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The classical Ramsey number $R(3,3,3,3) \leq 62$: The global arguments

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Abstract. We show that $R(3,3,3,3) \leq 62$, that is, any good edge coloring of a complete graph on 62 vertices with four colors must contain a monochromatic triangle. This paper gives the global arguments of the proof.

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1. Introduction

An edge coloring of a complete graph is called *good* provided that there do not exist monochromatic triangles. There are, up to isomorphism, exactly two good edge colorings with three colors on the complete graph with 16 vertices. (See [6].) These are called the untwisted and twisted colorings. The automorphism group on each of these colorings acts transitively on the vertices. By removing a single vertex from each of these colorings (along with all edges incident with the removed vertex), we get two non-isomorphic good edge colorings with three colors on the complete graph with 15 vertices. We refer to these as the untwisted and twisted colorings on 15 vertices. There are no others, up to isomorphism. (See [5].)

All of the arguments in this paper are based on our intimate knowledge of the untwisted and twisted colorings on 15 and 16 vertices with three colors, together with the knowledge that these are the only such good edge colorings possible. If we knew the good edge colorings on 16 vertices, but not on 15 vertices, then the arguments in this paper (with trivial and obvious modifications) would suffice

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only to prove that $R(3,3,3,3) \leq 64$, instead of $R(3,3,3,3) \leq 62$, as we prove here.

In this paper, we improve the known upper bound for the classical Ramsey number $R(3,3,3,3)$. It is trivial to see that

$$\begin{aligned} R(3,3,3,3) &\leq 4 \cdot (R(3,3,3;2) - 1) + 1 + 1 \\ &= 4 \cdot (17 - 1) + 1 + 1 \\ &= 66. \end{aligned}$$

(See [4].) In Folkman [3], it is shown that $R(3,3,3,3) \leq 65$. In Sanchez-Flores [10], it is shown that $R(3,3,3,3) \leq 64$. In this paper, we improve this to read $R(3,3,3,3) \leq 62$. That is, we show that for any coloring of a complete graph with 62 vertices using four colors, there must exist a monochromatic triangle. The best known lower bound for $R(3,3,3,3)$ was provided by Chung [1], who constructed two non-isomorphic monochromatic triangle free edge colorings, from the untwisted and twisted good colorings of K_{16} with three colors, using four colors of the complete graph with 50 vertices, thus showing that $R(3,3,3,3) \geq 51$.

The paper is organized as follows. In §2, we give a few preliminary definitions and facts, including some facts about the good colorings on the complete graphs K_{16} and K_{15} using three colors, which will be needed in in later sections. Suppose that we are given a good edge coloring, with four colors, on the complete graph on 62 vertices. Let u and v be two distinct vertices and let δ be any color. Suppose that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$. (According to Definition 2.2 in section §2, we define the set $S_\delta(u)$ to be the set of all vertices x such that the edge from u to x is of color δ.) Then the set $S_\delta(u) \cap S_\delta(v)$ is referred to as an *attaching set*. The structure of a potential attaching set is quite limited. The section ends with a statement, given without proof, of Theorem 2.5, which gathers together the results of the local arguments into a single theorem, This theorem limits the structure of potential attaching sets. More specifically, it states that every attaching set has cardinality 0, 1, 2 or 5. It further states that the structure of attaching sets of cardinality 5 are severely restricted. In §3, we give the global arguments, that is, arguments which require the consideration of multiple attaching sets at the same time. There we assume Theorem 2.5 for *every* attaching set in a given good coloring of K_{62} with four colors, and use that fact to derive a contradiction, thus proving that no such good coloring on K_{62} exists. This is the main result, Theorem 3.4, that is, $R(3,3,3,3) \leq 62$.

2. Preliminaries

Notation 2.1. *For convenience, we define*

$$[i_1, \ldots, i_n] = \left\{ \left(i_{f(1)}, \ldots, i_{f(n)}\right) \,\middle|\, f \text{ is a permutation on } \{1, \ldots, n\} \right\}.$$

We also write $u \overset{\alpha}{\text{———}} v$, where u and v are vertices in some edge colored graph and α is a color, to indicate that the edge connecting u and v is of color α.

