(188) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_2, x_4, x_{10}, x_0, x_6, y_6, y_0, y_{10}, y_4, y_2) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_2, x_6, x_0, x_{10}, x_4, y_6, y_2, y_4, y_{10}, y_0). \end{aligned}$$

Next, we note the following:

$$x_4 \overset{\delta}{-} y_0 \text{ by (188).}$$

$$x_1 \overset{\delta}{-} y_8 \text{ since } x_1, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(d).$$

$$x_0 \overset{\delta}{-} y_4 \text{ by (188).}$$

$$x_7 \overset{\delta}{-} y_7 \text{ since } x_7, y_7 \in S_\gamma(c) \cap S_\alpha(b) \cap S_\beta(d).$$

$$x_8 \overset{\delta}{-} y_8 \text{ since } x_8, y_8 \in S_\gamma(c) \cap S_\alpha(a) \cap S_\beta(b).$$

$$x_0 \overset{\alpha}{-} y_0 \text{ by (188).}$$

$$\text{Either } x_1 \overset{\beta}{-} y_1 \text{ or } x_1 \overset{\delta}{-} y_1 \text{ since } x_1, y_1 \in S_\gamma(c) \cap S_\alpha(a).$$

$$\text{Either } x_4 \overset{\alpha}{-} y_8 \text{ or } x_4 \overset{\delta}{-} y_8 \text{ since } x_4, y_8 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_4 \overset{\alpha}{-} y_4 \text{ or } x_4 \overset{\delta}{-} y_4 \text{ since } x_4, y_4 \in S_\gamma(c) \cap S_\beta(b).$$

$$\text{Either } x_1 \overset{\alpha}{-} y_7 \text{ or } x_1 \overset{\delta}{-} y_7 \text{ since } x_1, y_7 \in S_\gamma(c) \cap S_\beta(d).$$

$$\text{Either } x_8 \overset{\alpha}{-} y_4 \text{ or } x_8 \overset{\delta}{-} y_4 \text{ since } x_8, y_4 \in S_\gamma(c) \cap S_\beta(b).$$

Either $x_7 \overset{\alpha}{-} y_8$ or $x_7 \overset{\delta}{-} y_8$ since $x_7, y_8 \in S_\gamma(c) \cap S_\beta(d)$.

Note further that $x_4, x_1, x_8, x_0, x_7, y_0, y_8, y_1, y_4, y_7 \in S_\gamma(c)$. Note also that the following hold:

$$\begin{aligned} \{x_4, x_1, x_8, x_0, x_7\} \cap \{y_0, y_8, y_1, y_4, y_7\} &= \emptyset \\ P_{\beta, \alpha}(x_4, x_1, x_8, x_0, x_7) & \\ P_{\beta, \alpha}(y_0, y_8, y_1, y_4, y_7) & \end{aligned}$$

Since $\|S_\gamma(c)\| \geq 15$ by (183), we may apply Lemma 9 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_4, x_1, x_8, x_0, x_7)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_0, y_8, y_1, y_4, y_7)$$

to see that

$$(189) \quad \begin{aligned} &\text{either } M_{\beta, \alpha, \delta}^2(x_0, x_7, x_4, x_1, x_8, y_4, y_7, y_0, y_8, y_1) \\ &\text{or } M_{\beta, \alpha, \delta}^2(x_1, x_4, x_7, x_0, x_8, y_8, y_0, y_7, y_4, y_1). \end{aligned}$$

Now, note that $x_8 \overset{\delta}{-} y_8, y_1$ by (189). Note further that $c, y_{11}, y_1, y_8, d, x_1, x_8, x_5 \in S_\alpha(a)$. Note also that the following hold:

$$\begin{aligned} \{y_{11}, y_1, y_8\} \cap \{x_1, x_8, x_5\} &= \emptyset \\ P_{\beta, \gamma}(c, y_{11}, y_1, y_8, d) & \\ P_{\beta, \gamma}(d, x_1, x_8, x_5, c) & \end{aligned}$$

Since $\|S_\alpha(a)\| \geq 15$ by (181), we may apply Lemma 12 with

$$(u_1, \dots, u_8) = (c, y_{11}, y_1, y_8, d, x_1, x_8, x_5)$$

to see that

$$(190) \quad N_{\delta, \gamma, \beta}^2(c, y_{11}, y_1, y_8, d, x_1, x_8, x_5).$$

