RELATIVIZED RELATION ALGEBRAS

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§0. Introduction

In this paper we introduce a finitely axiomatizable equational class of relation type algebras which we call REL. This class includes all relation algebras and is closed under relativization to an arbitrary element. Thus REL contains every relativization of $\mathfrak{Re}\,U$, where $\mathfrak{Re}\,U$ is the relation algebra of all binary relations on the set U. The main theorem is the REL Representation Theorem 5.4, in which every complete atomic algebra in REL is shown to be isomorphic to a relativization of a representable relation algebra, and hence, since every algebra in REL has a perfect extension in REL, every algebra in REL is isomorphic to a subalgebra of a relativization of some $\mathfrak{Re}\,U$. This characterizes REL.

The paper is organized as follows. In §1 we define the class REL by giving a finite equational axiomatization. We also prove some elementary facts about algebras in REL that are needed for later sections. In §2 we show that REL is closed under relativization to arbitrary elements. In §3 we study a construction for adjoining converses, and in §4 we study a construction for adjoining identity elements. In §5 we use the results of §3 and §4 to prove the REL Representation Theorem, making use of the WA Representation Theorem (Theorem 5.20 of Maddux [2]).

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§1. Axioms and elementary properties

Definition 1.1. NREL *is defined to be the class of all relation type algebras*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smile}, 1' \rangle$$

which satisfy Ax1 *through* Ax8 *below.* REL *is defined to be the class of all algebras* $\mathfrak{A} \in$ NREL *which also satisfy* Ax9. *Also, we define* $|\mathfrak{A}| = A$.

Ax1. $\mathfrak{Bl}\,\mathfrak{A} = \langle A, +, \cdot, -, 0, 1 \rangle$ *is a Boolean algebra.*

Ax2. $(x \cdot y^{\smile})^{\smile} = x^{\smile} \cdot y$.

Ax3. $(x + y);z = x;z + y;z$ *and* $x;(y + z) = x;y + x;z$.

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Ax4. $1;0 = 0$ *and* $0;1 = 0$.

Ax5. $x^{\smile};y \cdot z = x^{\smile};[y \cdot x^{\smile\smile};z] \cdot z$ *and* $x;y^{\smile} \cdot z = [x \cdot z;y^{\smile\smile}];y^{\smile} \cdot z$.

Ax6. $1';x \leq x$ *and* $x;1' \leq x$.

Ax7. $1';1' = 1'$.

Ax8. $-1^{\smile}; -1^{\smile} \cdot 1' = 0$.

Ax9. $(x \cdot 1');(y;z) = (x \cdot 1');(y;z)$, $(x;(y \cdot 1'));z = x;((y \cdot 1');z)$, *and* $(x;y);(z \cdot 1') = x;(y;(z \cdot 1'))$.

Remark 1.2. In general, we do *not* have $-(x^{\smile}) = (-x)^{\smile}$ in NREL or in REL. We use the convention that $-x^{\smile} = -(x^{\smile})$. Thus, for example, Ax8 says that $-(1^{\smile}); -(1^{\smile}) \cdot 1' = 0$.

See Definitions 1.1 and 1.11 in Jónsson-Tarski [1] for the definitions of normal, monotonic, additive, completely additive, conjugate, and self-conjugate.

Theorem 1.3. *Let* $\mathfrak{A}$ *be a relation type algebra satisfying* Ax1 *and* Ax2. *Let* $x, y \in |\mathfrak{A}|$.

(i) $^{\smile}$ *is normal, i.e.,* $0^{\smile} = 0$.
(ii) $^{\smile}$ *is self-conjugate, i.e.,* $x \cdot y^{\smile} = 0$ *iff* $x^{\smile} \cdot y = 0$.
(iii) $^{\smile}$ *is completely additive. In particular,* $(x+y)^{\smile} = x^{\smile} + y^{\smile}$, *and if* $x \leq y$ *then* $x^{\smile} \leq y^{\smile}$.
(iv) $x^{\smile\smile} = x \cdot 1^{\smile}$. *In particular,* $1^{\smile\smile} = 1^{\smile}$ *and* $x^{\smile\smile} \leq x$.
(v) $x^{\smile\smile\smile} = x^{\smile}$.
(vi) $(x \cdot y)^{\smile} = x^{\smile} \cdot y^{\smile}$.
(vii) $x^{\smile} \leq y^{\smile}$ *iff* $x^{\smile\smile} \leq y^{\smile\smile}$.
(viii) $(-x)^{\smile} = -x^{\smile} \cdot 1^{\smile}$.
(ix) $x = x^{\smile\smile} + x \cdot -1^{\smile}$.

Proof.

(i).
$$\begin{aligned} 0^{\smile} &= (0 \cdot 0^{\smile})^{\smile} \\ &= 0^{\smile} \cdot 0 && \text{by Ax2} \\ &= 0 \end{aligned}$$

(ii). This follows from (i) and Ax2 by Theorem 1.16 of Jónsson-Tarski [1].

(iii). This follows directly from (ii) by Theorem 1.14 of Jónsson-Tarski [1].

(iv).
$$\begin{aligned} x^{\smile\smile} &= (1 \cdot x^{\smile})^{\smile} \\ &= 1^{\smile} \cdot x && \text{by Ax2} \end{aligned}$$

(v).
$$\begin{aligned} x^{\smile\smile\smile} &= (1^{\smile} \cdot x)^{\smile} && \text{by (iv)} \\ &= 1 \cdot x^{\smile} && \text{by Ax2} \\ &= x^{\smile} \end{aligned}$$

(vi).
$$\begin{aligned}(x\cdot y)^{\smile} &= (x\cdot y)^{\smile}\cdot 1\\ &= (x\cdot y\cdot 1^{\smile})^{\smile} && \text{by Ax2}\\ &= (x^{\smile\smile}\cdot y^{\smile\smile})^{\smile} && \text{by (iv)}\\ &= x^{\smile\smile\smile}\cdot y^{\smile} && \text{by Ax2}\\ &= x^{\smile}\cdot y^{\smile} && \text{by (v)}\end{aligned}$$

(vii). If $x^{\smile}\le y^{\smile}$, then $x^{\smile\smile}\le y^{\smile\smile}$ by (iii). If $x^{\smile\smile}\le y^{\smile\smile}$, then $x^{\smile}=x^{\smile\smile\smile}\le y^{\smile\smile\smile}=y^{\smile}$ by (iii) and (v).

(viii). This follows from

$$\begin{aligned}(-x)^{\smile}\cdot x^{\smile} &= (-x\cdot x)^{\smile} && \text{by (vi)}\\ &= 0^{\smile}\\ &= 0 && \text{by (i)}\end{aligned}$$

and

$$\begin{aligned}(-x)^{\smile}+x^{\smile} &= (-x+x)^{\smile} && \text{by (iii)}\\ &= 1^{\smile}\end{aligned}$$

(ix).
$$\begin{aligned}x &= x\cdot 1^{\smile}+x\cdot -1^{\smile} && \text{by (iii)}\\ &= x^{\smile\smile}+x\cdot -1^{\smile} && \text{by (iv)}\quad\square\end{aligned}$$

Theorem 1.4. *Let* $\mathfrak{A}$ *be a relation type algebra satisfying* Ax1 *and* Ax2. *Let* $x_0, x_1\in|\mathfrak{A}|$. *Then the following statements are equivalent.*

(i) $x_0^{\smile\smile}=x_1^{\smile}$,
(ii) $x_1^{\smile\smile}=x_0^{\smile}$,
(iii) $x_0^{\smile}\le x_1$ *and* $x_1^{\smile}\le x_0$,
(iv) *for any* $z\in|\mathfrak{A}|$, $x_0\cdot z^{\smile}=0$ *iff* $x_1\cdot z^{\smile\smile}=0$.

Proof. (i)⇒(ii). Suppose $x_0^{\smile\smile}=x_1^{\smile}$. Then

$$\begin{aligned}x_0^{\smile} &= x_0^{\smile\smile\smile} && \text{by Th.1.3(v)}\\ &= x_1^{\smile\smile} && \text{by (i)}\end{aligned}$$

(ii)⇒(iii). Suppose $x_1^{\smile\smile}=x_0^{\smile}$. Then

$$\begin{aligned}x_0^{\smile} &= x_1^{\smile\smile} && \text{by (ii)}\\ &= x_1\cdot 1^{\smile} && \text{Th.1.3(iv)}\\ &\le x_1\end{aligned}$$

and

$$\begin{aligned} x_1^{\smile} &= x_1^{\smile\smile\smile} && \text{by Th.1.3(v)} \\ &= x_0^{\smile\smile} && \text{by (ii)} \\ &= x_0 \cdot 1^{\smile} && \text{by Th.1.3(iv)} \\ &\leq x_0 \end{aligned}$$

(iii)⇒(i). Suppose that $x_0^{\smile} \leq x_1$ and $x_1^{\smile} \leq x_0$. Then

$$\begin{aligned} x_0^{\smile\smile} &\leq x_1^{\smile} && \text{by (iii) and Th.1.3(iii)} \\ &= x_1^{\smile\smile\smile} && \text{by Th.1.3(v)} \\ &\leq x_0^{\smile\smile} && \text{by (iii) and Th.1.3(iii)} \end{aligned}$$

Thus $x_0^{\smile\smile} = x_1^{\smile}$ as desired.
(ii)⇒(iv). Let $z \in |\mathfrak{A}|$ and suppose $x_1^{\smile\smile} = x_0^{\smile}$. Then

$$\begin{aligned} x_0 \cdot z^{\smile} = 0 \quad &\text{iff} \quad x_0^{\smile} \cdot z = 0 && \text{by Th.1.3(ii)} \\ &\text{iff} \quad x_1^{\smile\smile} \cdot z = 0 && \text{by (ii)} \\ &\text{iff} \quad x_1^{\smile} \cdot z^{\smile} = 0 && \text{by Th.1.3(ii)} \\ &\text{iff} \quad x_1 \cdot z^{\smile\smile} = 0 && \text{by Th.1.3(ii)} \end{aligned}$$

(iv)⇒(ii). Suppose $x_0 \cdot z^{\smile} = 0$ iff $x_1 \cdot z^{\smile\smile} = 0$ for any $z \in |\mathfrak{A}|$. Then

$$\begin{aligned} x_0^{\smile} \cdot z = 0 \quad &\text{iff} \quad x_0 \cdot z^{\smile} = 0 && \text{by Th.1.3(ii)} \\ &\text{iff} \quad x_1 \cdot z^{\smile\smile} = 0 && \text{by (iv)} \\ &\text{iff} \quad x_1^{\smile} \cdot z^{\smile} = 0 && \text{by Th.1.3(ii)} \\ &\text{iff} \quad x_1^{\smile\smile} \cdot z = 0 && \text{by Th.1.3(ii)} \end{aligned}$$

Thus $x_0^{\smile} = x_1^{\smile\smile}$ as desired. □

Remark 1.5. Let $\langle A, +, \cdot, -, 0, 1, {}^{\smile}\rangle$ satisfy Ax1 and Ax2. If we define 1' $= 0$ and $x;y = 0$ for all $x, y \in A$, then the algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smile}, 1'\rangle$ is easily seen to satisfy all of the REL axioms so that $\mathfrak{A} \in$ REL.

Theorem 1.6. *Let $\mathfrak{A}$ be a relation type algebra and let $x, y \in |\mathfrak{A}|$.*

(i) *If $x^{\smile};$(-) is conjugate to $x^{\smile\smile};$(-) for all $x \in |\mathfrak{A}|$ then $\mathfrak{A}$ satisfies the first equation in* Ax5*.*
(ii) *If (-)$;y^{\smile}$ is conjugate to (-)$;y^{\smile\smile}$ for all $y \in |\mathfrak{A}|$ then $\mathfrak{A}$ satisfies the second equation in* Ax5*.*

Proof. Let $x \in |\mathfrak{A}|$, and suppose $x^{\smile};$(-) is conjugate to $x^{\smile\smile};$(-). Then

$$x^{\smile};y \cdot z \leq x^{\smile};[y \cdot x^{\smile\smile};z]$$

by Theorem 1.15 of Jónsson-Tarski [1]. But $x^{\smile};$(-) is completely additive, and hence monotonic, by Theorem 1.14 of Jónsson-Tarski [1], so

$$x^{\smile};[y \cdot x^{\smile\smile};z] \cdot z \leq x^{\smile};y \cdot z$$

Thus the first equation in Ax5 holds in $\mathfrak{A}$. The proof of the second equation is similar. □

Theorem 1.7. *Let* $\mathfrak{A}$ *be a relation type algebra satisfying* Ax1, Ax3–Ax5, *and let* $x, y \in |\mathfrak{A}|$.

(i) $x^{\smile};(\text{-})$ *is conjugate to* $x^{\smile\smile};(\text{-})$.
(ii) $(\text{-});y^{\smile}$ *is conjugate to* $(\text{-});y^{\smile\smile}$.
(iii) $x^{\smile};(\text{-})$ *is completely additive.*
(iv) $(\text{-});y^{\smile}$ *is completely additive.*

Proof. By Ax3 and Ax4 we have $x^{\smile};0 \leq 1;0 = 0$ and similarly $x^{\smile\smile};0 = 0$, so $x^{\smile};(\text{-})$ and $x^{\smile\smile};(\text{-})$ are normal. By Ax5 we have

$$x^{\smile};y \cdot z \leq x^{\smile};[y \cdot x^{\smile\smile};z].$$

By Ax5 and Theorem 1.3(v) we also have

$$x^{\smile\smile};y \cdot z = x^{\smile\smile};[y \cdot x^{\smile\smile\smile};z] \cdot z \leq x^{\smile\smile};[y \cdot x^{\smile};z].$$

Thus condition (iii) of Theorem 1.15 in Jónsson-Tarski [1] is satisfied. By Theorem 1.15 of Jónsson-Tarski [1], (i) holds. Part (iii) then follows by Theorem 1.14 of Jónsson-Tarski [1]. The proofs of (ii) and (iv) are similar. □

Theorem 1.8. *Let* $\mathfrak{A} \in \mathsf{NREL}$ *and let* $x, y \in |\mathfrak{A}|$.

(i) $(x \cdot 1');(y \cdot 1') = x \cdot y \cdot 1'$.
(ii) $(x \cdot 1')^{\smile} = x \cdot 1' = x^{\smile} \cdot 1'$, $1'^{\smile} = 1'$, *and* $1' \leq 1^{\smile}$.
(iii) $(x \cdot 1');y = y \cdot (x \cdot 1');1$. *In particular,* $1';y = y \cdot 1';1$.
(iv) $x;(y \cdot 1') = x \cdot 1;(y \cdot 1')$. *In particular,* $x;1' = x \cdot 1;1'$.
(v) $(x^{\smile\smile};y^{\smile\smile})^{\smile} = (y^{\smile};x^{\smile})^{\smile\smile}$.
(vi) $x;y \cdot 1' = y^{\smile};x^{\smile} \cdot 1'$.
(vii) $(x \cdot 1'); -((x \cdot 1');1) = 0$.
(viii) $-(1;(x \cdot 1'));(x \cdot 1') = 0$.
(ix) $((x \cdot 1');1)^{\smile} = (1;(x \cdot 1'))^{\smile\smile}$.

Proof. (i) If $u, v \leq 1'$, then $u;v \leq u;1' \cdot 1';v \leq u \cdot v$ by Ax3 and Ax6. Thus we have

(1) $$u;v \leq u \cdot v \text{ for any } u, v \leq 1'.$$

Next, we show that

(2) $$u;u = u \text{ whenever } u \leq 1'.$$

Let $v = 1' \cdot -u$. Then $v \leq 1'$, $u \cdot v = 0$, and $u + v = 1'$. Then

$$\begin{aligned} u + v &= 1' \\ &= 1';1' && \text{by Ax7} \\ &= (u+v);(u+v) \\ &= u;u + v;v + u;v + v;u && \text{by Ax3} \\ &\leq u;u + v \cdot v + u \cdot v + v \cdot u && \text{by (1)} \\ &= u;u + v \end{aligned}$$

so $u = (u+v) \cdot u = (u;u+v) \cdot u = u;u \cdot u$. But $u;u \leq 1\text{'};u \leq u$ by Ax3 and Ax6, so $u = u;u$ as desired. Now we calculate

$$\begin{aligned}
(1\text{'} \cdot x);(1\text{'} \cdot y) &= (1\text{'} \cdot x);(1\text{'} \cdot y \cdot x + 1\text{'} \cdot y \cdot -x) \\
&= (1\text{'} \cdot x);(1\text{'} \cdot y \cdot x) + (1\text{'} \cdot x);(1\text{'} \cdot y \cdot -x) && \text{by Ax3} \\
&= (1\text{'} \cdot x);(1\text{'} \cdot y \cdot x) && \text{by (1)} \\
&= (1\text{'} \cdot x \cdot y + 1\text{'} \cdot x \cdot -y);(1\text{'} \cdot y \cdot x) \\
&= (1\text{'} \cdot x \cdot y);(1\text{'} \cdot y \cdot x) + (1\text{'} \cdot x \cdot -y);(1\text{'} \cdot y \cdot x) && \text{by Ax3} \\
&= 1\text{'} \cdot x \cdot y && \text{by (1) and (2)}
\end{aligned}$$

(ii). We have, by Theorem 1.3(ix), $x \cdot 1\text{'} = (x \cdot 1\text{'})^{\smile\smile} + (x \cdot 1\text{'}) \cdot -1^{\smile}$, but

$$\begin{aligned}
1\text{'} \cdot -1^{\smile} &= (1\text{'} \cdot -1^{\smile});(1\text{'} \cdot -1^{\smile}) \cdot 1\text{'} && \text{by (i)} \\
&\leq -1^{\smile};-1^{\smile} \cdot 1\text{'} && \text{by Ax3} \\
&= 0 && \text{by Ax8}
\end{aligned}$$

so that

$$x \cdot 1\text{'} = (x \cdot 1\text{'})^{\smile\smile} \tag{3}$$

Now we calculate

$$\begin{aligned}
x \cdot 1\text{'} &= (x \cdot 1\text{'});(x \cdot 1\text{'}) \cdot 1\text{'} && \text{by (i)} \\
&\leq (x \cdot 1\text{'})^{\smile\smile};1\text{'} \cdot 1\text{'} && \text{by Ax3 and (3)} \\
&= (x \cdot 1\text{'})^{\smile\smile};[1\text{'} \cdot (x \cdot 1\text{'})^{\smile\smile\smile};1\text{'}] \cdot 1\text{'} && \text{by Ax5} \\
&= (x \cdot 1\text{'});[1\text{'} \cdot (x \cdot 1\text{'})^{\smile};1\text{'}] \cdot 1\text{'} && \text{by Th.1.3(v) and (3)} \\
&\leq 1\text{'};[(x \cdot 1\text{'})^{\smile};1\text{'}] && \text{by Ax3} \\
&\leq (x \cdot 1\text{'})^{\smile} && \text{by Ax6}
\end{aligned}$$

Thus $x \cdot 1\text{'} \leq (x \cdot 1\text{'})^{\smile}$, so that $(x \cdot 1\text{'})^{\smile} \leq (x \cdot 1\text{'})^{\smile\smile} = x \cdot 1\text{'}$ by Theorem 1.3(iii) and (3). Therefore $(x \cdot 1\text{'})^{\smile} = x \cdot 1\text{'}$ as desired. Setting $x = 1$ yields $1\text{'}^{\smile} = 1\text{'}$. Then, by Theorem 1.3(v), $(x \cdot 1\text{'})^{\smile} = x^{\smile} \cdot 1\text{'}^{\smile} = x^{\smile} \cdot 1\text{'}$. Finally, $1\text{'} = 1\text{'}^{\smile} \leq 1^{\smile}$ by Theorem 1.3(iii).