Definition 2.2. *Let V be the vertex set of an edge colored complete graph. Let α be a color and let $v \in V$. Then we define*

$$S_\alpha(v) = \{\, x \in V \mid x \overset{\alpha}{-} v \,\}.$$

Definition 2.3. *Let $v_0, \dots, v_{15}$ be vertices of an edge coloring of a complete graph, and let α, β, and γ be colors. We define the following predicates:*

(1) $P_{\beta,\alpha}(v_1, \dots, v_5)$ *iff*$_{df}$
$v_1 \overset{\beta}{-} v_2 \overset{\beta}{-} v_3 \overset{\beta}{-} v_4 \overset{\beta}{-} v_5 \overset{\beta}{-} v_1$ *and* $v_1 \overset{\alpha}{-} v_3 \overset{\alpha}{-} v_5 \overset{\alpha}{-} v_2 \overset{\alpha}{-} v_4 \overset{\alpha}{-} v_1$. *(See Figure 1.)*

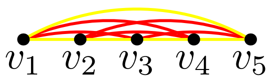

FIGURA 1. $P_{\beta,\alpha}(v_1, \dots, v_5)$

(2) $A^0_{\alpha,\beta,\gamma}(v_1, \dots, v_{10})$ *iff* $_{df}$
$P_{\beta,\alpha}(v_1, v_2, v_3, v_4, v_5)$ *and* $P_{\gamma,\beta}(v_6, v_7, v_8, v_9, v_{10})$ *and* $v_1 \overset{\beta}{-} v_6$ *and* $v_2 \overset{\beta}{-} v_7$ *and* $v_3 \overset{\beta}{-} v_8$ *and* $v_4 \overset{\beta}{-} v_9$ *and* $v_5 \overset{\beta}{-} v_{10}$ *and* $v_1 \overset{\alpha}{-} v_8 \overset{\alpha}{-} v_5 \overset{\alpha}{-} v_7 \overset{\alpha}{-} v_4 \overset{\alpha}{-} v_6 \overset{\alpha}{-} v_3 \overset{\alpha}{-} v_{10} \overset{\alpha}{-} v_2 \overset{\alpha}{-} v_9 \overset{\alpha}{-} v_1$ *and* $v_1 \overset{\gamma}{-} v_7 \overset{\gamma}{-} v_3 \overset{\gamma}{-} v_9 \overset{\gamma}{-} v_5 \overset{\gamma}{-} v_6 \overset{\gamma}{-} v_2 \overset{\gamma}{-} v_8 \overset{\gamma}{-} v_4 \overset{\gamma}{-} v_{10} \overset{\gamma}{-} v_1$.

(3) $A^1_{\alpha,\beta,\gamma}(v_1, \dots, v_{10})$ *iff*$_{df}$
$P_{\beta,\alpha}(v_1, v_2, v_3, v_4, v_5)$ *and* $P_{\gamma,\beta}(v_6, v_7, v_8, v_9, v_{10})$ *and* $v_1 \overset{\beta}{-} v_6$ *and* $v_2 \overset{\beta}{-} v_7$ *and* $v_3 \overset{\beta}{-} v_8$ *and* $v_4 \overset{\beta}{-} v_9$ *and* $v_5 \overset{\beta}{-} v_{10}$ *and* $v_1 \overset{\alpha}{-} v_8 \overset{\alpha}{-} v_5 \overset{\alpha}{-} v_7 \overset{\alpha}{-} v_4 \overset{\alpha}{-} v_{10} \overset{\alpha}{-} v_3 \overset{\alpha}{-} v_6 \overset{\alpha}{-} v_2 \overset{\alpha}{-} v_9 \overset{\alpha}{-} v_1$ *and* $v_1 \overset{\gamma}{-} v_7 \overset{\gamma}{-} v_3 \overset{\gamma}{-} v_9 \overset{\gamma}{-} v_5 \overset{\gamma}{-} v_6 \overset{\gamma}{-} v_4 \overset{\gamma}{-} v_8 \overset{\gamma}{-} v_2 \overset{\gamma}{-} v_{10} \overset{\gamma}{-} v_1$.

(4) $C^0_{\alpha,\beta,\gamma}(v_1, \dots, v_{15})$ *iff*$_{df}$
$A^0_{\alpha,\beta,\gamma}(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$ *and* $A^0_{\beta,\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ *and* $A^0_{\gamma,\alpha,\beta}(v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15})$.

(5) $C^1_{\alpha,\beta,\gamma}(v_1, \dots, v_{15})$ *iff*$_{df}$
$A^1_{\alpha,\beta,\gamma}(v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1, v_2, v_3, v_4, v_5)$ *and* $A^1_{\beta,\gamma,\alpha}(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ *and* $A^1_{\gamma,\alpha,\beta}(v_6, v_7, v_8, v_9, v_{10}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11})$.