From (190), we get

$$(191) \quad x_1 \overset{\beta}{-} y_1.$$

From (191) and (189), we see that

$$(192) \quad M_{\beta, \alpha, \delta}^2(x_0, x_7, x_4, x_1, x_8, y_4, y_7, y_0, y_8, y_1).$$

From (192), we get

$$(193) \quad x_4 \overset{\delta}{-} y_4.$$

From (193) and (188), we see that

$$(194) \quad M_{\beta, \alpha, \delta}^2(x_2, x_6, x_0, x_{10}, x_4, y_6, y_2, y_4, y_{10}, y_0).$$

Finally, we note the following:

$$x_0 \overset{\delta}{-} y_1 \text{ by (192).}$$

$x_{10} \xrightarrow{\delta} y_9$ by (187).

$x_3 \xrightarrow{\delta} y_0$ by (187).

$x_3 \xrightarrow{\delta} y_9$ by (187).

$x_0 \xrightarrow{\alpha} y_0$ by (192).

$x_0 \xrightarrow{\alpha} y_{10}$ by (194).

$x_{10} \xrightarrow{\beta} y_0$ by (194).

$x_1 \xrightarrow{\beta} y_1$ by (191).

$x_1 \xrightarrow{\beta} y_0$ by (192).

Note further $x_9, x_0, x_{10}, x_3, x_1, y_3, y_1, y_9, y_0, y_{10} \in S_\gamma(b)$. Note also that the following hold:

$$\begin{aligned} \{x_9, x_0, x_{10}, x_3, x_1\} \cap \{y_3, y_1, y_9, y_0, y_{10}\} &= \emptyset \\ P_{\beta, \alpha}(x_9, x_0, x_{10}, x_3, x_1) & \\ P_{\beta, \alpha}(y_3, y_1, y_9, y_0, y_{10}) & \end{aligned}$$

Since $\|S_\gamma(b)\| \geq 15$ by (182), we may apply Lemma 9 and Lemma 11 with

$$(u_3, u_4, u_0, u_1, u_2) = (x_9, x_0, x_{10}, x_3, x_1)$$

and

$$(v_3, v_4, v_0, v_1, v_2) = (y_3, y_1, y_9, y_0, y_{10})$$

to produce the desired contradiction. The proof is complete. \square

Theorem 12. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 4$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 4$. There are, up to isomorphism, exactly three possibilities consistent with Proposition 28. Let $S_\delta(u) \cap S_\delta(v) = \{w_0, w_1, w_2, w_3\}$.

The first possibility is $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3 \xrightarrow{\alpha} w_0$ and $w_0 \xrightarrow{\beta} w_2$ and $w_1 \xrightarrow{\beta} w_3$. In this case, we get a contradiction by Proposition 18.

The second possibility is $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3 \xrightarrow{\alpha} w_0$ and $w_0 \xrightarrow{\beta} w_2$ and $w_1 \xrightarrow{\gamma} w_3$. In this case, we get a contradiction by Proposition 29 and Proposition 30.

Finally, we consider the third possibility, namely, $w_0 \xrightarrow{\alpha} w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\alpha} w_3$ and $w_1 \xrightarrow{\beta} w_3 \xrightarrow{\beta} w_0 \xrightarrow{\beta} w_2$. In this case, we get a contradiction by Proposition 31, Proposition 32, Proposition 33, Proposition 34, Proposition 35, Proposition 36, Proposition 37, Proposition 38, Proposition 39, Proposition 40, Proposition 41, and Proposition 42.

The proof is complete. \square

3.13. Attaching Sets of Cardinality 3.

Proposition 43. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Suppose further that $\|S_\delta(u) \cap S_\delta(v)\| = 3$. If $x \in S_\delta(u) \cap S_\delta(v)$, then*

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 0, 0]. \end{aligned}$$

Proof. First note that for any $\eta \in \{\alpha, \beta, \gamma\}$ we have

$$\|S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)\| \leq R(3, 3; 2) - 1 = 6 - 1 = 5$$

since $S_\delta(u) \cap S_\delta(v) \cap S_\eta(x)$ has no edges of colors δ and η , so that the induced good edge coloring has only two colors. Note that

$$S_\delta(u) \cap S_\delta(v) \cap S_\delta(x) = \emptyset,$$

since otherwise we would have a monochromatic triangle. Thus, we have

$$\begin{aligned} S_\delta(u) \cap S_\delta(v) = \{x\} \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)) \\ \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)), \end{aligned}$$

so that

$$\begin{aligned} 3 = \|S_\delta(u) \cap S_\delta(v)\| = 1 + \|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\| \\ + \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} (\|S_\delta(u) \cap S_\delta(v) \cap S_\alpha(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\beta(x)\|, \|S_\delta(u) \cap S_\delta(v) \cap S_\gamma(x)\|) \\ \in [2, 0, 0] \cup [1, 1, 0]. \end{aligned}$$