(iii).

$$\begin{aligned}
y \cdot (x \cdot 1\text{'});1 &= y \cdot (x \cdot 1\text{'})^{\smile};1 && \text{by (ii)} \\
&= y \cdot (x \cdot 1\text{'})^{\smile};[1 \cdot (x \cdot 1\text{'})^{\smile\smile};y] && \text{by Ax5} \\
&= y \cdot (x \cdot 1\text{'});[(x \cdot 1\text{'});y] && \text{by (ii)} \\
&\leq y \cdot 1\text{'};[(x \cdot 1\text{'});y] && \text{by Ax3} \\
&\leq (x \cdot 1\text{'});y && \text{by Ax6}
\end{aligned}$$

For the other direction, we have

$$\begin{aligned}
(x \cdot 1\text{'});y &= (x \cdot 1\text{'});y \cdot (x \cdot 1\text{'});y \\
&\leq 1\text{'};y \cdot (x \cdot 1\text{'});1 && \text{by Ax3} \\
&\leq y \cdot (x \cdot 1\text{'});1 && \text{by Ax6}
\end{aligned}$$

(iv). Similar to (iii).

(v). First we note that

$$\begin{aligned}
(x^{\smile\smile};y^{\smile\smile})^{\smile}\cdot z=0 \quad &\text{iff}\quad x^{\smile\smile};y^{\smile\smile}\cdot z^{\smile}=0 && \text{by Th.1.3(ii)}\\
&\text{iff}\quad x^{\smile};z^{\smile}\cdot y^{\smile\smile}=0 && \text{by Th.1.7(i)}\\
&\text{iff}\quad y^{\smile\smile};z^{\smile\smile}\cdot x^{\smile}=0 && \text{by Th.1.7(ii)}\\
&\text{iff}\quad y^{\smile};x^{\smile}\cdot z^{\smile\smile}=0 && \text{by Th.1.7(i)}\\
&\text{iff}\quad (y^{\smile};x^{\smile})^{\smile}\cdot z^{\smile}=0 && \text{by Th.1.3(ii)}\\
&\text{iff}\quad (y^{\smile};x^{\smile})^{\smile\smile}\cdot z=0 && \text{by Th.1.3(ii)}
\end{aligned}$$

Setting first $z=-(x^{\smile\smile};y^{\smile\smile})^{\smile}$ and then $z=-(y^{\smile};x^{\smile})^{\smile\smile}$, we obtain $(x^{\smile\smile};y^{\smile\smile})^{\smile}=(y^{\smile};x^{\smile})^{\smile\smile}$, as desired.

(vi). First, we have

$$\begin{aligned}
&x;y\cdot 1'\\
&=(x^{\smile\smile}+x\cdot -1^{\smile});(y^{\smile\smile}+y\cdot -1^{\smile})\cdot 1' && \text{by Th.1.3(ix)}\\
&\le x^{\smile\smile};y^{\smile\smile}\cdot 1'+1^{\smile};-1^{\smile}\cdot 1'+-1^{\smile};1^{\smile}\cdot 1'+-1^{\smile};-1^{\smile}\cdot 1' && \text{by Ax3 and Th.1.3(iii)}\\
&=x^{\smile\smile};y^{\smile\smile}\cdot 1'+1^{\smile};[-1^{\smile}\cdot 1^{\smile\smile};1']\cdot 1'+[-1^{\smile}\cdot 1';1^{\smile\smile}];1^{\smile}\cdot 1' && \text{by Ax5 and Ax8}\\
&\le x^{\smile\smile};y^{\smile\smile}\cdot 1'+1^{\smile};[-1^{\smile}\cdot 1^{\smile}]\cdot 1'+[-1^{\smile}\cdot 1^{\smile}];1^{\smile}\cdot 1' && \text{by Ax6, Ax3, Th.1.3(iv)}\\
&=x^{\smile\smile};y^{\smile\smile}\cdot 1' && \text{by Ax4}\\
&\le x;y\cdot 1' && \text{by Th.1.3(iv)}
\end{aligned}$$

Thus

$$x;y\cdot 1'=x^{\smile\smile};y^{\smile\smile}\cdot 1',$$

so

$$\begin{aligned}
x;y\cdot 1' &=(x^{\smile\smile};y^{\smile\smile})^{\smile}\cdot 1' && \text{by (ii)}\\
&=(y^{\smile};x^{\smile})^{\smile\smile}\cdot 1' && \text{by (v)}\\
&=(y^{\smile};x^{\smile})^{\smile}\cdot 1' && \text{by (ii)}\\
&=y^{\smile};x^{\smile}\cdot 1' && \text{by (ii)}
\end{aligned}$$

(vii). Use (iii) with $y=-((x\cdot 1');1)$.

(viii). Use (iv) with x and y replaced by $-(1;(x\cdot 1'))$ and x, respectively.

(ix). We have

$$((x\cdot 1');1)^{\smile}=((x\cdot 1');1^{\smile})^{\smile}+((x\cdot 1');-1^{\smile})^{\smile}$$

by Ax3 and Th.1.3(iii). But

$$\begin{aligned}
((x\cdot 1');-1^{\smile})^{\smile} &\le (1';-1^{\smile})^{\smile} && \text{by Ax3, Th.1.3(iii)}\\
&\le (-1^{\smile})^{\smile} && \text{by Ax6 and Th.1.3(iii)}\\
&=-1^{\smile\smile}\cdot 1^{\smile} && \text{by Th.1.3(viii)}\\
&=-1^{\smile}\cdot 1^{\smile} && \text{by Th.1.3(iv)}\\
&=0
\end{aligned}$$

so

$$\begin{aligned}((x\cdot 1');1)^{\smile} &= ((x\cdot 1');1^{\smile})^{\smile} \\ &= ((x\cdot 1')^{\smile\smile};1^{\smile\smile})^{\smile} && \text{by (ii) and Th.1.3(iv)} \\ &= (1^{\smile};(x\cdot 1')^{\smile})^{\smile\smile} && \text{by (v)} \\ &\leq 1^{\smile};(x\cdot 1')^{\smile} && \text{by Th.1.3(iv)} \\ &= 1;(x\cdot 1') && \text{by Ax3 and (ii)}\end{aligned}$$

Thus $((x\cdot 1');1)^{\smile} \leq 1;(x\cdot 1')$. Similarly we have $(1;(x\cdot 1'))^{\smile} \leq (x\cdot 1');1$ so that, by Theorem 1.4, $((x\cdot 1');1)^{\smile} = (1;(x\cdot 1'))^{\smile\smile}$ as desired. □

Theorem 1.9. *Let $\mathfrak{A} \in \mathsf{NREL}$ and let $u \in |\mathfrak{A}|$ with $u \leq 1'$.*

(i) *$u;1;1 \leq u;1$ iff $u;x;y \leq u;(x;y)$ for all $x, y \in |\mathfrak{A}|$.*
(ii) *$u;(-(u;1);1) = 0$ iff $u;(x;y) \leq u;x;y$ for all $x, y \in |\mathfrak{A}|$.*
(iii) *$1;u;-(u;1) = 0$ iff $x;u;y \leq x;(u;y)$ for all $x, y \in |\mathfrak{A}|$.*
(iv) *$-(1;u);(u;1) = 0$ iff $x;(u;y) \leq x;u;y$ for all $x, y \in |\mathfrak{A}|$.*
(v) *$1;-(1;u);u = 0$ iff $x;y;u \leq x;(y;u)$ for all $x, y \in |\mathfrak{A}|$.*
(vi) *$1;(1;u) \leq 1;u$ iff $x;(y;u) \leq x;y;u$ for all $x, y \in |\mathfrak{A}|$.*

Proof. (i). Suppose that $u;x;y \leq u;(x;y)$ for all $x, y \in |\mathfrak{A}|$. Then $u;1;1 \leq u;(1;1) \leq u;1$ by Ax3. For the other direction suppose $u;1;1 \leq u;1$ and let $x, y \in |\mathfrak{A}|$. Then

$$\begin{aligned}u;x;y &= u;x;y\cdot u;x;y \\ &\leq 1';x;y\cdot u;1;1 && \text{by Ax3} \\ &\leq x;y\cdot u;1;1 && \text{by Ax6} \\ &\leq x;y\cdot u;1 && \text{by hyp.} \\ &= u;(x;y) && \text{by Th.1.8(iii)}\end{aligned}$$

(ii). Suppose $u;(x;y) \leq u;x;y$ for all $x, y \in |\mathfrak{A}|$. Then

$$\begin{aligned}u;(-(u;1);1) &\leq u;-(u;1);1 && \text{by hyp.} \\ &= 0;1 && \text{by Th.1.8(vii)} \\ &= 0 && \text{by Ax4}\end{aligned}$$

For the other direction suppose $u;(-(u;1);1) = 0$ and let $x, y \in |\mathfrak{A}|$. Then

$$\begin{aligned}u;(x;y) &= u;((x\cdot u;1 + x\cdot -(u;1));y) \\ &= u;((x\cdot u;1);y) + u;((x\cdot -(u;1));y) && \text{by Ax3} \\ &\leq 1';(u;x;y) + u;(-(u;1);1) && \text{by Th.1.8(iii) and Ax3} \\ &= 1';(u;x;y) && \text{by hyp.} \\ &\leq u;x;y && \text{by Ax6}\end{aligned}$$

(iii). Suppose $x;u;y \leq x;(u;y)$ for all $x, y \in |\mathfrak{A}|$. Then

$$\begin{aligned}1;u;-(u;1) &\leq 1;(u;-(u;1)) && \text{by hyp.} \\ &= 1;0 && \text{by Th.1.8(vii)} \\ &= 0 && \text{by Ax4}\end{aligned}$$

For the other direction suppose $1;u;-(u;1)=0$ and let $x,y\in|\mathfrak{A}|$. Then

$$\begin{aligned} x;u;y &= x;u;(y\cdot u;1+y\cdot -(u;1)) \\ &= x;u;(y\cdot u;1)+x;u;(y\cdot -(u;1)) && \text{by Ax3} \\ &\le x;1';(u;y)+1;u;-(u;1) && \text{by Ax3 and Th.1.8(iii)} \\ &= x;1';(u;y) && \text{by hyp.} \\ &\le x;(u;y) && \text{by Ax6 and Ax3} \end{aligned}$$

(iv). Similar to (iii).
(v). Similar to (ii).
(vi). Similar to (i). □

Theorem 1.10. *Let $\mathfrak{A}\in\mathsf{NREL}$ satisfy $1^\smile=1$ and suppose $u;1;1\le u;1$ for every $u\in|\mathfrak{A}|$ with $u\le 1'$. Then $\mathfrak{A}\in\mathsf{REL}$.*

Proof. We first show that $(x;y)^\smile = y^\smile;x^\smile$ for any $x,y\in|\mathfrak{A}|$.

$$\begin{aligned} (x;y)^\smile &= (x^{\smile\smile};y^{\smile\smile})^\smile && \text{by Th.1.3(iv) and } 1^\smile=1 \\ &= (y^\smile;x^\smile)^{\smile\smile} && \text{by Th.1.8(v)} \\ &= y^\smile;x^\smile && \text{by Th.1.3(iv) and } 1^\smile=1 \end{aligned} \tag{1}$$

By Theorem 1.9, to prove Ax9 we need only prove the left hand side of the equivalences (i) through (vi) of Theorem 1.9 for any $u\in|\mathfrak{A}|$ with $u\le 1'$.
(i). We have $u;1;1\le u;1$ by hypothesis.

(vi).
$$\begin{aligned} 1;(1;u) &= 1^\smile;(1^\smile;u^\smile) && \text{by } 1=1^\smile \text{ and Th.1.8(ii)} \\ &= 1^\smile;(u;1)^\smile && \text{by (1)} \\ &= (u;1;1)^\smile && \text{by (1)} \\ &\le (u;1)^\smile && \text{by (i) and Th.1.3(iii)} \\ &= 1^\smile;u^\smile && \text{by (1)} \\ &= 1;u && \text{by } 1^\smile=1 \text{ and Th.1.8(ii)} \end{aligned}$$

(ii).
$$\begin{aligned} u;(-(u;1);1)=0 \quad &\text{iff}\quad u;1\cdot -(u;1);1=0 && \text{by Th.1.8(iii)} \\ &\text{iff}\quad -(u;1)\cdot u;1;1=0 && \text{by Th.1.7(ii) and hyp.} \end{aligned}$$

But, by (i), $-(u;1)\cdot u;1;1\le -(u;1)\cdot u;1=0$.
(iii). First note that $(1;u)^\smile = u^\smile;1^\smile = u;1$ by (1), hypothesis, and Theorem 1.8(ii). Likewise $(u;1)^\smile = 1;u$. Then we have

$$\begin{aligned} 1;u;-(u;1)=0 \quad &\text{iff}\quad (u;1)^\smile;-(u;1)\cdot 1=0 \\ &\text{iff}\quad (u;1)^{\smile\smile};1\cdot -(u;1)=0 && \text{by Th.1.7(i)} \\ &\text{iff}\quad u;1;1\cdot -(u;1)=0 \end{aligned}$$

But $u;1;1 \cdot -(u;1) \leq u;1 \cdot -(u;1) = 0$ by (i).
(iv). Similar to (iii).
(v). Similar to (ii). $\square$

§2. Relativization

Definition 2.1. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smallsmile}, 1' \rangle$ *be a relation type algebra satisfying* Ax1 *and let* $r \in |\mathfrak{A}|$. *Then we define a relation type algebra*

$$\mathfrak{Rl}_r\mathfrak{A} = \langle \{x \in A : x \leq r\}, +, \cdot, {}_r-, 0, r, ;^r, {}^{\smallsmile r}, 1'_r \rangle$$

where ${}_r-x = -x \cdot r$, $x;^r y = x;y \cdot r$, $x^{\smallsmile r} = x^{\smallsmile} \cdot r$, *and* $1'_r = 1' \cdot r$. $\mathfrak{Rl}_r\mathfrak{A}$ *is called the* **relativization of $\mathfrak{A}$ to** r. *Clearly* $\mathfrak{Rl}_r\mathfrak{A}$ *satisfies* Ax1.

Theorem 2.2. *Let* $\mathfrak{A} \in \mathsf{NREL}$ *and let* $r \in |\mathfrak{A}|$. *Then* $\mathfrak{Rl}_r\mathfrak{A} \in \mathsf{NREL}$.

Proof. Ax1. As was noted in Definition 2.1, this axiom is obvious.
Ax2. Let $x, y \in |\mathfrak{A}|$ with $x, y \leq r$. Then

$$\begin{aligned}
(x \cdot y^{\smallsmile r})^{\smallsmile r} &= (x \cdot y^{\smallsmile} \cdot r)^{\smallsmile} \cdot r && \text{by Def.2.1} \\
&= (x \cdot y^{\smallsmile})^{\smallsmile} \cdot r && \text{by } x \leq r \\
&= x^{\smallsmile} \cdot y \cdot r && \text{by Ax2} \\
&= x^{\smallsmile} \cdot r \cdot y && \\
&= x^{\smallsmile r} \cdot y && \text{by Def.2.1}
\end{aligned}$$

Ax3. Let $x, y, z \in |\mathfrak{A}|$ with $x, y, z \leq r$. Then

$$\begin{aligned}
(x + y);^r z &= (x + y);z \cdot r && \text{by Def.2.1} \\
&= (x;z + y;z) \cdot r && \text{by Ax3} \\
&= x;z \cdot r + y;z \cdot r && \\
&= x;^r z + y;^r z && \text{by Def.2.1}
\end{aligned}$$

Similarly, $x;^r (y + z) = x;^r y + x;^r z$.

Ax4.

$$\begin{aligned}
r;^r 0 &= r;0 \cdot r && \text{by Def.2.1} \\
&\leq 1;0 && \text{by Ax3} \\
&= 0 && \text{by Ax4}
\end{aligned}$$

Similarly $0;^r r = 0$.
Ax5. Let $x, y, z \in |\mathfrak{A}|$ with $x, y, z \leq r$. Then

$$\begin{aligned}
x^{\smallsmile r};^r y \cdot z &= (x^{\smallsmile} \cdot r);y \cdot r \cdot z && \text{by Def.2.1} \\
&= (x \cdot r^{\smallsmile})^{\smallsmile};y \cdot z && \text{by Ax2} \\
&= (x \cdot r^{\smallsmile})^{\smallsmile};[y \cdot (x \cdot r^{\smallsmile})^{\smallsmile\smallsmile};z] \cdot z && \text{by Ax5} \\
&= (x^{\smallsmile} \cdot r);[y \cdot (x^{\smallsmile} \cdot r)^{\smallsmile};z] \cdot z && \text{by Ax2} \\
&= (x^{\smallsmile} \cdot r);[y \cdot (x \cdot r^{\smallsmile});z] \cdot z && \text{by Ax2} \\
&= (x^{\smallsmile} \cdot r);[y \cdot (x \cdot r^{\smallsmile} \cdot r);z \cdot r] \cdot r \cdot z && \\
&= (x^{\smallsmile} \cdot r);[y \cdot ((x^{\smallsmile} \cdot r)^{\smallsmile} \cdot r);z \cdot r] \cdot r \cdot z && \text{by Ax2} \\
&= x^{\smallsmile r};^r [y \cdot x^{\smallsmile r \smallsmile r};^r z] \cdot z && \text{by Def.2.1}
\end{aligned}$$

Similarly $x\,;^r\, y^{\smile r} \cdot z = [x \cdot z\,;^r\, y^{\smile r\,\smile r}]\,;^r\, y^{\smile r} \cdot z$.

Ax6. Let $x \in |\mathfrak{A}|$ with $x \leq r$. Then

$$\begin{aligned} 1\text{'}_r\,;^r\, x &= (1\text{'} \cdot r)\,;x \cdot r && \text{by Def.2.1} \\ &\leq 1\text{'}\,;x && \text{by Ax3} \\ &\leq x && \text{by Ax6} \end{aligned}$$

Similarly, $x\,;^r\, 1\text{'}_r \leq x$.