(6) $B^0_{\alpha,\beta,\gamma}(v_0, \dots, v_{15})$ *iff*$_{df}$
$C^0_{\alpha,\beta,\gamma}(v_1, \dots, v_{15})$ *and* $v_0 \overset{\alpha}{-} v_1, v_2, v_3, v_4, v_5$ *and* $v_0 \overset{\beta}{-} v_6, v_7, v_8, v_9, v_{10}$ *and* $v_0 \overset{\gamma}{-} v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$. *(See Figure 2.)*

(7) $B^1_{\alpha,\beta,\gamma}(v_0, \dots, v_{15})$ *iff*$_{df}$
$C^1_{\alpha,\beta,\gamma}(v_1, \dots, v_{15})$ *and* $v_0 \overset{\alpha}{-} v_1, v_2, v_3, v_4, v_5$ *and* $v_0 \overset{\beta}{-} v_6, v_7, v_8, v_9, v_{10}$ *and* $v_0 \overset{\gamma}{-} v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$. *(See Figure 3.)*

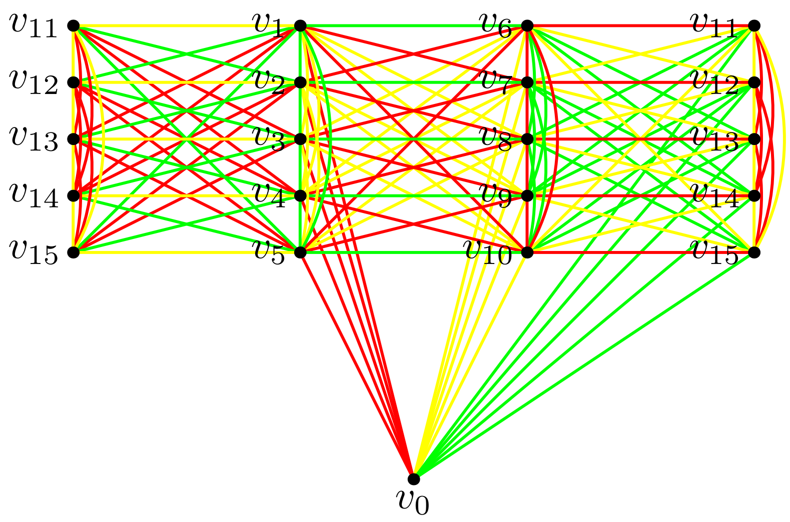

FIGURA 2. $B^0_{\alpha,\beta,\gamma}(v_1,\ldots,v_{15})$

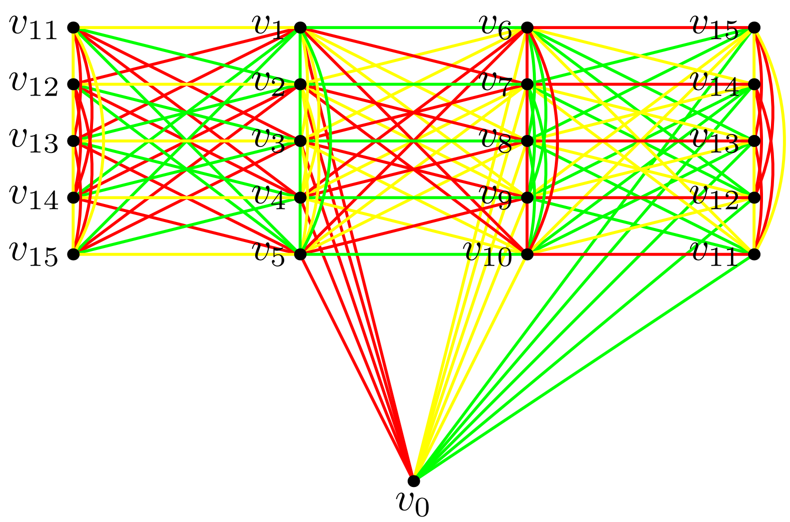

FIGURA 3. $B^1_{\alpha,\beta,\gamma}(v_1,\ldots,v_{15})$

Lemma 2.1. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. If $\|U\| = 16$, then there exist $x_0,\ldots,x_{15} \in U$ and some $i \in \{0,1\}$ such that $B^i_{\alpha,\beta,\gamma}(x_0,\ldots,x_{15})$*

Demostración. See Kalbfleisch and Stanton [6]. ☑

Lemma 2.2. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. Let $\|U\| = 16$ and let $a, b \in U$ with $a \overset{\gamma}{\text{---}} b$. Then, we have the following:*