The proposition now follows by an application of Proposition 4. \square

Theorem 13. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $\|V\| = 62$ and let $u, v \in V$ with $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. Then $\|S_\delta(u) \cap S_\delta(v)\| \neq 3$.*

Proof. Suppose not. Then $\|S_\delta(u) \cap S_\delta(v)\| = 3$. But there are no possibilities consistent with Proposition 43. \square

4. THE GLOBAL ARGUMENTS

Theorem 14. *Let V be the vertex set of a complete graph with a good edge coloring with four colors. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $u \neq v$ be such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ for some color δ . Then $\|S_\delta(u) \cap S_\delta(v)\| \in \{0, 1, 2, 5\}$. In addition, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then there exist $x_0, \dots, x_{15} \in S_\delta(u)$ and*

$y_0, \dots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 15\}$ and some $j \in \{0, 1\}$ and colors α, β , and γ , such that $B_{\alpha, \beta, \gamma}^j(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^j(y_0, \dots, y_{15})$ with

$$S_\delta(u) \cap S_\delta(v) = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} = \{y_{11}, y_{12}, y_{13}, y_{14}, y_{15}\}.$$

Furthermore, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then for each $w \in S_\delta(u) \cap S_\delta(v)$, we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted.

Proof. This is a trivial consequence of Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6, Theorem 7, Theorem 8, Theorem 9, Theorem 10, Theorem 11, Theorem 12, and Theorem 13. \square

Theorem 15. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Suppose that $\|V\| = 62$ and let $u, v, w, x, y \in V$ be pairwise distinct vertices which satisfy $u, v, w \stackrel{\delta}{\sim} x, y$ for some color δ . Then, for some $z \in \{u, v, w, x, y\}$ we have $\|S_\delta(z)\| \leq 15$.*

Proof. Suppose not. Then $\|S_\delta(z)\| = 16$ for all $z \in \{u, v, w, x, y\}$. Thus, by Theorem 14, we have $\|S_\delta(x) \cap S_\delta(y)\| \in \{0, 1, 2, 5\}$, so that, since $u, v, w \in S_\delta(x) \cap S_\delta(y)$, we must have

$$\|S_\delta(x) \cap S_\delta(y)\| = 5.$$

Next, we show that $\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5$. Suppose not. Then we may, without loss of generality, suppose that $\|S_\delta(u) \cap S_\delta(v)\| \neq 5$, so that, by Theorem 14, we have $\|S_\delta(u) \cap S_\delta(v)\| \leq 2$. But, $x, y \in S_\delta(u) \cap S_\delta(v)$, so that we must have $S_\delta(u) \cap S_\delta(v) = \{x, y\} \subseteq S_\delta(w)$. But then we have $(S_\delta(u) \sim S_\delta(w)) \cap (S_\delta(v) \sim S_\delta(w)) = \emptyset$. Thus, we see that

$$\begin{aligned} 62 &= \|V\| \\ &\geq \|S_\delta(u) \cup S_\delta(v) \cup S_\delta(w) \cup S_\delta(x) \cup S_\delta(y)\| \\ &= \|(S_\delta(x) \sim S_\delta(y)) \uplus (S_\delta(y) \sim S_\delta(x)) \uplus (S_\delta(x) \cap S_\delta(y)) \uplus \\ &\quad (S_\delta(u) \sim S_\delta(w)) \uplus (S_\delta(v) \sim S_\delta(w)) \uplus S_\delta(w)\| \\ &= \|S_\delta(x) \sim S_\delta(y)\| + \|S_\delta(y) \sim S_\delta(x)\| + \|S_\delta(x) \cap S_\delta(y)\| + \\ &\quad \|S_\delta(u) \sim S_\delta(w)\| + \|S_\delta(v) \sim S_\delta(w)\| + \|S_\delta(w)\| \\ &= 11 + 11 + 5 + \|S_\delta(u) \sim S_\delta(w)\| + \|S_\delta(v) \sim S_\delta(w)\| + 16 \\ &\geq 11 + 11 + 5 + 11 + 11 + 16 \\ &= 65, \end{aligned}$$

which is a contradiction. Thus, we have

$$\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5,$$

as desired.