Ax7.
$$\begin{aligned} 1\text{'}_r\,;^r\, 1\text{'}_r &= (1\text{'} \cdot r)\,;(1\text{'} \cdot r) \cdot r && \text{by Def.2.1} \\ &= (1\text{'} \cdot r) \cdot r && \text{by Th.1.8(i)} \\ &= 1\text{'}_r && \text{by Def.2.1} \end{aligned}$$

Ax8. ${}_r{-}r^{\smile r}\,;^r\, {}_r{-}r^{\smile r} \cdot 1\text{'}_r$

$$\begin{aligned} &= (r \cdot -(r^{\smile} \cdot r))\,;(r \cdot -(r^{\smile} \cdot r)) \cdot r \cdot 1\text{'} \cdot r && \text{by Def. 2.1} \\ &= (r \cdot (-r^{\smile} + -r))\,;(r \cdot (-r^{\smile} + -r)) \cdot 1\text{'} \cdot r && \\ &\leq (r \cdot -r^{\smile})\,;(r \cdot -r^{\smile}) \cdot 1\text{'} && \text{by Ax3} \\ &= (r \cdot -r^{\smile})^{\smile}\,;(r \cdot -r^{\smile})^{\smile} \cdot 1\text{'} && \text{by Th.1.8(vi)} \\ &= (r \cdot -r^{\smile})^{\smile}\,;[(r \cdot -r^{\smile})^{\smile} \cdot (r \cdot -r^{\smile})^{\smile\smile}\,;1\text{'}] \cdot 1\text{'} && \text{by Ax5} \\ &\leq 1\,;[(r \cdot -r^{\smile})^{\smile} \cdot (r \cdot -r^{\smile})^{\smile\smile}] && \text{by Ax3 and Ax6} \\ &\leq 1\,;[(r \cdot -r^{\smile})^{\smile} \cdot r \cdot -r^{\smile}] && \text{by Ax3 and Th.1.3(iv)} \\ &= 1\,;[r^{\smile} \cdot (-r^{\smile})^{\smile} \cdot r \cdot -r^{\smile}] && \text{by Th.1.3(vi)} \\ &= 1\,;0 && \\ &= 0 && \text{by Ax4} \quad \square \end{aligned}$$

Theorem 2.3. *Let* $\mathfrak{A} \in \mathsf{REL}$ *and let* $r \in |\mathfrak{A}|$. *Then* $\mathfrak{Rl}_r\mathfrak{A} \in \mathsf{REL}$.

Proof. By Theorem 2.2, we need only show that Ax9 holds in $\mathfrak{Rl}_r\mathfrak{A}$. Let $u, x, y \in |\mathfrak{A}|$ with $x, y \leq r$ and $u \leq 1\text{'}_r = 1\text{'} \cdot r$. Then

$$\begin{aligned} u;^r\, x;^r\, y &= (u;x \cdot r);y \cdot r && \text{by Def.2.1} \\ &= (u;1 \cdot x \cdot r);y \cdot r && \text{by Th.1.8(iii)} \\ &= (u;1 \cdot x);y \cdot r && \\ &= u;x;y \cdot r && \text{by Th.1.8(iii)} \\ &= u;(x;y) \cdot r && \text{by Ax9} \\ &= u;1 \cdot x;y \cdot r && \text{by Th.1.8(iii)} \\ &= u;1 \cdot (x;y \cdot r) \cdot r && \\ &= u;(x;y \cdot r) \cdot r && \text{by Th.1.8(iii)} \\ &= u;^r\, (x;^r\, y) && \text{by Def.2.1} \end{aligned}$$

Similarly $x ;^r y ;^r u = x ;^r (y ;^r u)$. Also,

$$
\begin{aligned}
x ;^r u ;^r y &= (x ; u \cdot r) ; y \cdot r && \text{by Def.2.1} \\
&= (1 ; u \cdot x \cdot r) ; y \cdot r && \text{by Th.1.8(iv)} \\
&= (1 ; u \cdot x) ; y \cdot r && \\
&= x ; u ; y \cdot r && \text{by Th.1.8(iv)} \\
&= x ; (u ; y) \cdot r && \text{by Ax9} \\
&= x ; (u ; 1 \cdot y) \cdot r && \text{by Th.1.8(iii)} \\
&= x ; (u ; 1 \cdot y \cdot r) \cdot r && \\
&= x ; (u ; y \cdot r) \cdot r && \text{by Th.1.8(iii)} \\
&= x ;^r (u ;^r y) && \text{by Def.2.1} \quad \square
\end{aligned}
$$

§3. Adjoining Converses

Theorem 3.1. *Let $\mathfrak{A} \in \mathsf{NREL}$ and let $r \in |\mathfrak{A}|$. Then for any $x \in |\mathfrak{A}|$ with $x \leq r^{\smile\smile}$, both of the maps*

$$
\begin{aligned}
x ; (\text{-}) \cdot r &: |\mathfrak{Rl}_r\mathfrak{A}| \to |\mathfrak{Rl}_r\mathfrak{A}| \\
(\text{-}) ; x \cdot r &: |\mathfrak{Rl}_r\mathfrak{A}| \to |\mathfrak{Rl}_r\mathfrak{A}|
\end{aligned}
$$

have conjugate maps from $|\mathfrak{Rl}_r\mathfrak{A}|$ to $|\mathfrak{Rl}_r\mathfrak{A}|$. Also, for any $x \in |\mathfrak{A}|$ with $x \leq r$, both of the maps

$$
\begin{aligned}
x ; (\text{-}) \cdot r^{\smile\smile} &: |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}| \to |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}| \\
(\text{-}) ; x \cdot r^{\smile\smile} &: |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}| \to |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|
\end{aligned}
$$

have conjugate maps from $|\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|$ to $|\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|$.

proof. Let $x \leq r^{\smile\smile} = r \cdot 1^{\smile}$ and define $\sigma, \tau : |\mathfrak{Rl}_r\mathfrak{A}| \to |\mathfrak{Rl}_r\mathfrak{A}|$ by $\sigma(y) = x^{\smile} ; y \cdot r$ and $\tau(y) = y ; x^{\smile} \cdot r$ for any $y \in |\mathfrak{Rl}_r\mathfrak{A}|$. Then for any $z \in |\mathfrak{Rl}_r\mathfrak{A}|$, we have

$$
\begin{aligned}
\sigma(y) \cdot z = 0 \quad &\text{iff} \quad x^{\smile} ; y \cdot r \cdot z = 0 && \text{by definition} \\
&\text{iff} \quad x^{\smile} ; y \cdot z = 0 && \text{since } z \leq r \\
&\text{iff} \quad x^{\smile\smile} ; z \cdot y = 0 && \text{by Th.1.7(i)} \\
&\text{iff} \quad (x \cdot 1^{\smile}) ; z \cdot y = 0 && \text{by Th.1.3(iv)} \\
&\text{iff} \quad x ; z \cdot y = 0 && \text{since } x \leq 1^{\smile} \\
&\text{iff} \quad x ; z \cdot r \cdot y = 0 && \text{since } y \leq r
\end{aligned}
$$

Thus σ is conjugate to $x ; (\text{-}) \cdot r$. Similarly, τ is conjugate to $(\text{-}) ; x \cdot r$.

Now, let $x \leq r$ and define $\sigma, \tau : |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}| \to |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|$ by $\sigma(y) = (y^{\smile};x)^{\smile} \cdot r^{\smile\smile}$ and $\tau(y) = (x;y^{\smile})^{\smile} \cdot r^{\smile\smile}$ for any $y \in |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|$. Then for any $z \in |\mathfrak{Rl}_{r^{\smile\smile}}\mathfrak{A}|$, we have

$$
\begin{array}{rlll}
\sigma(y) \cdot z = 0 & \text{iff} & (y^{\smile};x)^{\smile} \cdot r^{\smile\smile} \cdot z = 0 & \text{by definition} \\
 & \text{iff} & (y^{\smile};x)^{\smile} \cdot z = 0 & \text{since } z \leq r^{\smile\smile} \\
 & \text{iff} & y^{\smile};x \cdot z^{\smile} = 0 & \text{by Th.1.3(ii)} \\
 & \text{iff} & y^{\smile\smile};z^{\smile} \cdot x = 0 & \text{by Th.1.7(i)} \\
 & \text{iff} & x;z^{\smile\smile} \cdot y^{\smile\smile} & \text{by Th.1.7(ii)} \\
 & \text{iff} & x;(z \cdot 1^{\smile}) \cdot y \cdot 1^{\smile} = 0 & \text{by Th.1.3(iv)} \\
 & \text{iff} & x;z \cdot y = 0 & \text{since } y, z \leq r^{\smile\smile} \leq 1^{\smile} \\
 & \text{iff} & x;z \cdot r^{\smile\smile} \cdot y = 0 & \text{since } y \leq r^{\smile\smile}
\end{array}
$$

Thus σ is conjugate to $x;(\text{-}) \cdot r^{\smile\smile}$. Similarly, τ is conjugate to $(\text{-});x \cdot r^{\smile\smile}$. □

Remark 3.2. Let $\mathfrak{B}$ be a relation type algebra and let $s \in |\mathfrak{B}|$ satisfy $1^{\smile} \leq s$. Theorems 2.2 and 3.1 give necessary conditions for $\mathfrak{B}$ to be equal to $\mathfrak{Rl}_r\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{NREL}$ and some $r \in |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies $r^{\smile\smile} = s$. In particular, we must have $\mathfrak{B} \in \mathsf{NREL}$ and $s \in |\mathfrak{B}|$ must satisfy $(*)$ below.

$(*)$ For any $x \in |\mathfrak{B}|$ with $x \leq s$, both of the maps $x;(\text{-}) : |\mathfrak{B}| \to |\mathfrak{B}|$ and $(\text{-});x : |\mathfrak{B}| \to |\mathfrak{B}|$ have conjugate maps from $|\mathfrak{B}|$ to $|\mathfrak{B}|$. Also, for any $x \in |\mathfrak{B}|$, both of the maps $x;(\text{-}) \cdot s : |\mathfrak{Rl}_s\mathfrak{B}| \to |\mathfrak{Rl}_s\mathfrak{B}|$ and $(\text{-});x \cdot s : |\mathfrak{Rl}_s\mathfrak{B}| \to |\mathfrak{Rl}_s\mathfrak{B}|$ have conjugate maps from $|\mathfrak{Rl}_s\mathfrak{B}|$ to $|\mathfrak{Rl}_s\mathfrak{B}|$.

Similarly, if $\mathfrak{B}$ is a relation type algebra and $s \in |\mathfrak{B}|$ satisfies $1^{\smile} \leq s$, then Theorems 2.3 and 3.1 give necessary conditions for $\mathfrak{B}$ to be equal to $\mathfrak{Rl}_r\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{REL}$ and some $r \in |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies $r^{\smile\smile} = s$. In particular, we must have $\mathfrak{B} \in \mathsf{REL}$ and $s \in |\mathfrak{B}|$ must satisfy $(*)$ above.

The goal of this section is to show that the necessary conditions referred to here are also sufficient.

According to Definition 2.13 of Jónsson-Tarski [1], a relation type algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smile}, 1\text{'} \rangle$$

satisfying Ax1 is complete if $\langle A, +, \cdot, -, 0, 1 \rangle$ is a complete Boolean algebra and the operations ; and $^{\smile}$ are completely additive.

Theorem 3.3. *Let $\mathfrak{B} \in \mathsf{NREL}$ be complete. Then $\mathfrak{B}$ satisfies condition $(*)$ of Remark 3.2 for any $s \in |\mathfrak{B}|$ with $1^{\smile} \leq s$.*

proof. Use Theorem 1.14 of Jónsson-Tarski [1]. Note that the maps in question are all normal and completely additive. □

Definition 3.4. *Let $\mathfrak{B} \in \mathsf{NREL}$ and $s \in |\mathfrak{B}|$ with $1^{\smile} \leq s$ satisfy $(*)$ of Remark 3.2. Then for every $x \in |\mathfrak{B}|$ with $x \leq s$ we define maps $l_x^{\smile} : |\mathfrak{B}| \to |\mathfrak{B}|$ and $r_x^{\smile} : |\mathfrak{B}| \to |\mathfrak{B}|$ to be the conjugates of $x;(\text{-})$ and $(\text{-});x$, respectively. Also, for every $x \in |\mathfrak{B}|$ we define maps $\alpha_x : |\mathfrak{Rl}_s\mathfrak{B}| \to |\mathfrak{Rl}_s\mathfrak{B}|$ and $\beta_x : |\mathfrak{Rl}_s\mathfrak{B}| \to |\mathfrak{Rl}_s\mathfrak{B}|$ to be the conjugates of $x;(\text{-}) \cdot s$ and $(\text{-});x \cdot s$, respectively.*

Remark 3.5. The maps $l_x^{\smile}$, $r_x^{\smile}$ for $x \leq s$ and α_x, β_x for arbitrary $x \in |\mathfrak{B}|$ in Definition 3.4 are well defined by Theorem 1.13 of Jónsson-Tarski [1].

Theorem 3.6. *Let $\mathfrak{B} \in \mathsf{NREL}$ and $s \in |\mathfrak{B}|$ with $1^{\smile} \leq s$ satisfy $(*)$ of Remark 3.2 and let $x, y, z, w \in |\mathfrak{B}|$ with $x, y \leq s$. Then*

(i) $l^{\smile}_{x+y}(z) = l^{\smile}_x(z) + l^{\smile}_y(z)$.
(ii) $r^{\smile}_{x+y}(z) = r^{\smile}_x(z) + r^{\smile}_y(z)$.
(iii) $\alpha_{z+w}(x) = \alpha_z(x) + \alpha_w(x)$.
(iv) $\beta_{z+w}(x) = \beta_z(x) + \beta_w(x)$.
(v) $l^{\smile}_0(z) = 0$.
(vi) $r^{\smile}_0(z) = 0$.
(vii) $\alpha_0(z) = 0$.
(viii) $\beta_0(z) = 0$.
(ix) $l^{\smile}_x$ *is normal and completely additive.*
(x) $r^{\smile}_x$ *is normal and completely additive.*
(xi) α_z *is normal and completely additive.*
(xii) β_z *is normal and completely additive.*
(xiii) *If* $x \leq 1^{\smile}$, *then* $l^{\smile}_x(z) = x^{\smile};z$.
(xiv) *If* $x \leq 1^{\smile}$, *then* $r^{\smile}_x(z) = z;x^{\smile}$.
(xv) *If* $z \leq 1^{\smile}$, *then* $\alpha_z(x) = z^{\smile};x \cdot s$.
(xvi) *If* $z \leq 1^{\smile}$, *then* $\beta_z(x) = x;z^{\smile} \cdot s$.

proof.

(i). $l^{\smile}_{x+y}(z) \cdot w = 0$ iff $(x+y);w \cdot z = 0$ by Def.3.4
iff $x;w \cdot z = 0$ and $y;w \cdot z = 0$ by Ax3
iff $l^{\smile}_x(z) \cdot w = 0$ and $l^{\smile}_y(z) \cdot w = 0$ by Def.3.4
iff $(l^{\smile}_x(z) + l^{\smile}_y(z)) \cdot w = 0$

(ii). Similar to (i).

(iii) $\alpha_{z+w}(x) \cdot y = 0$ iff $(z+w);y \cdot s \cdot x = 0$ by Def.3.4
iff $z;y \cdot s \cdot x = 0$ and $w;y \cdot s \cdot x = 0$ by Ax3
iff $\alpha_z(x) \cdot y = 0$ and $\alpha_w(x) \cdot y = 0$ by Def.3.4
iff $(\alpha_z(x) + \alpha_w(x)) \cdot y = 0$

(iv). Similar to (iii).

(v). $l^{\smile}_0(z) = 0$ iff $l^{\smile}_0(z) \cdot 1 = 0$
iff $0;1 \cdot z = 0$ by Def.3.4
iff $0 \cdot z = 0$ by Ax4

(vi). Similar to (v).

(vii). $\alpha_0(x) = 0$ iff $\alpha_0(x) \cdot 1 = 0$
iff $0;1 \cdot s \cdot x = 0$ by Def.3.4
iff $0 \cdot s \cdot x = 0$ by Ax4

(viii). Similar to (vii).

(ix)–(xii). Use Definition 3.4 and Theorem 1.14 of Jónsson-Tarski [1].

(xiii). If $x \le 1^{\smile}$, then $x^{\smile\smile} = x$ by Theorem 1.3(iv). The conclusion follows from Theorem 1.7(i), Definition 3.4, and Remark 3.5.

(xiv). Similar to (xiii).

(xv). If $z \le 1^{\smile}$, then $z^{\smile\smile} = z$ by Theorem 1.3(iv). Thus, we have

$$\begin{array}{rlll} \alpha_z(x) \cdot y = 0 & \text{iff} & z;y \cdot s \cdot x = 0 & \text{by Def.3.4} \\ & \text{iff} & z^{\smile\smile};y \cdot x = 0 & \text{since } x \le s \\ & \text{iff} & z^{\smile};x \cdot y = 0 & \text{by Th.1.7(i)} \\ & \text{iff} & z^{\smile};x \cdot s \cdot y = 0 & \text{since } y \le s \end{array}$$

(xvi). Similar to (xv). □

Theorem 3.7. *Let $\mathfrak{B} \in \mathsf{NREL}$ and $s \in |\mathfrak{B}|$ with $1^{\smile} \le s$ satisfy $(*)$ of Remark 3.2 and let $x, y \in |\mathfrak{B}|$ with $x \le s$. Then*

(i) $l^{\smile}_x(y)^{\smile} = \alpha_y(x)^{\smile\smile}$,
(ii) $r^{\smile}_x(y)^{\smile} = \beta_y(x)^{\smile\smile}$.