(1) $\|\{ x \in U \mid a \overset{\alpha}{\text{---}} x \overset{\alpha}{\text{---}} b \}\| = 2$
(2) $\|\{ x \in U \mid a \overset{\alpha}{\text{---}} x \overset{\beta}{\text{---}} b \}\| = 1$
(3) $\|\{ x \in U \mid a \overset{\alpha}{\text{---}} x \overset{\gamma}{\text{---}} b \}\| = 2$

Demostración. Follows immediately by inspection from Lemma 2.1. ☑

Lemma 2.3. *Let U be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, and γ. If $\|U\| = 15$, then there exist $x_1, \ldots, x_{15} \in U$ and some $i \in \{0, 1\}$ such that $C^i_{\alpha,\beta,\gamma}(x_1, \ldots, x_{15})$*

Demostración. See Heinrich [5]. ☑

Remark 2.1. *In Lemma 2.1 and Lemma 2.3, $i = 0$ if the coloring is untwisted, and $i = 1$ if the coloring is twisted.*

Proposition 2.4. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $\|V\| = 62$ and let $x \in V$. Then*

$$\big(\|S_\alpha(x)\|, \|S_\beta(x)\|, \|S_\gamma(x)\|, \|S_\delta(x)\|\big) \\ \in [16,16,16,13] \cup [16,16,15,14] \cup [16,15,15,15].$$

Demostración. It is clear that

$$\begin{aligned} 62 &= \|V\| \\ &= \|\{ x \} \uplus S_\alpha(x) \uplus S_\beta(x) \uplus S_\gamma(x) \uplus S_\delta(x)\| \\ &= 1 + \|S_\alpha(x)\| + \|S_\beta(x)\| + \|S_\gamma(x)\| + \|S_\delta(x)\|. \end{aligned}$$

Also, for any $\eta \in \{\alpha, \beta, \gamma, \delta\}$, we see that the induced good edge coloring on the complete graph with vertex set $S_\eta(x)$ cannot contain any edges of color η, since otherwise we would have a monochromatic triangle of color η in V, contradicting the goodness of the original coloring. Thus, we have

$$\|S_\alpha(x)\|, \|S_\beta(x)\|, \|S_\gamma(x)\|, \|S_\delta(x)\| \leq R(3,3,3;2) - 1 = 17 - 1 = 16.$$

The proposition follows. ☑

The following theorem summarizes the results of the local arguments of [7], that is, arguments that proceed by consideration of a single attaching set. In these arguments, all potential attaching sets of cardinalities other than 0, 1, 2, and 5 are eliminated, and the structure of any attaching sets of cardinality 5 are severely restricted. This theorem is used repeatedly in the global arguments of the next section.

Theorem 2.5. *Let V be the vertex set of a complete graph with a good edge coloring with four colors. Suppose that $\|V\| = 62$ and let $u, v \in V$ with $u \neq v$ be such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ for some color δ. Then $\|S_\delta(u) \cap S_\delta(v)\| \in \{ 0, 1, 2, 5 \}$. In addition, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ and $y_0, \ldots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{ 11, 12, 13, 14, 15 \}$ and some $j \in \{ 0, 1 \}$ and colors α, β, and γ, such that $B^j_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^j_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ with*

$$S_\delta(u) \cap S_\delta(v) = \{ x_{11}, x_{12}, x_{13}, x_{14}, x_{15} \} = \{ y_{11}, y_{12}, y_{13}, y_{14}, y_{15} \}.$$

Furthermore, if $\|S_\delta(u) \cap S_\delta(v)\| = 5$, then for each $w \in S_\delta(u) \cap S_\delta(v)$, we have $\|S_\gamma(w)\| \leq 14$, and both $S_\alpha(w)$ and $S_\beta(w)$ are twisted.

Demostración. See [7]. ☑

Remark 2.2. *Note that if $\|S_\gamma(w)\| \leq 14$, then $S_\alpha(w), S_\beta(w) \geq 15$ for $\gamma \neq \alpha, \beta$, by Proposition 2.4, so that by Lemma 2.1, Lemma 2.3 and Remark 2.1, it makes sense to say that $S_\alpha(w)$ and $S_\beta(w)$ are twisted in the last sentence of Theorem 2.5.*

3. The Main Theorem

In this section, we present the global arguments, that is, arguments that consider more than one attaching set at a time. Essentially, Proposition 2.4 guarentees that there lots of neighborhoods of cardinality 16, and therefore lots of attaching sets. Theorem 2.5 applies to every attaching set, and will be our main tool.