We may now apply Theorem 14 to each pair in $\{u, v, w\}$ to see that there exist colors $\gamma, \gamma', \gamma'' \neq \delta$ such that

$$S_\delta(u) \cap S_\delta(v) \text{ contains no edges of color } \gamma \text{ and } \|S_\gamma(x)\| \leq 14,$$

$S_\delta(u) \cap S_\delta(w)$ contains no edges of color γ' and $\|S_{\gamma'}(x)\| \leq 14$,

and

$S_\delta(v) \cap S_\delta(w)$ contains no edges of color γ'' and $\|S_{\gamma''}(x)\| \leq 14$.

By Proposition 1, we see that $\gamma = \gamma' = \gamma''$.

Thus, there exists some $\gamma \neq \delta$, such that $\|S_\gamma(x)\| \leq 14$ and

$$\begin{aligned} \emptyset &= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(v) \cap S_\gamma(x)) \\ &= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(w) \cap S_\gamma(x)) \\ &= (S_\delta(v) \cap S_\gamma(x)) \cap (S_\delta(w) \cap S_\gamma(x)). \end{aligned}$$

Next, we show that $\|S_\delta(z) \cap S_\gamma(x)\| \leq 4$ for some $z \in \{u, v, w\}$. Suppose not. Then $\|S_\delta(z) \cap S_\gamma(x)\| = 5$ for all $z \in \{u, v, w\}$. Thus, since

$$\biguplus_{z \in \{u, v, w\}} (S_\delta(z) \cap S_\gamma(x)) \subseteq S_\gamma(x),$$

we must have $5 + 5 + 5 \leq 14$, which is impossible.

Thus, we may suppose, without loss of generality, that $\|S_\delta(u) \cap S_\gamma(x)\| \leq 4$. Let α and β be the two other colors. Then, since $u \overset{\delta}{\text{---}} x$, we must have

$$\begin{aligned} 16 &= \|S_\delta(u)\| \\ &= \|\{x\} \uplus (S_\delta(u) \cap S_\alpha(x)) \uplus (S_\delta(u) \cap S_\beta(x)) \uplus (S_\delta(u) \cap S_\gamma(x))\| \\ &= \|\{x\}\| + \|S_\delta(u) \cap S_\alpha(x)\| + \|S_\delta(u) \cap S_\beta(x)\| + \|S_\delta(u) \cap S_\gamma(x)\| \\ &\leq 1 + 5 + 5 + 4 \\ &= 15, \end{aligned}$$

which is impossible.

The proof is complete. \square

Theorem 16. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α , β , γ , and δ . Suppose that $u, v \in V$ with $S_\delta(u) = \{x_0, \dots, x_{15}\}$ and $S_\delta(v) = \{y_0, \dots, y_{15}\}$ with $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ such that $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Suppose further that $\|V\| = 62$. If $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. If $\|S_\beta(a)\| = 16$, then $\|S_\beta(d)\| = \|S_\beta(c)\| = 15$.*

Proof. First, note that the hypotheses of the theorem are preserved by the following symmetry Θ :

$$\begin{aligned} u &\longmapsto u \\ v &\longmapsto v \\ a &\longmapsto a \\ b &\longmapsto c \longmapsto e \longmapsto d \longmapsto b \\ x_0 &\longmapsto x_0 \\ x_3 &\longmapsto x_8 \longmapsto x_3 \end{aligned}$$

$$\begin{aligned}
 x_1 &\mapsto x_7 \mapsto x_5 \mapsto x_9 \mapsto x_1 \\
 x_2 &\mapsto x_{10} \mapsto x_4 \mapsto x_6 \mapsto x_2 \\
 & y_0 \mapsto y_0 \\
 & y_3 \mapsto y_8 \mapsto y_3 \\
 y_1 &\mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\
 y_2 &\mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2
 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
 \alpha &\mapsto \beta \mapsto \alpha \\
 \gamma &\mapsto \gamma \\
 \delta &\mapsto \delta
 \end{aligned}$$