Proof. (i). Let $z \in |\mathfrak{B}|$. Note that $z^{\smile\smile} = z \cdot 1^{\smile} \le s$ by Th.1.3(iv) and $1^{\smile} \le s$. Thus, we have

$$\begin{array}{rlll} l^{\smile}_x(y)^{\smile} \cdot z = 0 & \text{iff} & l^{\smile}_x(y) \cdot z^{\smile} = 0 & \text{by Th.1.3(ii)} \\ & \text{iff} & x;z^{\smile} \cdot y = 0 & \text{by Def.3.4 and } x \le s \\ & \text{iff} & y;z^{\smile\smile} \cdot x = 0 & \text{by Th.1.7(ii)} \\ & \text{iff} & y;z^{\smile\smile} \cdot s \cdot x = 0 & \text{since } x \le s \\ & \text{iff} & \alpha_y(x) \cdot z^{\smile\smile} = 0 & \text{by Def.3.4 and } x, z^{\smile\smile} \le s \\ & \text{iff} & \alpha_y(x)^{\smile\smile} \cdot z = 0 & \text{by Th.1.3(ii)} \end{array}$$

(ii). Similar to (i). □

Definition 3.8. *Let $\mathfrak{B} \in \mathsf{NREL}$ and $s \in |\mathfrak{B}|$ with $1^{\smile} \le s$ satisfy $(*)$ of Remark 3.2. We define a relation type algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\smile}, 1\text{'} \rangle$ as follows:*

$$\begin{aligned} &A = \{\langle x_0, x_1 \rangle \in |\mathfrak{B}| \times |\mathfrak{B}| : x_0^{\smile\smile} = x_1^{\smile} \ \ \& \ \ x_1 \le s\}, \\ &0 = \langle 0, 0 \rangle, \\ &1 = \langle 1, s \rangle, \\ &1\text{'} = \langle 1\text{'}, 1\text{'} \rangle, \\ &\langle x_0, x_1 \rangle + \langle y_0, y_1 \rangle = \langle x_0 + y_0, x_1 + y_1 \rangle, \\ &\langle x_0, x_1 \rangle \cdot \langle y_0, y_1 \rangle = \langle x_0 \cdot y_0, x_1 \cdot y_1 \rangle, \\ &- \langle x_0, x_1 \rangle = \langle -x_0, -x_1 \cdot s \rangle, \\ &\langle x_0, x_1 \rangle^{\smile} = \langle x_1, x_0 \cdot s \rangle, \end{aligned}$$

and

$$\langle x_0, x_1 \rangle ; \langle y_0, y_1 \rangle =$$
$$\left\langle x_0;y_0 + (y_1;x_1)^{\smile} + l^{\smile}_{x_1}(y_0) + r^{\smile}_{y_1}(x_0),\ y_1;x_1 \cdot s + (x_0;y_0)^{\smile} + \alpha_{y_0}(x_1) + \beta_{x_0}(y_1) \right\rangle.$$

Theorem 3.9. *The algebra $\mathfrak{A}$ defined in Definition 3.8 is well defined. That is, $A = |\mathfrak{A}|$ contains the constants and is closed under the operations.*

Proof. $0^{\smile\smile} = 0 = 0^{\smile}$ by Theorem 1.3(i) and $0 \leq s$ so $\langle 0, 0\rangle \in A$. $1^{\smile\smile} = (s \cdot 1^{\smile})^{\smile} = s^{\smile} \cdot 1 = s^{\smile}$ by Ax2 so $\langle 1, s\rangle \in A$. $1'^{\smile\smile} = 1' = 1'^{\smile}$ by Theorems 1.8(ii) and 1.3(iii), and $1' \leq 1^{\smile} \leq s$ by Theorem 1.8(ii). Thus $\langle 1', 1'\rangle \in A$.

Now, let $\langle x_0, x_1\rangle, \langle y_0, y_1\rangle \in A$. Then $x_0^{\smile\smile} = x_1^{\smile}$, $y_0^{\smile\smile} = y_1^{\smile}$, and $x_1, y_1 \leq s$.

$$\begin{aligned}
(x_0 + y_0)^{\smile\smile} &= x_0^{\smile\smile} + y_0^{\smile\smile} && \text{by Th.1.3(iii)}\\
&= x_1^{\smile} + y_1^{\smile}\\
&= (x_1 + y_1)^{\smile} && \text{by Th.1.3(iii)}
\end{aligned}$$

Also, $x_1 + y_1 \leq s + s = s$ so $\langle x_0 + y_0, x_1 + y_1\rangle \in A$.

$$\begin{aligned}
(x_0 \cdot y_0)^{\smile\smile} &= x_0^{\smile\smile} \cdot y_0^{\smile\smile} && \text{by Th.1.3(vi)}\\
&= x_1^{\smile} \cdot y_1^{\smile}\\
&= (x_1 \cdot y_1)^{\smile} && \text{by Th.1.3(vi)}
\end{aligned}$$

Also, $x_1 \cdot y_1 \leq s \cdot s = s$ so $\langle x_0 \cdot y_0, x_1 \cdot y_1\rangle \in A$. Since $1^{\smile} \leq s \leq 1$, we have, by Theorem 1.3(iii)(iv), $1^{\smile} = 1^{\smile\smile} \leq s^{\smile} \leq 1^{\smile}$ so $s^{\smile} = 1^{\smile}$. Thus

$$\begin{aligned}
(-x_0)^{\smile\smile} &= (-x_0^{\smile} \cdot 1^{\smile})^{\smile} && \text{by Th.1.3(viii)}\\
&= (-x_0^{\smile})^{\smile} \cdot 1^{\smile\smile} && \text{by Th.1.3(vi)}\\
&= (-x_0^{\smile})^{\smile} \cdot 1^{\smile} && \text{by Th.1.3(iv)}\\
&= -x_0^{\smile\smile} \cdot 1^{\smile} \cdot 1^{\smile} && \text{by Th.1.3(viii)}\\
&= -x_1^{\smile} \cdot 1^{\smile} \cdot s^{\smile} && \text{since } x_0^{\smile\smile} = x_1^{\smile} \text{ and } s^{\smile} = 1^{\smile}\\
&= (-x_1)^{\smile} \cdot s^{\smile} && \text{by Th.1.3(viii)}\\
&= (-x_1 \cdot s)^{\smile} && \text{by Th.1.3(vi)}
\end{aligned}$$

Thus $\langle -x_0, -x_1 \cdot s\rangle \in A$.

$$\begin{aligned}
x_1^{\smile\smile} &= x_0^{\smile\smile\smile} && \text{since } x_0^{\smile\smile} = x_1^{\smile}\\
&= x_0^{\smile} \cdot 1^{\smile} && \text{by Th.1.3(iv)}\\
&= x_0^{\smile} \cdot s^{\smile} && \text{since } s^{\smile} = 1^{\smile}\\
&= (x_0 \cdot s)^{\smile} && \text{by Th.1.3(vi)}
\end{aligned}$$

Thus $\langle x_1, x_0 \cdot s\rangle \in A$.

$$\begin{aligned}
&(x_0;y_0+(y_1;x_1)^\smile+l^\smile_{x_1}(y_0)+r^\smile_{y_1}(x_0))^{\smile\smile} \\
&=(x_0;y_0)^{\smile\smile}+(y_1;x_1)^{\smile\smile\smile}+l^\smile_{x_1}(y_0)^{\smile\smile}+r^\smile_{y_1}(x_0)^{\smile\smile} && \text{by Th.1.3(iii)}\\
&=(y_1;x_1)^{\smile\smile\smile}+(x_0;y_0)^{\smile\smile}+\alpha_{y_0}(x_1)^{\smile\smile\smile}+\beta_{x_0}(y_1)^{\smile\smile\smile} && \text{by Th.3.7}\\
&=(y_1;x_1)^\smile\cdot 1^\smile+(x_0;y_0)^{\smile\smile}+\alpha_{y_0}(x_1)^\smile+\beta_{x_0}(y_1)^\smile && \text{by Th1.3(v)(iv)}\\
&=(y_1;x_1)^\smile\cdot s^\smile+(x_0;y_0)^{\smile\smile}+\alpha_{y_0}(x_1)^\smile+\beta_{x_0}(y_1)^\smile && \text{since } s^\smile=1^\smile\\
&=(y_1;x_1\cdot s+(x_0;y_0)^\smile+\alpha_{y_0}(x_1)+\beta_{x_0}(y_1))^\smile && \text{by Th.1.3(iii)(vi)}
\end{aligned}$$

Also

$$y_1;x_1\cdot s+(x_0;y_0)^\smile+\alpha_{y_0}(x_1)+\beta_{x_0}(y_1)\le s+1^\smile+s+s=s.$$

Thus

$$\left\langle x_0;y_0+(y_1;x_1)^\smile+l^\smile_{x_1}(y_0)+r^\smile_{y_1}(x_0),\ y_1;x_1\cdot s+(x_0;y_0)^\smile+\alpha_{y_0}(x_1)+\beta_{x_0}(y_1)\right\rangle\in A.$$

□

Theorem 3.10. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8 and let*

$$\langle x_0,x_1\rangle,\langle y_0,y_1\rangle,\langle z_0,z_1\rangle\in A=|\mathfrak{A}|$$

Then $\langle x_0,x_1\rangle;\langle y_0,y_1\rangle\cdot\langle z_0,z_1\rangle=0$ iff all six of the following conditions hold in $\mathfrak{B}$:

(i) $x_0;y_0\cdot z_0=0$,
(ii) $x_1;z_0\cdot y_0=0$,
(iii) $z_0;y_1\cdot x_0=0$,
(iv) $y_1;x_1\cdot z_1=0$,
(v) $y_0;z_1\cdot x_1=0$,
(vi) $z_1;x_0\cdot y_1=0$.

Proof. Since $z_1\le s$, we have, by Definition 3.8,

$$\begin{aligned}
&\langle x_0,x_1\rangle;\langle y_0,y_1\rangle\cdot\langle z_0,z_1\rangle=\\
&\quad=\big\langle x_0;y_0\cdot z_0+(y_1;x_1)^\smile\cdot z_0+l^\smile_{x_1}(y_0)\cdot z_0+r^\smile_{y_1}(x_0)\cdot z_0,\\
&\qquad\qquad y_1;x_1\cdot z_1+(x_0;y_0)^\smile\cdot z_1+\alpha_{y_0}(x_1)\cdot z_1+\beta_{x_0}(y_1)\cdot z_1\big\rangle.
\end{aligned}$$

Clearly, this is equal to $\langle 0,0\rangle$ iff all eight of the terms occurring in the right hand side are equal to 0.

If $x_0;y_0\cdot z_0=0$, then

$$\begin{aligned}
(x_0;y_0)^\smile\cdot z_1&=(x_0;y_0)^\smile\cdot 1^\smile\cdot z_1 && \text{by Th.1.3(iii)}\\
&=(x_0;y_0)^\smile\cdot z_1^{\smile\smile} && \text{by Th.1.3(iv)}\\
&=(x_0;y_0)^\smile\cdot z_0^{\smile\smile\smile} && \text{since } z_0^{\smile\smile}=z_1^\smile\\
&=(x_0;y_0)^\smile\cdot z_0^\smile && \text{by Th.1.3(v)}\\
&=(x_0;y_0\cdot z_0)^\smile && \text{by Th.1.3(vi)}\\
&=0^\smile\\
&=0 && \text{by Th.1.3(i)}
\end{aligned}$$

Thus, since both of the conditions $x_0 ; y_0 \cdot z_0 = 0$ and $(x_0 ; y_0)^\smile \cdot z_1 = 0$ are included in the set of eight conditions, the condition $(x_0 ; y_0)^\smile \cdot z_1 = 0$ may be omitted as redundant. Likewise the condition $(y_1 ; x_1)^\smile \cdot z_0 = 0$ may be omitted since it follows from $y_1 ; x_1 \cdot z_1 = 0$. Thus $\langle x_0, x_1\rangle ; \langle y_0, y_1\rangle \cdot \langle z_0, z_1\rangle = 0$ iff the following six conditions hold:

(i′) $x_0 ; y_0 \cdot z_0 = 0$,
(ii′) $l^\smile_{x_1}(y_0) \cdot z_0 = 0$,
(iii′) $r^\smile_{y_1}(x_0) \cdot z_0 = 0$,
(iv′) $y_1 ; x_1 \cdot z_1 = 0$,
(v′) $\alpha_{y_0}(x_1) \cdot z_1 = 0$,
(vi′) $\beta_{x_0}(y_1) \cdot z_1 = 0$.

The following pairs of conditions are easily seen to be equivalent using only Definition 3.4: (i) iff (i′), (ii) iff (ii′), (iii) iff (iii′), (iv) iff (iv′), (v) iff (v′), (vi) iff (vi′). The theorem follows. □

Remark 3.11. Theorem 3.10 explains and motivates the form of Definition 3.8.

Theorem 3.12. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8. Then*

(i) $\langle 1, 1^\smile\rangle^{\smile\smile} = \langle s, s^\smile\rangle$ *holds in* $\mathfrak{A}$,
(ii) $\mathfrak{A} \in \mathsf{NREL}$,
(iii) *the map* $\varphi : |\mathfrak{B}| \to |\mathfrak{A}|$ *defined by* $\varphi(x) = \langle x, x^\smile\rangle$ *is an isomorphism:*

$$\varphi : \mathfrak{B} \cong \mathfrak{Rl}_{\langle 1,1^\smile\rangle}\mathfrak{A}.$$

Proof. First note that for any $x \in |\mathfrak{B}|$, $\langle x, x^\smile\rangle \in A$ since $x^{\smile\smile} = (x^\smile)^\smile$ and $x^\smile \leq 1^\smile \leq s$ by Theorem 1.3(iii).

(i).
$$\begin{aligned} \langle 1, 1^\smile\rangle^{\smile\smile} &= \langle 1^\smile, 1 \cdot s\rangle^\smile && \text{by Def.3.8} \\ &= \langle 1^\smile, s\rangle^\smile \\ &= \langle s, 1^\smile \cdot s\rangle && \text{by Def.3.8} \\ &= \langle s, 1^\smile\rangle && \text{since } 1^\smile \leq s \\ &= \langle s, s^\smile\rangle && \text{since } s^\smile = 1^\smile \end{aligned}$$

(ii). Let $\langle x_0, x_1\rangle, \langle y_0, y_1\rangle, \langle z_0, z_1\rangle \in A = |\mathfrak{A}|$.
Ax1. Obvious from Definition 3.8.

Ax2.
$$\begin{aligned} (\langle x_0, x_1\rangle \cdot \langle y_0, y_1\rangle^\smile)^\smile &= (\langle x_0, x_1\rangle \cdot \langle y_1, y_0 \cdot s\rangle)^\smile && \text{by Def.3.8} \\ &= \langle x_0 \cdot y_1, x_1 \cdot y_0 \cdot s\rangle^\smile && \text{by Def.3.8} \\ &= \langle x_0 \cdot y_1, x_1 \cdot y_0\rangle^\smile && \text{since } x_1 \leq s \\ &= \langle x_1 \cdot y_0, x_0 \cdot y_1 \cdot s\rangle && \text{by Def.3.8} \\ &= \langle x_1, x_0 \cdot s\rangle \cdot \langle y_0, y_1\rangle && \text{by Def.3.8} \\ &= \langle x_0, x_1\rangle^\smile \cdot \langle y_0, y_1\rangle && \text{by Def.3.8} \end{aligned}$$

Ax3.

$(\langle x_0, x_1\rangle + \langle y_0, y_1\rangle) ; \langle z_0, z_1\rangle$
$= (\langle x_0 + y_0, x_1 + y_1\rangle) ; \langle z_0, z_1\rangle$
$= \big\langle (x_0 + y_0) ; z_0 + (z_1 ; (x_1 + y_1))^\smile + l^\smile_{x_1+y_1}(z_0) + r^\smile_{z_1}(x_0 + y_0),$
$\quad z_1 ; (x_1 + y_1) \cdot s + ((x_0 + y_0) ; z_0)^\smile + \alpha_{z_0}(x_1 + y_1) + \beta_{x_0+y_0}(z_1)\big\rangle$
by Def.3.8
$= \big\langle x_0 ; z_0 + (z_1 ; x_1)^\smile + l^\smile_{x_1}(z_0) + r^\smile_{z_1}(x_0),\ z_1 ; x_1 \cdot s + (x_0 ; z_0)^\smile + \alpha_{z_0}(x_1) + \beta_{x_0}(z_1)\big\rangle +$
$\quad + \big\langle y_0 ; z_0 + (z_1 ; y_1)^\smile + l^\smile_{y_1}(z_0) + r^\smile_{z_1}(y_0),\ z_1 ; y_1 \cdot s + (y_0 ; z_0)^\smile + \alpha_{z_0}(y_1) + \beta_{y_0}(z_1)\big\rangle$
by Def.3.8, Ax3, Th.1.3(iii), Th.3.6(i)(x)(xi)(iv)
$= \langle x_0, x_1\rangle ; \langle z_0, z_1\rangle + \langle y_0, y_1\rangle ; \langle z_0, z_1\rangle .$

Similarly, $\langle x_0, x_1\rangle ; (\langle y_0, y_1\rangle + \langle z_0, z_1\rangle) = \langle x_0, x_1\rangle ; \langle y_0, y_1\rangle + \langle x_0, x_1\rangle ; \langle z_0, z_1\rangle$.

Ax4.
$\langle 1, s\rangle ; \langle 0, 0\rangle$
$= \langle 1 ; 0 + (0 ; s)^\smile + l^\smile_s(0) + r^\smile(1), 0 ; s \cdot s + (1 ; 0)^\smile + \alpha_0(s) + \beta_1(0)\rangle$ by Def.3.8
$= \langle 0, 0\rangle$ by Ax4, Th.1.3(i), Th.3.6(ix)(vi)(vii)(xii)

Similarly $\langle 0, 0\rangle ; \langle 1, s\rangle = \langle 0, 0\rangle$.

Ax5. By Theorem 1.6 it suffices to show that $\langle x_0, x_1\rangle^\smile ;$(-) is conjugate to $\langle x_0, x_1\rangle^{\smile\smile} ;$(-) and that (-)$; \langle y_0, y_1\rangle^\smile$ is conjugate to (-)$; \langle y_0, y_1\rangle^{\smile\smile}$. Note that $\langle x_0, x_1\rangle^\smile = \langle x_1, x_0 \cdot s\rangle$ and $\langle x_0, x_1\rangle^{\smile\smile} = \langle x_0 \cdot s, x_1 \cdot s\rangle = \langle x_0 \cdot s, x_1\rangle$. Thus we must show that $\langle x_1, x_0 \cdot s\rangle ; \langle y_0, y_1\rangle \cdot \langle z_0, z_1\rangle = 0$ iff $\langle x_0 \cdot s, x_1\rangle ; \langle z_0, z_1\rangle \cdot \langle y_0, y_1\rangle = 0$. By Theorem 3.10 this reduces to checking the equivalence of the conjunction of the conditions

(i) $x_1 ; y_0 \cdot z_0 = 0$,
(ii) $(x_0 \cdot s) ; z_0 \cdot y_0 = 0$,
(iii) $z_0 ; y_1 \cdot x_1 = 0$,
(iv) $y_1 ; (x_0 \cdot s) \cdot z_1 = 0$,
(v) $y_0 ; z_1 \cdot x_0 \cdot s = 0$,
(vi) $z_1 ; x_1 \cdot y_1 = 0$,

with the conjunction of the six conditions

(i′) $(x_0 \cdot s) ; z_0 \cdot y_0 = 0$,
(ii′) $x_1 ; y_0 \cdot z_0 = 0$,
(iii′) $y_0 ; z_1 \cdot x_0 \cdot s = 0$,
(iv′) $z_1 ; x_1 \cdot y_1 = 0$,
(v′) $z_0 ; y_1 \cdot x_1 = 0$,
(vi′) $y_1 ; (x_0 \cdot s) \cdot z_1 = 0$.

Since the conditions are identical, except for order, we have that $\langle x_0, x_1\rangle^\smile ;$(-) is conjugate to $\langle x_0, x_1\rangle^{\smile\smile} ;$(-). Similarly, (-)$; \langle y_0, y_1\rangle^\smile$ is conjugate to (-)$; \langle y_0, y_1\rangle^{\smile\smile}$.