Theorem 3.1. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Suppose that $\|V\| = 62$ and let $u, v, w, x, y \in V$ be pairwise distinct vertices which satisfy $u, v, w \stackrel{\delta}{\text{——}} x, y$ for some color δ. Then, for some $z \in \{ u, v, w, x, y \}$ we have $\|S_\delta(z)\| \leq 15$.*

Demostración. Suppose not. Then $\|S_\delta(z)\| = 16$ for all $z \in \{ u, v, w, x, y \}$. Thus, by Theorem 2.5, we have $\|S_\delta(x) \cap S_\delta(y)\| \in \{ 0, 1, 2, 5 \}$, so that, since $u, v, w \in S_\delta(x) \cap S_\delta(y)$, we must have

$$\|S_\delta(x) \cap S_\delta(y)\| = 5.$$

Next, we show that $\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5$. Suppose not. Then we may, without loss of generality, suppose that $\|S_\delta(u) \cap S_\delta(v)\| \neq 5$, so that, by Theorem 2.5, we have $\|S_\delta(u) \cap S_\delta(v)\| \leq 2$. But, $x, y \in S_\delta(u) \cap S_\delta(v)$, so that we must have $S_\delta(u) \cap S_\delta(v) = \{ x, y \} \subseteq S_\delta(w)$.

But then we have $\big(S_\delta(u) \sim S_\delta(w)\big) \cap \big(S_\delta(v) \sim S_\delta(w)\big) = \emptyset$. Thus, we see that

$$\begin{aligned}
62 &= \big\|V\big\| \\
&\geq \big\|S_\delta(u) \cup S_\delta(v) \cup S_\delta(w) \cup S_\delta(x) \cup S_\delta(y)\big\| \\
&= \big\|\big(S_\delta(x) \sim S_\delta(y)\big) \uplus \big(S_\delta(y) \sim S_\delta(x)\big) \uplus \big(S_\delta(x) \cap S_\delta(y)\big) \uplus \\
&\quad \big(S_\delta(u) \sim S_\delta(w)\big) \uplus \big(S_\delta(v) \sim S_\delta(w)\big) \uplus S_\delta(w)\big\| \\
&= \big\|S_\delta(x) \sim S_\delta(y)\big\| + \big\|S_\delta(y) \sim S_\delta(x)\big\| + \big\|S_\delta(x) \cap S_\delta(y)\big\| + \\
&\quad \big\|S_\delta(u) \sim S_\delta(w)\big\| + \big\|S_\delta(v) \sim S_\delta(w)\big\| + \big\|S_\delta(w)\big\| \\
&= 11 + 11 + 5 + \big\|S_\delta(u) \sim S_\delta(w)\big\| + \big\|S_\delta(v) \sim S_\delta(w)\big\| + 16 \\
&\geq 11 + 11 + 5 + 11 + 11 + 16 \\
&= 65,
\end{aligned}$$

which is a contradiction. Thus, we have

$$\|S_\delta(u) \cap S_\delta(v)\| = \|S_\delta(u) \cap S_\delta(w)\| = \|S_\delta(v) \cap S_\delta(w)\| = 5,$$

as desired.

We may now apply Theorem 2.5 to each pair in $\{\, u, v, w \,\}$ to see that there exist colors $\gamma, \gamma', \gamma'' \neq \delta$ such that

$$S_\delta(u) \cap S_\delta(v) \text{ contains no edges of color } \gamma \text{ and } \|S_\gamma(x)\| \leq 14,$$

$$S_\delta(u) \cap S_\delta(w) \text{ contains no edges of color } \gamma' \text{ and } \|S_{\gamma'}(x)\| \leq 14,$$

and

$$S_\delta(v) \cap S_\delta(w) \text{ contains no edges of color } \gamma'' \text{ and } \|S_{\gamma''}(x)\| \leq 14.$$

By Proposition 2.4, we see that $\gamma = \gamma' = \gamma''$.

Thus, there exists some $\gamma \neq \delta$, such that $\|S_\gamma(x)\| \leq 14$ and

$$\begin{aligned}
\emptyset &= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(v) \cap S_\gamma(x)) \\
&= (S_\delta(u) \cap S_\gamma(x)) \cap (S_\delta(w) \cap S_\gamma(x)) \\
&= (S_\delta(v) \cap S_\gamma(x)) \cap (S_\delta(w) \cap S_\gamma(x)).
\end{aligned}$$

Next, we show that $\|S_\delta(z) \cap S_\gamma(x)\| \leq 4$ for some $z \in \{\, u, v, w \,\}$. Suppose not. Then $\|S_\delta(z) \cap S_\gamma(x)\| = 5$ for all $z \in \{\, u, v, w \,\}$. Thus, since

$$\biguplus_{z \in \{\, u,v,w \,\}} \big(S_\delta(z) \cap S_\gamma(x)\big) \subseteq S_\gamma(x),$$

we must have $5 + 5 + 5 \leq 14$, which is impossible.