First, we show that if $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = 15$. Suppose not. That is, suppose that $\|S_\alpha(a)\| = 15$ but $\|S_\alpha(b)\| \neq 16$. By Theorem 14, we see that $\|S_\gamma(b)\| \leq 14$, so that, by Proposition 1, we must have $\|S_\alpha(b)\| \geq 15$, so that, in fact,

$$\|S_\alpha(b)\| = \|S_\alpha(a)\| = 16.$$

Applying Theorem 14 once again, this time to a and b , and using the fact that $x_5, y_5, d \in S_\alpha(a) \cap S_\alpha(b)$ are pairwise distinct elements, we see that

$$\|S_\alpha(a) \cap S_\alpha(b)\| = 5.$$

Furthermore, note that $x_5 \overset{\delta}{\dashv} y_5$, since $x_5, y_5 \in S_\alpha(a) \cap S_\beta(c) \cap S_\gamma(d)$. Note also that $d \overset{\gamma}{\dashv} x_5, x_6$. Thus, by Theorem 14, applied to a and b , we see that

$$\|S_\beta(d)\| \leq 14.$$

But by Theorem 14, applied to u and v , we see that

$$\|S_\gamma(d)\| \leq 14,$$

which is impossible, by Proposition 1.

Thus, we have shown that

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(b)\| = 15.$$

Repeated applications of the symmetry Θ give

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(c)\| = 15,$$

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(e)\| = 15,$$

and

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(d)\| = 15.$$

The proof is complete. \square

Theorem 17. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Then there exists some color α and vertices $u, v \in V$, such that $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$ and $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$ with both $S_\alpha(u)$ and $S_\alpha(v)$ twisted.*

Proof. By Proposition 1, for each $z \in V$ there exists some color η such that $\|S_\eta(z)\| = 16$. Thus, we have

$$\|\{ (z, \eta) \mid \|S_\eta(z)\| = 16 \}\| \geq 62.$$

Thus, there must exist some color δ such that

$$\|\{ z \mid \|S_\delta(z)\| = 16 \}\| \geq 16.$$

Let $z_0, z_1, z_2, z_3, z_4, z_5 \in V$ be pairwise distinct vertices such that $\|S_\delta(z_i)\| = 16$ for all $i \in \{0, 1, 2, 3, 4, 5\}$.

First, we show that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$ for some $i, j \in \{0, 1, 2, 3, 4, 5\}$. Suppose not. Then, by Theorem 14, we see that $\|S_\delta(z_i) \cap S_\delta(z_j)\| \leq 2$ for any distinct $i, j \in \{0, 1, 2, 3, 4, 5\}$. Thus, we have

$$\begin{aligned} 62 &= \|V\| \\ &\geq \|S_\delta(z_0) \cup S_\delta(z_1) \cup S_\delta(z_2) \cup S_\delta(z_3) \cup S_\delta(z_4) \cup S_\delta(z_5)\| \\ &\geq 16 + 14 + 12 + 10 + 8 + 6 \\ &= 66, \end{aligned}$$

which is impossible. Thus, we see that there exist some $i, j \in \{0, 1, 2, 3, 4, 5\}$, such that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$. Let $x = z_i$ and $y = z_j$.

Thus, we have $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$. By Theorem 14, applied to x and y , we see that there exist colors α, β , and γ , such that $\|S_\gamma(w)\| \leq 14$ for all $w \in S_\delta(x) \cap S_\delta(y)$ and that all of the edges in $S_\delta(x) \cap S_\delta(y)$ are colored with the colors α and β .

By Proposition 1, we see that for any $w \in S_\delta(x) \cap S_\delta(y)$, we have either $\|S_\alpha(w)\| = 16$ or $\|S_\beta(w)\| = 16$ (or both). But $\|S_\delta(x) \cap S_\delta(y)\| = 5$, so that there must exist pairwise distinct $u, v, w \in S_\delta(x) \cap S_\delta(y)$ such that either $\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16$ or $\|S_\beta(u)\| = \|S_\beta(v)\| = \|S_\beta(w)\| = 16$. Without loss of generality, we may suppose that

$$\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16.$$

Since $\{u, v, w\} \subseteq S_\delta(x) \cap S_\delta(y)$, we see that all of the edges in $\{u, v, w\}$ must be colored with the colors α and β . Since all three edges in $\{u, v, w\}$ cannot be the same color, at least one such edge must be of color β . Without loss of generality, we may suppose that

$$u \xrightarrow{\beta} v.$$

By Theorem 14, applied to x and y , we see that there exist $w_0, w_1, w_2 \in S_\delta(x) \cap S_\delta(y)$ such that

$$P_{\beta, \alpha}(u, w_0, w_1, w_2, v).$$

Note that $w_1 \xrightarrow{\alpha} u, v$ and that w_1 is the only such element of $S_\delta(x) \cap S_\delta(y)$. Since $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$, we may apply Lemma 6(1), to see that there exists some $z \in S_\delta(x) \sim S_\delta(y)$ and some $z' \in S_\delta(y) \sim S_\delta(x)$ such that $w_1, z, z' \in S_\alpha(u) \cap S_\alpha(v)$. Clearly, the elements w_1, z, z' are pairwise distinct, so that, since $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$, we see, by Theorem 14, applied to u and v , that $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$.