Ax6. We must show that $\langle 1', 1'\rangle ; \langle x_0, x_1\rangle \leq \langle x_0, x_1\rangle$, but this is equivalent to

$$\langle 1', 1'\rangle ; \langle x_0, x_1\rangle \cdot \langle -x_0, -x_1 \cdot s\rangle = \langle 0, 0\rangle$$

since $\mathfrak{A}$ satisfies Ax1. By Theorem 3.10, this is equivalent to the conjunction of the following six conditions

(i) $1';x_0 \cdot -x_0 = 0$,
(ii) $1';-x_0 \cdot x_0 = 0$,
(iii) $-x_0;x_1 \cdot 1' = 0$,
(iv) $x_1;1' \cdot -x_1 \cdot s = 0$,
(v) $x_0;(-x_1 \cdot s) \cdot 1' = 0$,
(vi) $(-x_1 \cdot s);1' \cdot x_1 = 0$.

Conditions (i), (ii), (iv), and (vi) follow directly from Ax6. For (iii), we have

$$\begin{aligned}
-x_0;x_1 \cdot 1' &= x_1^{\smile};(-x_0)^{\smile} \cdot 1' && \text{by Th.1.8(vi)}\\
&= x_1^{\smile};[(-x_0)^{\smile} \cdot x_1^{\smile\smile};1'] \cdot 1' && \text{by Ax5}\\
&= x_1^{\smile};[-x_0^{\smile} \cdot 1^{\smile} \cdot x_1^{\smile\smile};1] \cdot 1' && \text{by Th.1.3(viii)}\\
&\le x_1^{\smile};[-x_1^{\smile\smile} \cdot 1^{\smile} \cdot x_1^{\smile\smile}] \cdot 1' && \text{by Ax6, Ax3}\\
&= x_1^{\smile};0 \cdot 1' &&\\
&= 0 && \text{by Ax4}
\end{aligned}$$

The proof of (v) is similar. Thus $\langle 1', 1'\rangle ; \langle x_0, x_1\rangle \le \langle x_0, x_1\rangle$. Similarly $\langle x_0, x_1\rangle ; \langle 1', 1'\rangle \le \langle x_0, x_1\rangle$.

Ax7. By Ax6 in $\mathfrak{A}$, we have $\langle 1', 1'\rangle ; \langle 1', 1'\rangle \le \langle 1', 1'\rangle$. But

$$\begin{aligned}
\langle 1', 1'\rangle ; \langle 1', 1'\rangle &= \langle 1';1' + \dots, 1';1' + \dots\rangle && \text{by Def.3.8}\\
&= \langle 1' + \dots, 1' + \dots\rangle && \text{by Ax7}\\
&\ge \langle 1', 1'\rangle
\end{aligned}$$

Thus $\langle 1', 1'\rangle ; \langle 1', 1'\rangle = \langle 1', 1'\rangle$.

Ax8.
$$\begin{aligned}
\langle 1, s\rangle^{\smile} ; -\langle 1, s\rangle^{\smile} \cdot \langle 1', 1'\rangle &= -\langle s, s\rangle ; -\langle s, s\rangle \cdot \langle 1', 1'\rangle && \text{by Def.3.8}\\
&= \langle -s, -s \cdot s\rangle ; \langle -s, -s \cdot s\rangle \cdot \langle 1', 1'\rangle && \text{by Def.3.8}\\
&= \langle -s, 0\rangle ; \langle -s, 0\rangle \cdot \langle 1', 1'\rangle
\end{aligned}$$

But, by Theorem 3.10, $\langle -s, 0\rangle ; \langle -s, 0\rangle \cdot \langle 1', 1'\rangle = \langle 0, 0\rangle$ iff the following six conditions hold in $\mathfrak{B}$:

(i) $-s;-s \cdot 1' = 0$,
(ii) $0;1' \cdot -s = 0$,
(iii) $1';0 \cdot -s = 0$,
(iv) $0;0 \cdot 1' = 0$,
(v) $-s;1' \cdot 0 = 0$,
(vi) $1';-s \cdot 0 = 0$.

Now (i) holds since $-s;-s \cdot 1' \le -1^{\smile};-1^{\smile} \cdot 1' = 0$ by Ax3, $1^{\smile} \le s$, and Ax8. The rest ((ii) through (vi)) follow obviously by Ax3 and Ax4. Therefore $\mathfrak{A} \in \mathsf{NREL}$ as desired.

(iii). Let $\varphi : |\mathfrak{B}| \to |\mathfrak{A}|$ be defined by $\varphi(x) = \langle x, x^{\smile}\rangle$. Then $\varphi(x) = \langle x, x^{\smile}\rangle \le \langle 1, 1^{\smile}\rangle$ by Theorem 1.3(iii). Thus $\varphi : |\mathfrak{B}| \to |\mathfrak{Rl}_{\langle 1,1^{\smile}\rangle}\mathfrak{A}|$.

$$\begin{aligned}
\varphi(0) &= \langle 0, 0^{\smile}\rangle \\
&= \langle 0, 0\rangle && \text{by Th.1.3(i)} \\
\varphi(1) &= \langle 1, 1^{\smile}\rangle \\
\varphi(1') &= \langle 1', 1'^{\smile}\rangle \\
&= \langle 1', 1'\rangle && \text{by Th.1.8(ii)} \\
&= \langle 1', 1' \cdot 1^{\smile}\rangle && \text{by Th.1.8(ii)} \\
&= \langle 1', 1'\rangle \,;\, \langle 1, 1^{\smile}\rangle && \text{by Def.3.8} \\
\varphi(x+y) &= \langle x+y, (x+y)^{\smile}\rangle \\
&= \langle x+y, x^{\smile} + y^{\smile}\rangle && \text{by Th.1.3(iii)} \\
&= \langle x, x^{\smile}\rangle + \langle y, y^{\smile}\rangle && \text{by Def.3.8} \\
&= \varphi(x) + \varphi(y) \\
\varphi(x \cdot y) &= \langle x \cdot y, (x \cdot y)^{\smile}\rangle \\
&= \langle x \cdot y, x^{\smile} \cdot y^{\smile}\rangle && \text{by Th.1.3(vi)} \\
&= \langle x, x^{\smile}\rangle \cdot \langle y, y^{\smile}\rangle && \text{by Def.3.8} \\
&= \varphi(x) \cdot \varphi(y) \\
\varphi(-x) &= \langle -x, (-x)^{\smile}\rangle \\
&= \langle -x, -x^{\smile} \cdot 1^{\smile}\rangle && \text{by Th.1.3(viii)} \\
&= \langle -x, -x^{\smile} \cdot s \cdot 1^{\smile}\rangle && \text{since } 1^{\smile} \le s \\
&= \langle -x, -x^{\smile} \cdot s\rangle \cdot \langle 1, 1^{\smile}\rangle && \text{by Def.3.8} \\
&= -\langle x, x^{\smile}\rangle \cdot \langle 1, 1^{\smile}\rangle && \text{by Def.3.8} \\
&= -\varphi(x) \cdot \langle 1, 1^{\smile}\rangle \\
\varphi(x^{\smile}) &= \langle x^{\smile}, x^{\smile\smile}\rangle \\
&= \langle x^{\smile}, x \cdot s \cdot 1^{\smile}\rangle && \text{by } 1^{\smile} \le s \text{ and Th.1.3(iv)} \\
&= \langle x^{\smile}, x \cdot s\rangle \cdot \langle 1, 1^{\smile}\rangle && \text{by Def.3.8} \\
&= \langle x, x^{\smile}\rangle^{\smile} \cdot \langle 1, 1^{\smile}\rangle && \text{by Def.3.8} \\
&= \varphi(x)^{\smile} \cdot \langle 1, 1^{\smile}\rangle
\end{aligned}$$

Now, consider the condition $\varphi(x;y) = \varphi(x);\varphi(y) \cdot \langle 1, 1^{\smile}\rangle$. By the definition of φ, this says that

$$\langle x;y, (x;y)^{\smile}\rangle = \langle x, x^{\smile}\rangle \,;\, \langle y, y^{\smile}\rangle \cdot \langle 1, 1^{\smile}\rangle$$

whenever $x, y \in |\mathfrak{B}|$. Noting that

$$\begin{aligned}
\langle z_0, z_1\rangle \le \langle 1, 1^{\smile}\rangle \quad &\text{iff} \quad z_1 \le 1^{\smile} \\
&\text{iff} \quad z_1^{\smile\smile} = z_1 && \text{by Th.1.3(iv)} \\
&\text{iff} \quad z_0^{\smile\smile\smile} = z_1 && \text{since } z_0^{\smile\smile} = z_1^{\smile} \\
&\text{iff} \quad z_0^{\smile} = z_1 && \text{Th.1.3(vi)} \\
&\text{iff} \quad \langle z_0, z_1\rangle = \varphi(z_0)
\end{aligned}$$

for any $\langle z_0, z_1\rangle \in |\mathfrak{A}|$, we see that the map $\varphi : |\mathfrak{B}| \to |\mathfrak{Rl}_{\langle 1,1^\smile\rangle}\mathfrak{A}|$ is surjective. Clearly φ is also one-to-one since for $x, y \in |\mathfrak{B}|$, if $\langle x, x^\smile\rangle = \langle y, y^\smile\rangle$ then $x = y$. Thus we need only show that

$$(**) \qquad \langle x;y, (x;y)^\smile\rangle \cdot \langle z, z^\smile\rangle = 0 \quad \text{iff} \quad \langle x, x^\smile\rangle \,;\, \langle y, y^\smile\rangle \cdot \langle z, z^\smile\rangle = 0$$

for any $x, y, z \in |\mathfrak{B}|$.

By Definition 3.8 and Theorem 1.3(vi),

$$\langle x;y, (x;y)^\smile\rangle \cdot \langle z, z^\smile\rangle = \langle x;y \cdot z, (x;y)^\smile \cdot z^\smile\rangle = \langle x;y \cdot z, (x;y \cdot z)^\smile\rangle$$

so the left hand side of $(**)$ is equivalent to $x;y \cdot z = 0$ by Theorem 1.3(i). The right hand side of $(**)$, by Theorem 3.10, is equivalent to the conjunction of the following six conditions:

(i) $x;y \cdot z = 0$,
(ii) $x^\smile;z \cdot y = 0$,
(iii) $z;y^\smile \cdot x = 0$,
(iv) $y^\smile;x^\smile \cdot z^\smile = 0$,
(v) $y;z^\smile \cdot x^\smile = 0$,
(vi) $z^\smile;x \cdot y^\smile = 0$.

Since the left hand side of $(**)$ is equivalent to (i) we need only show that (i) implies all of (ii) through (vi). To that end, suppose (i) holds.

(ii).
$$\begin{aligned} x^{\smile\smile};y \cdot z &\le x;y \cdot z && \text{by Ax3, Th.1.3(iv)} \\ &= 0 && \text{by (i)} \end{aligned}$$

(iii).
$$\begin{aligned} x^\smile;y^{\smile\smile} \cdot z &\le x;y \cdot z && \text{by Ax3, Th.1.3(iv)} \\ &= 0 && \text{by (i)} \end{aligned}$$

Thus, by Theorem 1.7(ii), $z;y^\smile \cdot x = 0$.

(v).
$$\begin{aligned} x^\smile;z^{\smile\smile} \cdot y &\le x^\smile;z \cdot y && \text{by Ax3, Th.1.3(iv)} \\ &= 0 && \text{by (ii)} \end{aligned}$$

Thus, by Theorem 1.7(ii), $y;z^\smile \cdot x = 0$.

(vi).
$$\begin{aligned} z^{\smile\smile};y^\smile \cdot x &\le z;y^\smile \cdot x && \text{by Ax3, Th.1.3(iv)} \\ &= 0 && \text{by (iii)} \end{aligned}$$

Thus, by Theorem 1.7(i), $z^\smile;x \cdot y^\smile = 0$.

(iv).
$$\begin{aligned} z^\smile;x^{\smile\smile} \cdot y^\smile &\le z^\smile;x \cdot y^\smile && \text{by Ax3, Th.1.3(iv)} \\ &= 0 && \text{by (vi)} \end{aligned}$$

Thus, by Theorem 1.7(ii), $y^\smile;x^\smile \cdot z^\smile = 0$. □

Theorem 3.13. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8. If $\mathfrak{B} \in \mathsf{REL}$ and $u \in |\mathfrak{B}|$ with $u \le 1'$ in $\mathfrak{B}$, then*

(i) $\langle u, u\rangle \,;\, \langle 1, s\rangle = \langle u;1, s;u\rangle$,
(ii) $\langle 1, s\rangle \,;\, \langle u, u\rangle = \langle 1;u, u;s\rangle$,
(iii) $-(\langle u, u\rangle \,;\, \langle 1, s\rangle) = \langle -(u;1), -(1;u)\cdot s\rangle$,
(iv) $-(\langle 1, s\rangle \,;\, \langle u, u\rangle) = \langle -(1;u), -(u;1)\cdot s\rangle$.

Proof. (i). First note that

$$\begin{aligned} (u;1)^{\smile\smile} &= (1;u)^{\smile\smile\smile} && \text{Th.1.8(ix)}\\ &= (1;u)^{\smile} && \text{Th.1.3(v)}\\ &= (s;u)^{\smile} + (-s;u)^{\smile} && \text{by Ax3, Th.1.3(iii)}\\ &= (s;u)^{\smile} \end{aligned}$$

since

$$\begin{aligned} (-s;u)^{\smile} &\le (-s)^{\smile} && \text{by Ax6, Ax3, Th.1.3(iii)}\\ &\le (-1^{\smile})^{\smile} && \text{by Th.1.3(iii) and } 1^{\smile} \le s\\ &= 0 && \text{by Th.1.3(iii)} \end{aligned}$$

Thus $\langle u;1, s;u\rangle \in A = |\mathfrak{A}|$. By Definition 3.8, we have

$$\langle u, u\rangle \,;\, \langle 1, s\rangle = \langle u;1 + \ldots, s;u + \ldots\rangle \ge \langle u;1, s;u\rangle\,.$$

Thus, we need only show that $\langle u, u\rangle \,;\, \langle 1, s\rangle \le \langle u;1, s;u\rangle$, which is equivalent to the condition

$$\langle u, u\rangle \,;\, \langle 1, s\rangle \cdot \langle -(u;1), -(s;u)\cdot s\rangle = 0.$$

By Theorem 3.10 this is equivalent to the conjunction of the following six conditions:

(i′) $u;1 \cdot -(u;1) = 0$,
(ii′) $u; -(u;1)\cdot 1 = 0$,
(iii′) $-(u;1);s \cdot u = 0$,
(iv′) $s;u \cdot -(s;u)\cdot s = 0$,
(v′) $1;(-(s;u)\cdot s)\cdot u = 0$,
(vi′) $(-(s;u)\cdot s);u \cdot s = 0$.

(i′). Trivial.

(ii′). Use Theorem 1.8(vii).

(iii′).
$$\begin{aligned} -(u;1);s\cdot u = 0 \quad &\text{iff} \quad s^{\smile};(-(u;1))^{\smile}\cdot u = 0 && \text{by Th.1.8(vi)}\\ &\text{iff} \quad (-(u;1))^{\smile\smile};s^{\smile\smile}\cdot u = 0 && \text{by Th.1.8(vi)}\\ &\text{iff} \quad u;s^{\smile}\cdot(-(u;1)^{\smile\smile} = 0 && \text{by Th.1.7(ii)}\\ &\text{iff} \quad u;s^{\smile}\cdot -(u;1)\cdot 1^{\smile} = 0 && \text{by Th.1.3(iv)}\\ &\text{iff} \quad u;1\cdot s^{\smile}\cdot -(u;1)\cdot 1^{\smile} = 0 && \text{by Th.1.8(iii)} \end{aligned}$$

(iv′). Trivial.

(v′). $1;(-(s;u)\cdot s)\cdot u=0$

$$
\begin{aligned}
&\text{iff}\quad (-(s;u)\cdot s)^{\smile};1^{\smile}\cdot u=0 && \text{by Th.1.8(vi)}\\
&\text{iff}\quad 1^{\smile\smile};(-(s;u)\cdot s)^{\smile\smile}\cdot u=0 && \text{by Th.1.8(vi)}\\
&\text{iff}\quad 1^{\smile};u\cdot(-(s;u)\cdot s)^{\smile\smile}=0 && \text{by Th.1.7(i)}\\
&\text{iff}\quad 1^{\smile};u\cdot -(s;u)\cdot s\cdot 1^{\smile}=0 && \text{by Th.1.3(iv)}\\
&\text{iff}\quad 1;u\cdot 1^{\smile}\cdot -(1;u\cdot s)\cdot s\cdot 1^{\smile}=0 && \text{by Th.1.8(iv)}\\
&\text{iff}\quad 1;u\cdot(-(1;u)+-s)\cdot s\cdot 1^{\smile}=0\\
&\text{iff}\quad 1;u\cdot-(1;u)\cdot s\cdot 1^{\smile}=0\quad\text{and}\quad 1;u\cdot -s\cdot s\cdot 1^{\smile}=0
\end{aligned}
$$

(vi′). $(-(s;u)\cdot s);u\cdot s=0$ iff $s;u\cdot-(s;u)\cdot s=$ by Theorem 1.8(ii) and Theorem 1.7(ii). Thus $\langle u,u\rangle\,;\langle 1,s\rangle=\langle u;1,s;u\rangle$.

(ii). Likewise $\langle 1,s\rangle\,;\langle u,u\rangle=\langle 1;u,u;s\rangle$.

(iii).
$$
\begin{aligned}
-(\langle u,u\rangle\,;\langle 1,s\rangle)&=-\langle u;1,s;u\rangle && \text{by (i)}\\
&=\langle -(u;1),-(s;u)\cdot s\rangle && \text{by Def.3.8}\\
&=\langle -(u;1),-(1;u\cdot s)\cdot s\rangle && \text{by Th.1.8(iv)}\\
&=\langle -(u;1),(-(1;u)+-s)\cdot s\rangle\\
&=\langle -(u;1),-(1;u)\cdot s\rangle
\end{aligned}
$$

(iv). Similar to (iii). □

Theorem 3.14. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8. If $\mathfrak{B}\in\mathsf{REL}$, then $\mathfrak{A}\in\mathsf{REL}$.*

Proof. Let $u\in|\mathfrak{B}|$ with $u\le 1'$ in $\mathfrak{B}$. By Theorem 1.9, to show Ax9 holds it suffices to prove the following six statements:

(i) $\langle u,u\rangle\,;\langle 1,s\rangle\,;\langle 1,s\rangle\le\langle u,u\rangle\,;\langle 1,s\rangle$,
(ii) $\langle u,u\rangle\,;(-(\langle u,u\rangle\,;\langle 1,s\rangle);\langle 1,s\rangle)=\langle 0,0\rangle$,
(iii) $\langle 1,s\rangle\,;\langle u,u\rangle\,;-(\langle u,u\rangle\,;\langle 1,s\rangle)=\langle 0,0\rangle$,
(iv) $-(\langle 1,s\rangle\,;\langle u,u\rangle);(\langle u,u\rangle\,;\langle 1,s\rangle)=\langle 0,0\rangle$,
(v) $\langle 1,s\rangle\,;-(\langle 1,s\rangle\,;\langle u,u\rangle);\langle u,u\rangle=\langle 0,0\rangle$,
(vi) $\langle 1,s\rangle\,;(\langle 1,s\rangle\,;\langle u,u\rangle)\le\langle 1,s\rangle\,;\langle u,u\rangle$.