Thus, we may suppose, without loss of generality, that $\|S_\delta(u)\cap S_\gamma(x)\| \le 4$. Let α and β be the two other colors. Then, since $u \overset{\delta}{\text{———}} x$, we must have

$$\begin{aligned} 16 &= \big\|S_\delta(u)\big\| \\ &= \big\|\{\,x\,\} \uplus \big(S_\delta(u)\cap S_\alpha(x)\big) \uplus \big(S_\delta(u)\cap S_\beta(x)\big) \uplus \big(S_\delta(u)\cap S_\gamma(x)\big)\big\| \\ &= \big\|\{\,x\,\}\big\| + \big\|S_\delta(u)\cap S_\alpha(x)\big\| + \big\|S_\delta(u)\cap S_\beta(x)\big\| + \big\|S_\delta(u)\cap S_\gamma(x)\big\| \\ &\le 1+5+5+4 \\ &= 15, \end{aligned}$$

which is impossible.

The proof is complete. ☑

Theorem 3.2. *Let V be the vertex set of a complete graph with a good edge coloring, colored with the pairwise distinct colors α, β, γ, and δ. Suppose that $u, v \in V$ with $S_\delta(u) = \{\,x_0,\dots,x_{15}\,\}$ and $S_\delta(v) = \{\,y_0,\dots,y_{15}\,\}$ with $B^1_{\alpha,\beta,\gamma}(x_0,\dots,x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0,\dots,y_{15})$ such that $S_\delta(u)\cap S_\delta(v) = \{\,a,b,c,d,e\,\}$ where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$. Suppose further that $\|V\| = 62$. If $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. If $\|S_\beta(a)\| = 16$, then $\|S_\beta(d)\| = \|S_\beta(c)\| = 15$.*

Demostración. First, note that the hypotheses of the theorem are preserved by the following symmetry Θ:

$$\begin{gathered} u \longmapsto u \\ v \longmapsto v \\ a \longmapsto a \\ b \longmapsto c \longmapsto e \longmapsto d \longmapsto b \\ x_0 \longmapsto x_0 \\ x_3 \longmapsto x_8 \longmapsto x_3 \\ x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1 \\ x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2 \\ y_0 \longmapsto y_0 \\ y_3 \longmapsto y_8 \longmapsto y_3 \\ y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1 \\ y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2 \end{gathered}$$

Note that the symmetry Θ acts on the colors as follows:

$$\begin{gathered} \alpha \longmapsto \beta \longmapsto \alpha \\ \gamma \longmapsto \gamma \\ \delta \longmapsto \delta \end{gathered}$$

First, we show that if $\|S_\alpha(a)\| = 16$, then $\|S_\alpha(b)\| = 15$. Suppose not. That is, suppose that $\|S_\alpha(a)\| = 16$ but $\|S_\alpha(b)\| \ne 15$. By Theorem 2.5, we see that

$\|S_\gamma(b)\| \leq 14$, so that, by Proposition 2.4, we must have $\|S_\alpha(b)\| \geq 15$, so that, in fact,

$$\|S_\alpha(b)\| = \|S_\alpha(a)\| = 16.$$

Applying Theorem 2.5 once again, this time to a and b, and using the fact that $x_5, y_5, d \in S_\alpha(a) \cap S_\alpha(b)$ are pairwise distinct elements, we see that

$$\|S_\alpha(a) \cap S_\alpha(b)\| = 5.$$

Furthermore, note that $x_5 \overset{\delta}{\text{———}} y_5$, since $x_5, y_5 \in S_\alpha(a) \cap S_\beta(c) \cap S_\gamma(d)$. Note also that $d \overset{\gamma}{\text{———}} x_5, y_5$. Thus, by Theorem 2.5, applied to a and b, we see that

$$\|S_\beta(d)\| \leq 14.$$

But by Theorem 2.5, applied to u and v, we see that

$$\|S_\gamma(d)\| \leq 14,$$

which is impossible, by Proposition 2.4.