To see that $S_\alpha(u)$ and $S_\alpha(v)$ are twisted, we need only apply Theorem 14 to x and y , noting that $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$ and that the edges in $S_\delta(x) \cap S_\delta(y)$ are colored with the colors α and β .

The proof is complete. □

Theorem 18. $R(3, 3, 3, 3) \leq 62$.

Proof. Suppose not. Then there exists a good edge coloring, using four colors, on a complete graph with vertex set V where $\|V\| = 62$. By Theorem 17, there exists some color δ and vertices $u, v \in V$ such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ and $\|S_\delta(u) \cap S_\delta(v)\| = 5$ with both $S_\delta(u)$ and $S_\delta(v)$ twisted.

By Theorem 14, there exist $x_0, \dots, x_{15} \in S_\delta(u)$ and $y_0, \dots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 14\}$ and colors α, β , and γ , such that $B_{\alpha, \beta, \gamma}^1(x_0, \dots, x_{15})$ and $B_{\alpha, \beta, \gamma}^1(y_0, \dots, y_{15})$ with $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$, where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$.

First, note that the entire situation so far is preserved by the following symmetry Θ :

$$\begin{aligned}
 u &\mapsto u \\
 v &\mapsto v \\
 a &\mapsto a \\
 b &\mapsto c \mapsto e \mapsto d \mapsto b \\
 x_0 &\mapsto x_0 \\
 x_3 &\mapsto x_8 \mapsto x_3 \\
 x_1 &\mapsto x_7 \mapsto x_5 \mapsto x_9 \mapsto x_1 \\
 x_2 &\mapsto x_{10} \mapsto x_4 \mapsto x_6 \mapsto x_2 \\
 y_0 &\mapsto y_0 \\
 y_3 &\mapsto y_8 \mapsto y_3 \\
 y_1 &\mapsto y_7 \mapsto y_5 \mapsto y_9 \mapsto y_1 \\
 y_2 &\mapsto y_{10} \mapsto y_4 \mapsto y_6 \mapsto y_2
 \end{aligned}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{aligned}
 \alpha &\mapsto \beta \mapsto \alpha \\
 \gamma &\mapsto \gamma \\
 \delta &\mapsto \delta
 \end{aligned}$$

By Theorem 14, applied to u and v , we see that $\|S_\gamma(w)\| \leq 14$ for all $w \in \{a, b, c, d, e\}$.

Now, Suppose that $\|S_\alpha(a)\| = 16$. Then, by Theorem 16, applied to u and v , we have $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. But $\|S_\gamma(b)\| = \|S_\gamma(e)\| = 14$, so that, by Proposition 1, we have $\|S_\delta(b)\| = \|S_\delta(e)\| = \|S_\beta(b)\| = \|S_\beta(e)\| = 16$. Since $b, e, a \stackrel{\delta}{\sim} u, v$, we see, by Theorem 15, that $\|S_\delta(a)\| \leq 15$, so that, since $\|S_\delta(a)\| \leq$

14, we see, by Proposition 1, that $\|S_\beta(a)\| = 16$. Likewise, since $b, e, c \stackrel{\delta}{\sim} u, v$, we see, by Theorem 14, that $\|S_\delta(c)\| \leq 15$, so that, since $\|S_\delta(c)\| \leq 14$, we see, by Proposition 1, that $\|S_\beta(c)\| = 16$. But, by Theorem 16, applied to u and v , we cannot have $\|S_\beta(a)\| = \|S_\beta(c)\| = 16$, thus giving a contradiction.

Thus, we have shown that

$$\|S_\alpha(a)\| \leq 15.$$

An application of the symmetry to this gives

$$\|S_\beta(a)\| \leq 15.$$

Since $\|S_\gamma(a)\| \leq 14$, we get a contradiction, by Proposition 1.

The theorem is proved. \square

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