(i). The statement, by Theorem 3.13(i)(iii), is equivalent to

$$\langle u;1,s;u\rangle\,;\langle 1,s\rangle\cdot\langle -(1;u),-(1;u)\cdot s\rangle=\langle 0,0\rangle\,.$$

By Theorem 3.10, we must prove the following six conditions:

(i′) $u;1;1\cdot-(u;1)=0$,
(ii′) $s;u;-(u;1)\cdot 1=0$,
(iii′) $-(u;1);s\cdot u;1=0$,
(iv′) $s;(s;u)\cdot-(1;u)\cdot s=0$,
(v′) $1;(-(1;u)\cdot s)\cdot s;u=0$,
(vi′) $(-(1;u)\cdot s);(u;1)\cdot s=0$.

(i′). $u;1;1\cdot -(u;1)\le u;1\cdot -(u;1)=0$ by Theorem 1.9(i).

(ii′). $s;u;-(u;1)\cdot 1\le 1;u;-(u;1)=0$ by Ax3 and Theorem 1.9(iii).

(iii′). By Ax3, $-(u;1);s\cdot u;1\le -(u;1);1\cdot u;1$. But $-(u;1);1\cdot u;1=0$ iff $1\cdot u;(-(u;1);1)=0$ by Theorems 1.7(i) and 1.8(ii). The latter statement holds by Theorem 1.9(ii).

(iv′). $s;(s;u)\cdot -(1;u)\cdot s\le 1;(1;u)\cdot -(1;u)\cdot s\le 1;u\cdot -(u;1)\cdot s=0$ by Ax3 and Theorem 1.9(vi).

(v′). $1;(-(1;u)\cdot s)\cdot s;u\le 1;-(1;u)\cdot 1;u$ by Ax3. But, by Theorems 1.7(ii) and 1.8(ii), $1;-(1;u)\cdot 1;u=0$ iff $1\cdot 1;-(1;u);u=0$, which holds by Theorem 1.9(v).

(vi′). $(-(1;u)\cdot s);(u;1)\cdot s\le -(u;1)\cdot u;1=0$ by Ax3 and Theorem 1.9(iv).

Therefore (i) is true in $\mathfrak{A}$.

(ii). By Theorems 3.12(ii), 1.7(i), and 1.8(ii), the statement is equivalent to

$$\langle u,u\rangle\,;\,\langle 1,s\rangle\cdot -(\langle u,u\rangle\,;\,\langle 1,s\rangle);\langle 1,s\rangle=\langle 0,0\rangle$$

which, by Theorem 3.13, is equivalent to

$$\langle -(u;1),-(1;u)\cdot s\rangle\,;\,\langle 1,s\rangle\cdot\langle u;1,s;u\rangle=\langle 0,0\rangle\,.$$

By Theorem 3.10, we must prove the following six statements:

(i′) $-(u;1);1\cdot u;1=0$,
(ii′) $(-(1;u)\cdot s);(u;1)\cdot 1=0$,
(iii′) $u;1;s\cdot -(u;1)=0$,
(iv′) $s;(-(1;u)\cdot s)\cdot s;u=0$,
(v′) $1;(s;u)\cdot -(1;u)\cdot s=0$,
(vi′) $s;u;-(u;1)\cdot s=0$.

(i′). By Theorems 1.7(i) and 1.8(ii), $-(u;1);1\cdot u;1=0$ iff $1\cdot u;(-(u;1);1)=0$, which holds by Theorem 1.9(ii).

(ii′). By Ax3 and Theorem 1.9(iv), $(-(1;u)\cdot s);(u;1)\cdot 1\le -(1;u);(u;1)=0$.

(iii′). By Ax3 and Theorem 1.9(i), $u;1;s\cdot -(u;1)\le u;1;1\cdot -(u;1)\le u;1\cdot -(u;1)=0$.

(iv′). By Ax3, $s;(-(1;u)\cdot s)\cdot s;u\le 1;-(1;u)\cdot 1;u$. But, by Theorems 1.7(ii) and 1.8(ii), $1;-(1;u)\cdot 1;u=0$ iff $1;-(1;u);u=0$, which holds by Theorem 1.9(v).

(v′). By Ax3 and Theorem 1.9(vi), $1;(s;u)\cdot -(1;u)\cdot s\le 1;1;u\cdot -(1;u)\le 1;u\cdot -(1;u)=0$.

(vi′). By Ax3 and Theorem 1.9(iii), $s;u;-(u;1)\cdot s\le 1;u;-(u;1)=0$.

Therefore (ii) is true in $\mathfrak{A}$.

(iii). By Theorem 3.13, the statement is equivalent to

$$\langle 1;u,u;s\rangle\,;\,\langle u;1,-(1;u)\cdot s\rangle\cdot\langle 1,s\rangle=\langle 0,0\rangle\,.$$

By Theorem 3.10, we must prove the following six statements:

(i′) $1;u\cdot -(u;1)\cdot 1=0$,
(ii′) $u;s;1\cdot -(u;1)=0$,
(iii′) $1;(-(1;u)\cdot s)\cdot 1;u=0$,
(iv′) $(-(1;u)\cdot s);(u;s)\cdot s=0$,
(v′) $-(u;1);s\cdot u;s=0$,
(vi′) $s;(1;u)\cdot -(1;u)\cdot s=0$.

(i′). This is essentially Theorem 1.9(iii).

(ii′). By Ax3 and Theorem 1.9(i), $u;s;1\cdot-(u;1) \leq u;1;1\cdot-(u;1) \leq u;1\cdot-(u;1) = 0$.

(iii′). By Ax3, $1;(-(1;u)\cdot s)\cdot 1;u \leq 1;-(1;u)\cdot 1;u$. But by Theorem 1.7(ii) and 1.8(ii), $1;-(u;1)\cdot(1;u) = 0$ iff $1\cdot 1;-(1;u);u = 0$, which holds by Theorem 1.9(v).

(iv′). By Ax3 and Theorem 1.9(iv), $(-(1;u)\cdot s);(u;s)\cdot s \leq -(1;u);(u;1) = 0$.

(v′). By Ax3, $-(u;1);s\cdot u;s \leq -(u;1);1\cdot u;1$. But by Theorems 1.7(i) and 1.8(ii), $-(u;1);1\cdot u;1 = 0$ iff $1\cdot u;(-(u;1);1) = 0$, which holds by Theorem 1.9(ii).

(vi′). By Ax3 and Theorem 1.9(vi), $s;(1;u)\cdot-(1;u)\cdot s \leq 1;(1;u)\cdot-(1;u) \leq 1;u\cdot-(1;u) = 0$.

Therefore (iii) is true in $\mathfrak{A}$.

(iv). Similar to (iii).

(v). Similar to (ii).

(vi). Similar to (i).

Therefore $\mathfrak{A} \in \mathsf{REL}$ as desired.

□

Theorem 3.15. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8 and let $r = \langle 1, 1^{\smile}\rangle \in A = |\mathfrak{A}|$. Then $\mathfrak{A}$ satisfies $r + r^{\smile} = 1$ and $(r\cdot -r^{\smile});(r\cdot -r^{\smile})\cdot r^{\smile}\cdot -r = 0$.*

Proof. First we calculate $r^{\smile}$, $-r$, and $-r^{\smile}$ in $\mathfrak{A}$. Using Definition 3.8 we have

$$\begin{aligned} r^{\smile} &= \langle 1, 1^{\smile}\rangle^{\smile} = \langle 1^{\smile}, 1\cdot s\rangle = \langle 1^{\smile}, s\rangle, \\ -r &= -\langle 1, 1^{\smile}\rangle = \langle 0, -1^{\smile}\cdot s\rangle, \\ -r^{\smile} &= -\langle 1^{\smile}, s\rangle = \langle -1^{\smile}, -s\cdot s\rangle = \langle -1^{\smile}, 0\rangle, \end{aligned}$$

and hence $r + r^{\smile} = \langle 1, 1^{\smile}\rangle + \langle 1^{\smile}, s\rangle = \langle 1, s\rangle = 1$. Also, we have

$$\begin{aligned} &(r\cdot -r^{\smile});(r\cdot -r^{\smile})\cdot r^{\smile}\cdot -r \\ &= (\langle 1, 1^{\smile}\rangle\cdot\langle -1^{\smile}, 0\rangle);(\langle 1, 1^{\smile}\rangle\cdot\langle -1^{\smile}, 0\rangle)\cdot\langle 1^{\smile}, s\rangle\cdot\langle 0, -1^{\smile}\cdot s\rangle \\ &= \langle -1^{\smile}, 0\rangle\,;\langle -1^{\smile}, 0\rangle\cdot\langle 0, -1^{\smile}\cdot s\rangle. \end{aligned}$$

Thus, by Theorem 3.10, $(r\cdot -r^{\smile});(r\cdot -r^{\smile})\cdot r^{\smile}\cdot -r = 0$ iff all six of the following conditions are satisfied:

(i) $-1^{\smile};-1^{\smile}\cdot 0 = 0$.
(ii) $0;(-1^{\smile}\cdot s)\cdot -1^{\smile} = 0$.
(iii) $0;0\cdot -1^{\smile} = 0$.
(iv) $0;0\cdot -1^{\smile}\cdot s = 0$.
(v) $-1^{\smile};(-1^{\smile}\cdot s)\cdot 0 = 0$.
(vi) $(-1^{\smile}\cdot s);-1^{\smile}\cdot 0 = 0$.

Clearly, by Ax3 and Ax4, all six of the conditions hold. □

Theorem 3.16. *Let $\mathfrak{B}$, s, and $\mathfrak{A}$ be as in Definition 3.8 and let $\mathfrak{C} \in \mathsf{NREL}$ be such that $\mathfrak{B} \cong \mathfrak{Rl}_r\mathfrak{C}$ for some $r \in |\mathfrak{C}|$. Assume that $s \leq r^{\smile\smile}$ in $\mathfrak{C}$ and suppose that*

$$(r\cdot -r^{\smile});(r\cdot -r^{\smile})\cdot r^{\smile}\cdot -r = 0$$

in $\mathfrak{C}$*. Then there is a unique isomorphism* $\varphi : \mathfrak{Rl}_{r+s^{\smile}}\mathfrak{C} \cong \mathfrak{A}$ *such that* $\varphi(x) = \langle x, x^{\smile}\rangle$ *for any* $x \in |\mathfrak{B}| = |\mathfrak{Rl}_r\mathfrak{C}|$.

Proof. We omit the proof. $\square$

§4. Adjoining Identity Elements

Definition 4.1. *Let* $\mathfrak{B} \in \mathsf{NREL}$ *and let* $\mathfrak{V}$ *be a Boolean algebra. Let* $\pi_0, \pi_1 : |\mathfrak{V}| \to |\mathfrak{B}|$ *be normal and additive such that* $\pi_0(a)^{\smile\smile} = \pi_1(a)^{\smile}$ *for any* $a \in |\mathfrak{V}|$ *and* $1' \cdot \pi_0(1) = 1' \cdot \pi_1(1) = 0$*. Let* $\mathsf{d}, \mathsf{r} : |\mathfrak{Rl}_{1'}\mathfrak{B}| \to |\mathfrak{V}|$ *be conjugate to* $\pi_0(\text{-})^{\smile\smile}, \pi_1(\text{-})^{\smile\smile} : |\mathfrak{V}| \to |\mathfrak{Rl}_{1'}\mathfrak{B}|$*, respectively. Then we define a relation type algebra* $\mathfrak{A}$ *by:*

$$
\begin{aligned}
&\mathfrak{Bl}\mathfrak{A} = \mathfrak{V} \times \mathfrak{Bl}\mathfrak{B},\\
&1' = \langle 1, 1'\rangle\,,\\
&\langle a, x\rangle^{\smile} = \langle a, x^{\smile}\rangle\,,\\
&\langle a, x\rangle\,;\,\langle b, y\rangle = \langle a \cdot b + \mathsf{d}(x \cdot y^{\smile}) + \mathsf{r}(x^{\smile} \cdot y),\ x\,;y + \pi_0(a) \cdot y + \pi_1(b) \cdot x\rangle\,.
\end{aligned}
$$

Theorem 4.2. *Let* $\mathfrak{B}$*,* $\mathfrak{V}$*,* π_0*,* π_1*,* d*, and* r *be as in Definition 4.1. Then for any* $x \in |\mathfrak{B}|$ *we have* $\mathsf{r}(x^{\smile\smile}) = \mathsf{d}(x^{\smile})$.

Proof. Let $a \in |\mathfrak{V}|$ be arbitrary. Then

$$
\begin{aligned}
\mathsf{r}(x^{\smile\smile}) \cdot a = 0 \quad &\text{iff} \quad \pi_1(a)^{\smile\smile} \cdot x^{\smile\smile} = 0 && \text{by Def.4.1}\\
&\text{iff} \quad \pi_0(a)^{\smile} \cdot x^{\smile\smile} = 0 && \text{by Def.4.1 and Th.1.3(v)}\\
&\text{iff} \quad \pi_0(a)^{\smile\smile} \cdot x^{\smile} = 0 && \text{by Th.1.3(ii)}\\
&\text{iff} \quad \mathsf{d}(x^{\smile}) \cdot a = 0 && \text{by Def.4.1}
\end{aligned}
$$

$\square$

Theorem 4.3. *Let* $\mathfrak{B}$*,* $\mathfrak{V}$*,* π_0*,* π_1*,* d*,* r*, and* $\mathfrak{A}$ *be as in Definition 4.1. Then* $\mathfrak{A} \in \mathsf{NREL}$.

Proof. Ax1. Obvious from Definition 4.1.

Ax2.

$$
\begin{aligned}
(\langle a, x\rangle \cdot \langle b, y\rangle^{\smile})^{\smile} &= (\langle a, x\rangle \cdot \langle b, y^{\smile}\rangle)^{\smile} && \text{by Def.4.1}\\
&= \langle a \cdot b, x \cdot y^{\smile}\rangle^{\smile} && \text{by Def.4.1}\\
&= \langle a \cdot b, (x \cdot y^{\smile})^{\smile}\rangle && \text{by Def.4.1}\\
&= \langle a \cdot b, x^{\smile} \cdot y\rangle && \text{by Ax2}\\
&= \langle a, x^{\smile}\rangle \cdot \langle b, y\rangle && \text{by Def.4.1}\\
&= \langle a, x\rangle^{\smile} \cdot \langle b, y\rangle && \text{by Def.4.1}
\end{aligned}
$$

Ax3.
$$\begin{aligned}
&(\langle a,x\rangle+\langle b,y\rangle)\,;\langle c,z\rangle\\
&=\langle a+b,x+y\rangle\,;\langle c,z\rangle\\
&\qquad\text{by Def.4.1}\\
&=\langle (a+b)\cdot c+\mathsf{d}((x+y)\cdot z^{\smile})+\mathsf{r}((x+y)^{\smile}\cdot z),\\
&\qquad (x+y);z+\pi_0(a+b)\cdot z+\pi_1(c)\cdot(x+y)\rangle\\
&\qquad\text{by Def.4.1}\\
&=\langle a\cdot c+\mathsf{d}(x\cdot z^{\smile})+\mathsf{r}(x^{\smile}\cdot z),x;z+\pi_0(a)\cdot z+\pi_1(c)\cdot x\rangle\\
&\quad+\langle b\cdot c+\mathsf{d}(y\cdot z^{\smile})+\mathsf{r}(y^{\smile}\cdot z),y;z+\pi_0(b)\cdot z+\pi_1(c)\cdot y\rangle\\
&\qquad\text{by Def.4.1, Ax3, Th.1.3(iii)}\\
&=\langle a,x\rangle\,;\langle c,z\rangle+\langle b,y\rangle\,;\langle c,z\rangle\\
&\qquad\text{by Def.4.1}
\end{aligned}$$

Likewise $\langle a,x\rangle\,;(\langle b,y\rangle+\langle c,z\rangle)=\langle a,x\rangle\,;\langle b,y\rangle+\langle a,x\rangle\,;\langle c,z\rangle$.

Ax4. By Definition 4.1, Ax4, and Theorem 1.3(i),

$$\langle 1,1\rangle\,;\langle 0,0\rangle=\langle 1\cdot 0+\mathsf{d}(1\cdot 0^{\smile})+\mathsf{r}(1^{\smile}\cdot 0),1;0+\pi_0(1)\cdot 0+\pi_1(0)\cdot 1\rangle=\langle 0,0\rangle\,.$$

Likewise $\langle 0,0\rangle\,;\langle 1,1\rangle=\langle 0,0\rangle$.

Ax5. By Definition 4.1, we have, for any $\langle a,x\rangle\,,\langle b,y\rangle\,,\langle c,z\rangle\in|\mathfrak{A}|$,

$$\begin{aligned}
&\langle a,x\rangle\,;\langle b,y\rangle\cdot\langle c,z\rangle=\\
&\quad=\langle a\cdot b\cdot c+\mathsf{d}(x\cdot y^{\smile})\cdot c+\mathsf{r}(x^{\smile}\cdot y)\cdot c,x;y\cdot z+\pi_0(a)\cdot y\cdot z+\pi_1(b)\cdot x\cdot z\rangle\,.
\end{aligned}$$

Thus, $\langle a,x\rangle\,;\langle b,y\rangle\cdot\langle c,z\rangle=\langle 0,0\rangle$ iff the following six conditions are satisfied:

(i) $a\cdot b\cdot c=0$,
(ii) $\mathsf{d}(x\cdot y^{\smile})\cdot c=0$,
(iii) $\mathsf{r}(x^{\smile}\cdot y)\cdot c=0$,
(iv) $x;y\cdot z=0$,
(v) $\pi_0(a)\cdot y\cdot z=0$,
(vi) $\pi_1(b)\cdot x\cdot z=0$.

Note that conditions (ii) and (iii) are equivalent by Theorem 4.2 since $(x\cdot y^{\smile})^{\smile}=x^{\smile}\cdot y$ by Ax2. Also, it will be convenient to note two other equivalent forms of (ii) and (iii) which follow from Definition 4.1 and Theorem 1.3(iii)(iv), $\pi_0(c)\cdot x\cdot y^{\smile}=0$ and $\pi_1(c)\cdot x^{\smile}\cdot y=0$, respectively. Now, $\langle a,x^{\smile}\rangle\,;\langle b,y\rangle\cdot\langle c,z\rangle=\langle 0,0\rangle$ is equivalent, by the above, to the following five conditions:

(i′) $a\cdot b\cdot c=0$,
(iii′) $\pi_1(c)\cdot x^{\smile\smile}\cdot y=0$,
(iv′) $x^{\smile};y\cdot z=0$,
(v′) $\pi_0(a)\cdot y\cdot z=0$,
(vi′) $\pi_1(b)\cdot x^{\smile}\cdot z=0$.