Thus, we have shown that

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(b)\| = 15.$$

Repeated applications of the symmetry Θ give

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(c)\| = 15,$$

$$\text{if } \|S_\alpha(a)\| = 16, \text{ then } \|S_\alpha(e)\| = 15,$$

and

$$\text{if } \|S_\beta(a)\| = 16, \text{ then } \|S_\beta(d)\| = 15.$$

The proof is complete. ☑

Theorem 3.3. *Let V be the vertex set of a complete graph with a good edge coloring, colored with four colors. Then there exists some color α and vertices $u, v \in V$, such that $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$ and $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$ with both $S_\alpha(u)$ and $S_\alpha(v)$ twisted.*

Demostración. By Proposition 2.4, for each $z \in V$ there exists some color η such that $\|S_\eta(z)\| = 16$. Thus, we have

$$\Big\|\big\{\, (z, \eta) \;\big|\; \|S_\eta(z)\| = 16 \,\big\}\Big\| \geq 62.$$

Thus, there must exist some color δ such that

$$\Big\|\big\{\, z \;\big|\; \|S_\delta(z)\| = 16 \,\big\}\Big\| \geq 16.$$

Let $z_0, z_1, z_2, z_3, z_4, z_5 \in V$ be pairwise distinct vertices such that $\|S_\delta(z_i)\| = 16$ for all $i \in \{0, 1, 2, 3, 4, 5\}$.

First, we show that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$ for some $i, j \in \{0,1,2,3,4,5\}$. Suppose not. Then, by Theorem 2.5, we see that $\|S_\delta(z_i) \cap S_\delta(z_j)\| \le 2$ for any distinct $i, j \in \{0,1,2,3,4,5\}$. Thus, we have

$$\begin{aligned} 62 &= \big\|V\big\| \\ &\ge \big\|S_\delta(z_0) \cup S_\delta(z_1) \cup S_\delta(z_2) \cup S_\delta(z_3) \cup S_\delta(z_4) \cup S_\delta(z_5)\big\| \\ &\ge 16 + 14 + 12 + 10 + 8 + 6 \\ &= 66, \end{aligned}$$

which is impossible. Thus, we see that there exist some $i, j \in \{0,1,2,3,4,5\}$, such that $\|S_\delta(z_i) \cap S_\delta(z_j)\| = 5$. Let $x = z_i$ and $y = z_j$.

Thus, we have $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$. By Theorem 2.5, applied to x and y, we see that there exist colors α, β, and γ, such that $\|S_\gamma(w)\| \le 14$ for all $w \in S_\delta(x) \cap S_\delta(y)$ and that all of the edges in $S_\delta(u) \cap S_\delta(v)$ are colored with the colors α and β.

By Proposition 2.4, we see that for any $w \in S_\delta(u) \cap S_\delta(v)$, we have either $\|S_\alpha(w)\| = 16$ or $\|S_\beta(w)\| = 16$ (or both). But $\|S_\delta(x) \cap S_\delta(y)\| = 5$, so that there must exist pairwise distinct $u, v, w \in S_\delta(x) \cap S_\delta(y)$ such that either $\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16$ or $\|S_\beta(u)\| = \|S_\beta(v)\| = \|S_\beta(w)\| = 16$. Without loss of generality, we may suppose that

$$\|S_\alpha(u)\| = \|S_\alpha(v)\| = \|S_\alpha(w)\| = 16.$$

Since $\{u, v, w\} \subseteq S_\delta(x) \cap S_\delta(y)$, we see that all of the edges in $\{u, v, w\}$ must be colored with the colors α and β. Since all three edges in $\{u, v, w\}$ cannot be the same color, at least one such edge must be of color β. Without loss of generality, we may suppose that

$$u \overset{\beta}{\text{——}} v.$$

By Theorem 2.5, applied to x and y, we see that there exist $w_0, w_1, w_2 \in S_\delta(x) \cap S_\delta(y)$ such that

$$P_{\beta,\alpha}(u, w_0, w_1, w_2, v).$$

Note that $w_1 \overset{\alpha}{\text{——}} u, v$ and that w_1 is the only such element of $S_\delta(x) \cap S_\delta(y)$. Since $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$, we may apply Lemma 2.2(1), to see that there exists some $z \in S_\delta(x) \sim S_\delta(y)$ and some $z' \in S_\delta(y) \sim S_\delta(x)$ such that $w_1, z, z' \in S_\alpha(u) \cap S_\alpha(v)$. Clearly, the elements w_1, z, z' are pairwise distinct, so that, since $\|S_\alpha(u)\| = \|S_\alpha(v)\| = 16$, we see, by Theorem 2.5, applied to u and v, that $\|S_\alpha(u) \cap S_\alpha(v)\| = 5$. To see that $S_\alpha(u)$ and $S_\alpha(v)$ are twisted, we need only apply Theorem 2.5 to x and y, noting that $\|S_\delta(x)\| = \|S_\delta(y)\| = 16$ and $\|S_\delta(x) \cap S_\delta(y)\| = 5$ and that the edges in $S_\delta(x) \cap S_\delta(y)$ are colored with the colors α and β.