On the other hand, $\langle a,x^{\smile\smile}\rangle\,;\langle c,z\rangle\cdot\langle b,y\rangle=\langle 0,0\rangle$ iff all five of the following conditions hold:

(i″) $a\cdot c\cdot b=0$,

(iii″) $\pi_1(b) \cdot x^{\smile\smile\smile} \cdot z = 0$,
(iv″) $x^{\smile\smile} ; z \cdot y = 0$,
(v″) $\pi_0(a) \cdot z \cdot y = 0$,
(vi″) $\pi_1(c) \cdot x^{\smile\smile} \cdot y = 0$.

But (i′) iff (i″), (iii′) iff (vi″), (iv′) iff (iv″) by Ax5, (v′) iff (v″), and (vi′) iff (iii″) by Theorem 1.3(v). Thus $\langle a, x\rangle^{\smile} ; \langle b, y\rangle \cdot \langle c, z\rangle = \langle 0, 0\rangle$ iff $\langle a, x\rangle^{\smile\smile} ; \langle c, z\rangle \cdot \langle b, y\rangle = \langle 0, 0\rangle$. Likewise $\langle a, x\rangle ; \langle b, y\rangle^{\smile} \cdot \langle c, z\rangle = \langle 0, 0\rangle$ iff $\langle c, z\rangle ; \langle b, y\rangle^{\smile\smile} \cdot \langle a, x\rangle = \langle 0, 0\rangle$. Thus, by Theorem 1.7, Ax5 holds in $\mathfrak{A}$.

Ax6. $\langle 1, 1\text{'}\rangle ; \langle b, y\rangle \le \langle b, y\rangle$ iff $\langle 1, 1\text{'}\rangle ; \langle b, y\rangle \cdot \langle -b, -y\rangle = \langle 0, 0\rangle$ which, by the proof of Ax5, is equivalent to the conjunction of the following five conditions:

(i) $1 \cdot b \cdot -b = 0$,
(ii) $\pi_0(-b) \cdot 1\text{'} \cdot y^{\smile} = 0$,
(iv) $1\text{'} ; y \cdot -y = 0$,
(v) $\pi_0(1) \cdot y \cdot -y = 0$,
(vi) $\pi_1(b) \cdot 1\text{'} \cdot -y = 0$.

(i). Obvious.

(ii). By Definition 4.1, $\pi_0(-b) \cdot 1\text{'} \cdot y^{\smile} \le \pi_0(1) \cdot 1\text{'} = 0$.

(iv). Obvious.

(v). Obvious.

(vi). By Definition 4.1, $\pi_1(b) \cdot 1\text{'} \cdot -y \le \pi_1(1) \cdot 1\text{'} = 0$. Similarly, $\langle a, x\rangle ; \langle 1, 1\text{'}\rangle \le \langle a, x\rangle$.

Ax7. By Definition 4.1 and Ax7,

$$
\begin{aligned}
&\langle 1, 1\text{'}\rangle ; \langle 1, 1\text{'}\rangle \\
&= \langle 1 \cdot 1 + \mathsf{d}(1\text{'} \cdot 1\text{'}^{\smile}) + \mathsf{r}(1\text{'}^{\smile} \cdot 1\text{'}), 1\text{'} ; 1\text{'} + \pi_0(1) \cdot 1\text{'} + \pi_1(1) \cdot 1\text{'}\rangle \\
&= \langle 1, 1\text{'}\rangle .
\end{aligned}
$$

Ax8. $\langle 1, 1\rangle^{\smile} ; -\langle 1, 1\rangle^{\smile} \cdot \langle 1, 1\text{'}\rangle$

$= -\langle 1, 1^{\smile}\rangle ; -\langle 1, 1^{\smile}\rangle \cdot \langle 1, 1\text{'}\rangle$	by Def.4.1
$= \langle 0, -1^{\smile}\rangle ; \langle 0, -1^{\smile}\rangle \cdot \langle 1, 1\text{'}\rangle$	by Def.4.1
$= \langle 0 \cdot 0 + \mathsf{d}(-1^{\smile} \cdot (-1^{\smile})^{\smile}) + \mathsf{r}((-1^{\smile})^{\smile} \cdot -1^{\smile}),$ $-1^{\smile} ; -1^{\smile} + \pi_0(0) \cdot -1^{\smile} + \pi_1(0) \cdot -1^{\smile}\rangle \cdot \langle 1, 1\text{'}\rangle$	by Def.4.1
$= \langle 0 + \mathsf{d}(0) + \mathsf{r}(0), -1^{\smile} ; -1^{\smile}\rangle \cdot \langle 1, 1\text{'}\rangle$	by Def.4.1, Th.1.3(viii)(iv)
$= \langle 0, -1^{\smile} ; -1^{\smile}\rangle \cdot \langle 1, 1\text{'}\rangle$	by Def.4.1
$= \langle 0, -1^{\smile} ; -1^{\smile} \cdot 1\text{'}\rangle$	by Def.4.1
$= \langle 0, 0\rangle$	by Ax8 □

Theorem 4.4. *Let $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, r, and $\mathfrak{A}$ be as in Definition 4.1. Assume*

further that π_0 *and* π_1 *satisfy the following conditions for any* $a \in |\mathfrak{V}|$*:*

$$\begin{array}{ll} 1' ; \pi_0(1) = 0 & \pi_1(1) ; 1' = 0 \\ -\pi_0(1) ; 1 \leq -\pi_0(1) & 1 ; -\pi_1(1) \leq -\pi_1(1) \\ \pi_0(a) ; 1 \leq \pi_0(a) & 1 ; \pi_1(a) \leq \pi_1(a) \\ \pi_0(a) \cdot \pi_0(-a) = 0 & \pi_1(a) \cdot \pi_1(-a) = 0 \\ \pi_1(a) \cdot -\pi_0(-a) = 0 & -\pi_1(a) \cdot \pi_0(-a) = 0 \end{array}$$

If $\mathfrak{B} \in \mathsf{REL}$*, then* $\mathfrak{A} \in \mathsf{REL}$*.*

Remark 4.5. The equations $\pi_0(1) \cdot 1' = \pi_1(1) \cdot 1' = 0$ in Definition 4.1 are actually redundant in the hypothesis of Theorem 4.4. To prove $\pi_0(1) \cdot 1' = 0$, for example, we calculate

$$\begin{array}{rll} \pi_0(a) \cdot 1' &= 1' ; (\pi_0(1) \cdot 1') & \text{by Th.1.8(i)} \\ &\leq 1' ; \pi_0(1) & \text{by Ax3} \\ &= 0 & \text{by hyp.} \end{array}$$

Proof of Theorem 4.4. By Theorem 4.3, $\mathfrak{A} \in \mathsf{NREL}$. Thus, by Theorem 1.9, we need only prove the following six statements for any $\langle a, u\rangle \in |\mathfrak{A}|$ with $u \leq 1'$ in $\mathfrak{B}$:

(i) $\langle a, u\rangle ; \langle 1, 1\rangle ; \langle 1, 1\rangle \leq \langle a, u\rangle ; \langle 1, 1\rangle$,
(ii) $\langle a, u\rangle ; (-(\langle a, u\rangle ; \langle 1, 1\rangle) ; \langle 1, 1\rangle) = \langle 0, 0\rangle$,
(iii) $\langle 1, 1\rangle ; \langle a, u\rangle ; -(\langle u, a\rangle ; \langle 1, 1\rangle) = \langle 0, 0\rangle$,
(iv) $-(\langle 1, 1\rangle ; \langle a, u\rangle) ; (\langle a, u\rangle ; \langle 1, 1\rangle) = \langle 0, 0\rangle$,
(v) $\langle 1, 1\rangle ; -(\langle 1, 1\rangle ; \langle a, u\rangle) ; \langle a, u\rangle = \langle 0, 0\rangle$,
(vi) $\langle 1, 1\rangle ; (\langle 1, 1\rangle ; \langle a, u\rangle) \leq \langle 1, 1\rangle ; \langle a, u\rangle$.

First, we show that $\mathsf{d}(1' \cdot x) = \mathsf{r}(1' \cdot x) = 0$ for any $x \in |\mathfrak{B}|$:

$$\begin{array}{rlll} \mathsf{d}(1' \cdot x) \cdot 1 = 0 & \text{iff} & \pi_0(1)^{\smile\smile} \cdot 1' \cdot x = 0 & \text{by Def.4.1} \\ & \text{iff} & \pi_0(1) \cdot 1^{\smile} \cdot 1' \cdot x = 0 & \text{by Th.1.3(iv)} \end{array}$$

which is true by Definition 4.1 since $\pi_0(1) \cdot 1' = 0$. Likewise, $\mathsf{r}(1' \cdot x) = 0$.

Now we calculate

$$\begin{array}{rll} \langle a, u\rangle ; \langle 1, 1\rangle &= \langle a \cdot 1 + \mathsf{d}(u \cdot 1^{\smile}) + \mathsf{r}(u^{\smile} \cdot 1), & \\ & \quad u ; 1 + \pi_0(a) \cdot 1 + \pi_1(1) \cdot u\rangle & \text{by Def.4.1} \\ &= \langle a, u ; 1 + \pi_0(a)\rangle & \text{by Th.1.8(ii), Def.4.1} \end{array}$$

Similarly, $\langle 1, 1\rangle ; \langle a, u\rangle = \langle a, 1 ; u + \pi_1(a)\rangle$.

(i). The statement is equivalent to

$$\langle a, u ; 1 + \pi_0(a)\rangle ; \langle 1, 1\rangle \leq \langle a, u ; 1 + \pi_0(a)\rangle \,,$$

which, in turn, is equivalent to

$$\langle a, u ; 1 + \pi_0(a)\rangle ; \langle 1, 1\rangle \cdot \langle -a, -(u ; 1 + \pi_0(a))\rangle = \langle 0, 0\rangle \,.$$

By the proof of Theorem 4.3 (Ax5) we must prove the following conditions:

(i′) $a \cdot 1 \cdot -a = 0$,
(ii′) $\pi_0(-a) \cdot (u;1 + \pi_0(a)) \cdot 1^{\smile} = 0$,
(iv′) $u;1 + \pi_0(a));1 \cdot -(u;1 + \pi_0(a)) = 0$,
(v′) $\pi_0(a) \cdot 1 \cdot -(u;1 + \pi_0(a)) = 0$,
(vi′) $\pi_1(1) \cdot (u;1 + \pi_0(a)) \cdot -(u;1 + \pi_0(a)) = 0$.

(i′). Obvious.

(ii′). By hypothesis, $\pi_0(-a)\cdot\pi_0(a) = 0$. Thus we need only prove that $\pi_0(-a)\cdot u;1 = 0$. But by Theorems 1.7(i) and 1.8(ii), $\pi_0(-a) \cdot u;1 = 0$ iff $1 \cdot u;\pi_0(-a) = 0$, which holds since $u;\pi_0(-a) \leq 1\text{'};\pi_0(1) = 0$ by Definition 4.1, Ax3, and hypothesis.

(iv′). $(u;1 + \pi_0(a));1 = u;1;1 + \pi_0(a);1$ by Ax3
$\leq u;1 + \pi_0(a)$ by Th.1.9(i), hyp.

(v′). Obvious.

(vi′). Obvious.

(ii). The statement is equivalent to

$$\langle a, u\rangle\, ; (\langle -a, -(u;1 + \pi_0(a))\rangle\, ; \langle 1, 1\rangle) = \langle 0, 0\rangle$$

which, by Theorems 4.3, 1.7(i), and 1.8(ii), is equivalent to

$$\langle a, u\rangle\, ; \langle 1, 1\rangle \cdot \langle -a, -(u;1 + \pi_0(a))\rangle\, ; \langle 1, 1\rangle = \langle 0, 0\rangle\,,$$

which is equivalent to

$$\langle -a, -(u;1 + \pi_0(a)\rangle\, ; \langle 1, 1\rangle \cdot \langle a, u;1 + \pi_0(a)\rangle = \langle 0, 0\rangle\,.$$

By the proof of Theorem 4.3 (Ax5), we must prove the following five conditions:

(i′) $-a \cdot 1 \cdot a = 0$,
(ii′) $\pi_0(a) \cdot -(u;1 + \pi_0(a)) \cdot 1^{\smile} = 0$,
(iv′) $-(u;1 + \pi_0(a));1 \cdot (u;1 + \pi_0(a)) = 0$,
(v′) $\pi_0(-a) \cdot 1 \cdot (u;1 + \pi_0(a)) = 0$,
(vi′) $\pi_1(1) \cdot -(u;1 + \pi_0(a)) \cdot (u;1 + \pi_0(a)) = 0$.

(i′). Obvious.

(ii′). Obvious.

(iv′). By Ax3,

$$\begin{aligned} -(u;1 + \pi_0(a));1 \cdot (u;1 + \pi_0(a)) &= (-(u;1) \cdot -\pi_0(a));1 \cdot (u;1 + \pi_0(a)) \\ &\leq -(u;1);1 \cdot u;1 + -\pi_0(a);1 \cdot \pi_0(a). \end{aligned}$$

But, by Theorem 1.7(i)(ii), $-(u;1);1 \cdot u;1 = 0$ iff $1 \cdot u;(-(u;1);1) = 0$, which holds by Theorem 1.9(ii).

Thus, we need only show that $-\pi_0(a);1 = -\pi_0(a)$. Since $\pi_0(a) \cdot \pi_0(-a) = 0$ and $\pi_0(a) + \pi_0(-a) = \pi_0(1)$, we have $-\pi_0(a) = \pi_0(-a) + -\pi_0(1)$. But then

$$\begin{aligned} -\pi_0(a);1 &= \pi_0(-a);1 + -\pi_0(1);1 && \text{by Ax3} \\ &\leq \pi_0(-a) + -\pi_0(1) && \text{by hyp.} \\ &= -\pi_0(a) \end{aligned}$$

(v′). As in subcase (ii′) of case (i), we have $\pi_0(-a) \cdot u;1 = 0$. By hypothesis, we also have $\pi_0(-a) \cdot \pi_0(a) = 0$.

(vi′). Obvious.

(iii). The statement is equivalent to

$$\langle a, 1;u + \pi_1(a)\rangle \, ; \langle -a, -(u;1 + \pi_0(a))\rangle \cdot \langle 1, 1\rangle = \langle 0, 0\rangle \, .$$

By the proof of Theorem 4.3 (Ax5), we must prove the following five conditions:

(i′) $a \cdot -a \cdot 1 = 0$,
(iii′) $\pi_1(1) \cdot (1;u + \pi_1(a))^{\smile} \cdot -(u;1 + \pi_0(a)) = 0$,
(iv′) $(1;u + \pi_1(a)); -(u;1 + \pi_0(a)) \cdot 1 = 0$,
(v′) $\pi_0(a) \cdot -(u;1 + \pi_0(a)) \cdot 1 = 0$,
(vi′) $\pi_1(-a) \cdot (1;u + \pi_1(a)) \cdot 1 = 0$.

(i′). Obvious.

(iii′).
$$\begin{aligned} (1;u + \pi_1(a))^{\smile} &= (1;u)^{\smile} + \pi_1(a)^{\smile} && \text{by Th.1.3(iii)} \\ &= (u;1)^{\smile\smile} + \pi_0(a)^{\smile\smile} && \text{by Th.1.8(ix), hyp., Th.1.3(v)} \\ &\leq u;1 + \pi_0(a) && \text{by Th.1.3(iv)} \end{aligned}$$

(iv′).
$$\begin{aligned} &(1;u + \pi_1(a)); -(u;1 + \pi_0(a)) \cdot 1 \\ &= 1;u;(-(u;1) \cdot -\pi_0(a)) + \pi_1(a);(-(u;1) \cdot -\pi_0(a)) && \text{by Ax3} \\ &\leq 1;u; -(u;1) + \pi_1(a); -\pi_0(a) && \text{by Ax3} \\ &= 0 + 0 && \text{by Th.1.9(iii), hyp.} \end{aligned}$$

(v′). Obvious.

(vi′).
$$\begin{aligned} \pi_1(-a) \cdot (1;u + \pi_1(a)) \cdot 1 &= \pi_1(-a) \cdot 1;u + \pi_1(-a) \cdot \pi(a) \\ &= \pi_1(-a) \cdot 1;u + 0 && \text{by hyp.} \end{aligned}$$

Thus, we need only show that $\pi_1(-a) \cdot 1;u = 0$. Bu this is similar to the proof of $\pi_0(-a) \cdot u;1 = 0$ in subcase (ii′) of case (i).

(iv). Similar to (iii).

(v). Similar to (ii).

(vi). Similar to (i).