The proof is complete. ☑

Theorem 3.4. *$R(3, 3, 3, 3) \le 62$.*

Demostración. Suppose not. Then there exists a good edge coloring, using four colors, on a complete graph with vertex set V where $\|V\| = 62$. By Theorem 3.3, there exists some color δ and vertices $u, v \in V$ such that $\|S_\delta(u)\| = \|S_\delta(v)\| = 16$ and $\|S_\delta(u) \cap S_\delta(v)\| = 5$ with both $S_\delta(u)$ and $S_\delta(v)$ twisted.

By Theorem 2.5, there exist $x_0, \ldots, x_{15} \in S_\delta(u)$ and $y_0, \ldots, y_{15} \in S_\delta(v)$ with $x_i = y_i$ for all $i \in \{11, 12, 13, 14, 15\}$ and colors α, β, and γ, such that $B^1_{\alpha,\beta,\gamma}(x_0, \ldots, x_{15})$ and $B^1_{\alpha,\beta,\gamma}(y_0, \ldots, y_{15})$ with $S_\delta(u) \cap S_\delta(v) = \{a, b, c, d, e\}$, where $a = x_{13} = y_{13}$, $b = x_{14} = y_{14}$, $c = x_{15} = y_{15}$, $d = x_{11} = y_{11}$, and $e = x_{12} = y_{12}$.

First, note that the entire situation so far is preserved by the following symmetry Θ:

$$u \longmapsto u$$
$$v \longmapsto v$$
$$a \longmapsto a$$
$$b \longmapsto c \longmapsto e \longmapsto d \longmapsto b$$
$$x_0 \longmapsto x_0$$
$$x_3 \longmapsto x_8 \longmapsto x_3$$
$$x_1 \longmapsto x_7 \longmapsto x_5 \longmapsto x_9 \longmapsto x_1$$
$$x_2 \longmapsto x_{10} \longmapsto x_4 \longmapsto x_6 \longmapsto x_2$$
$$y_0 \longmapsto y_0$$
$$y_3 \longmapsto y_8 \longmapsto y_3$$
$$y_1 \longmapsto y_7 \longmapsto y_5 \longmapsto y_9 \longmapsto y_1$$
$$y_2 \longmapsto y_{10} \longmapsto y_4 \longmapsto y_6 \longmapsto y_2$$

Note that the symmetry Θ acts on the colors as follows:

$$\alpha \longmapsto \beta \longmapsto \alpha$$
$$\gamma \longmapsto \gamma$$
$$\delta \longmapsto \delta$$

By Theorem 2.5, applied to u and v, we see that $\|S_\gamma(w)\| \le 14$ for all $w \in \{a, b, c, d, e\}$.

Now, Suppose that $\|S_\alpha(a)\| = 16$. Then, by Theorem 3.2, applied to u and v, we have $\|S_\alpha(b)\| = \|S_\alpha(e)\| = 15$. But $\|S_\gamma(b)\| = \|S_\gamma(e)\| = 14$, so that, by Proposition 2.4, we have $\|S_\delta(b)\| = \|S_\delta(e)\| = \|S_\beta(b)\| = \|S_\beta(e)\| = 16$. Since $b, e, a \overset{\delta}{\longrightarrow} u, v$, we see, by Theorem 3.1, that $\|S_\delta(a)\| \le 15$, so that, since $\|S_\delta(a)\| \le 14$, we see, by Proposition 2.4, that $\|S_\beta(a)\| = 16$. Likewise, since $b, e, c \overset{\delta}{\longrightarrow} u, v$, we see, by Theorem 2.5, that $\|S_\delta(c)\| \le 15$, so that, since $\|S_\delta(c)\| \le 14$, we see, by Proposition 2.4, that $\|S_\beta(c)\| = 16$. But, by Theorem 3.2, applied to u and v, we cannot have $\|S_\beta(a)\| = \|S_\beta(c)\| = 16$, thus giving a contradiction.

Thus, we have shown that

$$\|S_\alpha(a)\| \leq 15.$$

An application of the symmetry to this gives

$$\|S_\beta(a)\| \leq 15.$$

Since $\|S_\gamma(a)\| \leq 14$, we get a contradiction, by Proposition 2.4.

The theorem is proved. ☑

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