Thus $\mathfrak{A} \in \mathsf{REL}$ as desired. □

Theorem 4.5. *Let* $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, r, *and* $\mathfrak{A}$ *be as in Definition 4.1. Then the map* $\varphi : |\mathfrak{B}| \to |\mathfrak{Rl}_{\langle 0,1\rangle}\mathfrak{A}|$ *defined by* $\varphi(x) = \langle 0, x\rangle$ *is an isomorphism:*

$$\varphi : \mathfrak{B} \cong \mathfrak{Rl}_{\langle 0,1\rangle}\mathfrak{A}.$$

Proof. Clearly φ preserves the Boolean operations by Definition 4.1. Also, the bijectivity of φ is clear from Definition 4.1. By Definition 4.1 we also have

$$\begin{aligned}
\varphi(1') &= \langle 0, 1'\rangle \\
&= \langle 1, 1'\rangle \cdot \langle 0, 1\rangle \\
&= 1' \cdot \langle 0, 1\rangle \\
\varphi(x^{\smile}) &= \langle 0, x^{\smile}\rangle \\
&= \langle 0, x^{\smile}\rangle \cdot \langle 0, 1\rangle \\
&= \langle 0, x\rangle^{\smile} \cdot \langle 0, 1\rangle \\
&= \varphi(x)^{\smile} \cdot \langle 0, 1\rangle
\end{aligned}$$

Using Definition 4.1 and the fact that π_0 and π_1 are normal, we have

$$\begin{aligned}
\varphi(x;y) &= \langle 0, x;y\rangle \\
&= \langle 0, x;y + \pi_0(0)\cdot y + \pi_1(0)\cdot x\rangle \\
&= \langle 0\cdot 0 + \mathsf{d}(x\cdot y^{\smile}) + \mathsf{r}(x^{\smile}\cdot y), x;y + \pi_0(0)\cdot y + \pi_1(0)\cdot x\rangle \cdot \langle 0, 1\rangle \\
&= \langle 0, x\rangle \,;\, \langle 0, y\rangle \cdot \langle 0, 1\rangle \\
&= \varphi(x);\varphi(y)\cdot \langle 0, 1\rangle
\end{aligned}$$

Thus $\varphi : \mathfrak{B} \cong \mathfrak{Rl}_{\langle 0,1\rangle}\mathfrak{A}$. □

Theorem 4.6. *Let* $\mathfrak{A}' \in \mathsf{NREL}$ *and let* $r \in |\mathfrak{A}'|$ *satisfy* $-r \leq 1'$. *We define* $\mathfrak{B}$ *and* $\mathfrak{V}$ *by*

$$\begin{aligned}
\mathfrak{B} &= \mathfrak{Rl}_r\mathfrak{A}', \\
\mathfrak{V} &= \mathfrak{Bl}(\mathfrak{Rl}_{-r}\mathfrak{A}').
\end{aligned}$$

We define $\pi_0, \pi_1 : |\mathfrak{V}| \to |\mathfrak{B}|$ *by*

$$\begin{aligned}
\pi_0(a) &= a;r, \\
\pi_1(a) &= r;a,
\end{aligned}$$

where ; *is taken in* $\mathfrak{A}'$. *We also define* $\mathsf{d}, \mathsf{r} : |\mathfrak{Rl}_{1^{\smile}}\mathfrak{B}| \to |\mathfrak{V}|$ *by*

$$\begin{aligned}
\mathsf{d}(x) &= x;x^{\smile}\cdot -r, \\
\mathsf{r}(x) &= x^{\smile};x\cdot -r,
\end{aligned}$$

where $-$, $^{\smile}$, *and* ; *are taken in* $\mathfrak{A}'$. *Then* $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, *and* r *satisfy all the conditions of Definition 4.1. Let* $\mathfrak{A}$ *be the algebra constructed in Definition 4.1. Then the map* $\Theta : |\mathfrak{A}| \to |\mathfrak{A}'|$ *defined by* $\Theta\langle a, x\rangle = a + x$ *is an isomorphism:*

$$\Theta : \mathfrak{A} \cong \mathfrak{A}'.$$

Proof. First note that

$$\begin{aligned}\pi_0(x) &= a;x \\ &\leq 1';r && \text{by Ax3 and } a \leq -r \leq 1' \\ &\leq r && \text{by Ax6}\end{aligned}$$

so that $\pi_0(a) \in |\mathfrak{B}|$ for any $a \in |\mathfrak{V}|$. Similarly, $\pi_1(a) \in |\mathfrak{B}|$ for any $a \in |\mathfrak{V}|$. π_0 and π_1 are both normal and additive by Ax3 and Ax4. Note that

$$\begin{aligned}r^{\smile} \cdot -r &= r^{\smile} \cdot 1' \cdot -r \\ &= (r \cdot 1'^{\smile})^{\smile} \cdot -r && \text{by Ax2} \\ &= r \cdot 1' \cdot -r && \text{by Th.1.8(ii)} \\ &= 0\end{aligned}$$

Thus $r^{\smile} \leq r$. Now, we calculate, for $a \in |\mathfrak{V}|$,

$$\begin{aligned}\pi_0(a)^{\smile r \smile r} &= ((a;r)^{\smile} \cdot r)^{\smile} \cdot r \\ &= ((a;1 \cdot r)^{\smile} \cdot r)^{\smile} \cdot r && \text{by Th.1.8(iii)} \\ &= ((a;1)^{\smile} \cdot r^{\smile} \cdot r)^{\smile} \cdot r && \text{by Th.1.3(vi)} \\ &= (((a;1)^{\smile} \cdot r) \cdot r^{\smile})^{\smile} \cdot r \\ &= ((a;1)^{\smile} \cdot r)^{\smile} \cdot r \cdot r && \text{by Ax2} \\ &= (a;1)^{\smile\smile} \cdot r^{\smile} \cdot r \\ &= (1;a)^{\smile\smile\smile} \cdot r^{\smile} \cdot r && \text{by Th.1.8(ix)} \\ &= (1;a)^{\smile} \cdot r^{\smile} \cdot r && \text{by Th.1.3(v)} \\ &= (1;a \cdot r)^{\smile} \cdot r && \text{by Th.1.3(vi)} \\ &= (r;a)^{\smile} \cdot r && \text{by Th.1.8(iv)} \\ &= \pi_1(a)^{\smile r}\end{aligned}$$

By Definition 2.1, $1'_r \cdot \pi_0(-r) = 1' \cdot r \cdot -r;r = 0$. Likewise, $1'_r \cdot \pi_1(-r) = 0$. Now, let $a \in |\mathfrak{V}|$ and $x \in |\mathfrak{Rl}_{1^{\smile}}\mathfrak{B}| = |\mathfrak{Rl}_{r^{\smile}r}\mathfrak{A}|$. Note that $r^{\smile r} = r^{\smile} \cdot r = r^{\smile}$ since $r^{\smile} \leq r$. Thus $x \leq r^{\smile} \leq 1^{\smile}$ by Theorem 1.3(iii), so that $x^{\smile\smile} = x \cdot 1^{\smile} = x$ by Theorem 1.3(iv). Note also that $a \leq -r \leq 1'$. Thus

$$\begin{aligned}\mathsf{d}(x) \cdot a = 0 \quad &\text{iff} \quad x;x^{\smile} \cdot -r \cdot a = 0 \\ &\text{iff} \quad x;x^{\smile} \cdot a = 0 \\ &\text{iff} \quad a;x^{\smile\smile} \cdot x = 0 \\ &\text{iff} \quad a;x \cdot x = 0 \\ &\text{iff} \quad a;1 \cdot x \cdot x = 0 && \text{by Th.1.8(iii)} \\ &\text{iff} \quad a;1 \cdot r \cdot x = 0 \\ &\text{iff} \quad a;r \cdot x = 0 && \text{by Th.1.8(iii)} \\ &\text{iff} \quad \pi_0(a) \cdot x = 0 \\ &\text{iff} \quad \pi_0(a) \cdot x^{\smile\smile} = 0 \\ &\text{iff} \quad \pi_0(a)^{\smile\smile} \cdot x = 0 && \text{by Th.1.3(ii)}\end{aligned}$$

Similarly, r is conjugate to $\pi_1(\text{-})^{\smile\smile}$. Thus $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, and r satisfy all of the conditions of Definition 4.1.

Now, let $\mathfrak{A}$ be the algebra constructed in Definition 4.1 and let $\Theta : |\mathfrak{A}| \to |\mathfrak{A}'|$ be defined by $\Theta\langle a, x\rangle = a + x$. Then we have

$$\begin{aligned}
\Theta\langle 0,0\rangle &= 0+0=0, \\
\Theta\langle -r,r\rangle &= -r+r=1, \\
\Theta(\langle a,x\rangle + \langle b,y\rangle) &= \Theta\langle a+b, x+y\rangle && \text{by Def.4.1} \\
&= a+b+x+y \\
&= a+x+b+y \\
&= \Theta\langle a,x\rangle + \Theta\langle b,y\rangle, \\
\Theta(\langle a,x\rangle \cdot \langle b,y\rangle) &= \Theta\langle a\cdot b, x\cdot y\rangle && \text{by Def.4.1} \\
&= a\cdot b + x\cdot y \\
&= a\cdot b + x\cdot y + a\cdot y + x\cdot b && \text{since } a,b \le -r \text{ and } x,y \le r \\
&= (a+x)\cdot(b+y) \\
&= \Theta\langle a,x\rangle \cdot \Theta\langle b,y\rangle, \\
\Theta(-\langle a,x\rangle) &= \Theta\langle -a\cdot -r, -x\cdot r\rangle && \text{by Def.4.1} \\
&= -a\cdot -r + -x\cdot r \\
&= -(a+r) + -(x+-r) \\
&= -((a+r)\cdot(x+-r)) \\
&= -(a\cdot x + a\cdot -r + r\cdot x + r\cdot -r) \\
&= -(0+a+x+0) \\
&= -(a+x) \\
&= -\Theta\langle a,x\rangle, \\
\Theta\langle -r, 1\text{'}\cdot r\rangle &= -r + 1\text{'}\cdot r = 1\text{'} && \text{since } -s \le 1\text{'}, \\
\Theta(\langle a,x\rangle^{\smile}) &= \Theta\langle a, x^{\smile}\cdot r\rangle && \text{by Def.4.1} \\
&= a + x^{\smile}\cdot r \\
&= a + x^{\smile} && \text{since } x^{\smile} \le r^{\smile} \le r \\
&= a^{\smile} + x^{\smile} && \text{by Th.1.8(ii), since } a \le -r \le 1\text{'} \\
&= (a+x)^{\smile} && \text{by Th.1.3(iii)} \\
&= (\Theta\langle a,x\rangle)^{\smile}
\end{aligned}$$

$$\begin{aligned}
&\Theta(\langle a,x\rangle \,;\, \langle b,y\rangle) \\
&= \Theta\langle a\cdot b + \mathsf{d}(x\cdot y^{\smile}) + \mathsf{r}(x^{\smile}\cdot y), x;y + \pi_0(a)\cdot y + \pi_1(b)\cdot x\rangle && \text{by Def.4.1} \\
&= a\cdot b + \mathsf{d}(x\cdot y^{\smile}) + \mathsf{r}(x^{\smile}\cdot y) + x;y + \pi_0(a)\cdot y + \pi_1(b)\cdot x \\
&= a\cdot b + \mathsf{d}(x\cdot y^{\smile}) + x;y + \pi_0(a)\cdot y + \pi_1(b)\cdot x \\
&\qquad \text{by Th.4.2, since } (x\cdot y^{\smile})^{\smile} = (x^{\smile}\cdot y)^{\smile\smile} \\
&= a\cdot b + (x\cdot y^{\smile});(x\cdot y^{\smile})\cdot -r + x;y + a;r\cdot y + r;b\cdot x
\end{aligned}$$

$= a \cdot b + (x \cdot y^{\smile});(x^{\smile} \cdot y) \cdot -r + x;y + a;r \cdot y + r;b \cdot x$	by Ax2
$= a \cdot b + x;y + a;r \cdot y + r;b \cdot x$	by Ax3
$= a \cdot b + x;y + a;1 \cdot r \cdot y + 1;b \cdot r \cdot x$	by Th.1.8(iii)(iv)
$= a \cdot b + x;y + a;1 \cdot y + 1;b \cdot x$	since $x, y \leq r$
$= a \cdot b + x;y + a;y + x;b$	by Th.1.8(iii)(iv)
$= a;b + x;y + a;y + x;b$	by Th.1.8(i)
$= (a + x);(b + y)$	by Ax3
$= \Theta \langle a, x \rangle ; \Theta \langle b, y \rangle$	

Since Θ is clearly bijective, the theorem follows. □

Theorem 4.7. *Let r, $\mathfrak{A}'$, $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, and r be as in Theorem 4.6 and assume that $\mathfrak{A}' \in \mathsf{REL}$. Then the conditions of Theorem 4.4 are satisfied.*

Proof. Recalling that $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{A}'$, $\mathfrak{V} = \mathfrak{Bl}(\mathfrak{Rl}_{-r}\mathfrak{A}')$, $\pi_0(a) = a;r$, and $\pi_1(a) = r;a$, we can rewrite the conditions of Theorem 4.4 as follows (here we assume $a \leq -r$ and all operations are taken in $\mathfrak{A}'$):

$(1' \cdot r);(-r;r) \cdot r = 0$	$r; -r;(1' \cdot r) \cdot r = 0$
$(-(-r;r) \cdot r);r \cdot r \leq -(-r;r) \cdot r$	$r;(-(r; -r) \cdot r) \cdot r \leq -(r; -r) \cdot r$
$a;r;r \cdot r \leq a;r$	$r;(r;a) \cdot r \leq r;a$
$a;r \cdot (-a \cdot -r);r = 0$	$r;a \cdot r;(-a \cdot -r) = 0$
$r;a;(-(a;r) \cdot r) \cdot r = 0$	$(-(r;a) \cdot r);(a;r) = 0$

We will prove the five equations on the left. The five equations on the right will follow by symmetry. For the first equation, we calculate

$$\begin{aligned}(1' \cdot r);(-r;r) \cdot r &= (1' \cdot r); -r;r \cdot r && \text{by Ax9}\\ &= (1' \cdot r \cdot -r);r \cdot r && \text{by Th.1.8(i), since } -r \leq 1'\\ &= 0;r \cdot r\\ &= 0 && \text{by Ax4}\end{aligned}$$

Before proving the second equation on the left, we first note that for any $a \leq -r$, we have

$$\begin{aligned}a;1 &= a;(r + -r)\\ &= a;r + a; -r && \text{by Ax3}\\ &= a;r + a && \text{by Th.1.8(i)}\end{aligned}$$

Thus $-(a;1) = -(a;r + a) = -(a;r) \cdot -a$. In particular, $-(-r;1) = -(-r;r) \cdot r$. With this in mind, we see that the second equation on the left reduces to

$$-(-r;1);r \cdot r \leq -(-r;1),$$

which in turn is equivalent to

$$-(-r;1);r \cdot r \cdot -r;1 = 0.$$

By Theorems 1.7(i) and 1.8(ii), this is equivalent to

$$-r;(-(-r;1);r \cdot r) \cdot 1 = 0.$$

But

$$\begin{aligned} -r;(-(-r;1);r \cdot r) \cdot 1 &\le -r;(-(-r;1);r) && \text{by Ax3} \\ &= -r; -(-r;1);r && \text{by Ax9} \\ &= 0;r && \text{by Th.1.8(vii)} \\ &= 0 && \text{by Ax3 and Ax4} \end{aligned}$$

Thus, we have proven the second equation on the left.

For the third equation on the left, we have (recall that $a \le -r \le 1'$)

$$\begin{aligned} a;r;r \cdot r \le a;r \quad &\text{iff} \quad a;r;r \cdot r \cdot -(a;r) = 0 \\ &\text{iff} \quad a;(r;r) \cdot r \cdot -(a;r) = 0 && \text{by Ax9} \\ &\text{iff} \quad a;(r \cdot -(a;r)) \cdot r;r = 0 && \text{by Th.1.7(i), Th.1.8(ii)} \end{aligned}$$

But

$$\begin{aligned} a;(r \cdot -(a;r)) \cdot r;r &\le a;(-a \cdot -(a;r)) && \text{by Ax3, since } r \le -a \\ &= a; -(a;1) \\ &= 0 && \text{by Th.1.8(vii)} \end{aligned}$$

For the fourth equation on the left, we have

$$\begin{aligned} a;r \cdot (-a \cdot -r);r = 0 \quad &\text{iff} \quad a;((-a \cdot -r);r) \cdot r = 0 && \text{by Th.1.7(i), Th.1.8(ii)} \\ &\text{iff} \quad a;(-a \cdot -r);r \cdot r = 0 && \text{by Ax9} \\ &\text{iff} \quad (a \cdot -a \cdot -r);r \cdot r = 0 && \text{by Th.1.8(i)} \\ &\text{iff} \quad 0;r \cdot r = 0 \\ &\text{iff} \quad 0 = 0 && \text{by Ax3 and Ax4} \end{aligned}$$

For the fifth equation on the left, we have

$$\begin{aligned} r;a;(-(a;r) \cdot r) \cdot r &\le r;a;(-(a;r) \cdot -a) && \text{by Ax3, since } r \le -a \\ &= r;a; -(a;1) \\ &= r;(a; -(a;1)) && \text{by Ax9} \\ &= r;0 && \text{by Th.1.8(vii)} \\ &= 0 && \text{by Ax3 and Ax4} \end{aligned}$$

The five equations on the right are proved similarly. $\square$

Theorem 4.8. *Let* $\mathfrak{B} \in \mathsf{NREL}$ *and let* $p_0, p_1 \in |\mathfrak{B}|$ *satisfy* $p_0^{\smile} = p_1^{\smile\smile}$ *and* $1' \cdot p_0 = 1' \cdot p_1 = 0$. *Then there exists an* $\mathfrak{A} \in \mathsf{NREL}$ *(unique up to isomorphism) such that*

(i) $\mathfrak{B} = \mathfrak{Rl}_{-w}\mathfrak{A}$ *for some* $w \leq 1'$ *with* w *an atom of* $\mathfrak{A}$,
(ii) $w;1 \cdot -w = p_0$ *holds in* $\mathfrak{A}$,
(iii) $1;w \cdot -w = p_1$ *holds in* $\mathfrak{A}$.

Proof. Let $\mathfrak{V}$ be a two-element Boolean algebra with $|\mathfrak{V}| = \{0,1\}$. We define $\pi_0, \pi_1 : |\mathfrak{V}| \to |\mathfrak{B}|$ by

$$\pi_0(0) = \pi_1(0) = 0,$$
$$\pi_0(1) = p_0,$$
$$\pi_1(1) = p_1.$$

Then π_0 and π_1 are completely additive and normal. By Theorem 1.3(i)(iii), the maps

$$\pi_0(\text{-})^{\smile\smile}, \pi_1(\text{-})^{\smile\smile} : |\mathfrak{V}| \to |\mathfrak{Rl}_{1^{\smile}}\mathfrak{B}|$$

are also completely additive and normal. Also, $\mathfrak{V}$ is complete since it is finite. Thus, by Theorem 1.14 of Jónsson-Tarski [1] the maps $\pi_0(\text{-})^{\smile\smile}$ and $\pi_1(\text{-})^{\smile\smile}$ have conjugates

$$\mathsf{d}, \mathsf{r} : |\mathfrak{Rl}_{1^{\smile}}\mathfrak{B}| \to |\mathfrak{V}|$$

respectively.

Note that $\pi_0(0)^{\smile\smile} = 0 = \pi_1(0)^{\smile}$ by Theorem 1.3(i) and $\pi_0(1)^{\smile\smile} = p_0^{\smile\smile} = p_1^{\smile} = \pi_1(1)^{\smile}$ by hypothesis. Also, $1' \cdot \pi_0(1) = 1' \cdot p_0 = 0$ and $1' \cdot \pi_1(1) = 1' \cdot p_1 = 0$ by hypothesis. Thus $\mathfrak{B}$, $\mathfrak{V}$, π_0, π_1, d, and r are as in Definition 4.1. Let $w = \langle 1, 0\rangle \in |\mathfrak{A}|$, where $\mathfrak{A}$ is as in Definition 4.1. Then $w \leq \langle 1, 1'\rangle = 1'$ is an atom of $\mathfrak{A}$. By Theorem 4.3, $\mathfrak{A} \in \mathsf{NREL}$. By Theorem 4.5, the map $\varphi : |\mathfrak{B}| \to |\mathfrak{Rl}_{\langle 0,1\rangle}\mathfrak{A}|$ defined by $\varphi(x) = \langle 0, x\rangle$ is an isomorphism $\mathfrak{B} \cong \mathfrak{Rl}_{\langle 0,1\rangle}\mathfrak{A} = \mathfrak{Rl}_{-w}\mathfrak{A}$. Now we calculate (in $\mathfrak{A}$)

$$\begin{aligned} w;1 &= \langle 1, 0\rangle \,;\, \langle 1, 1\rangle \\ &= \langle 1 \cdot 1 + \mathsf{d}(0 \cdot 1^{\smile}) + \mathsf{r}(0^{\smile} \cdot 1), 0;1 + \pi_0(1) + \pi_1(1) \cdot 0\rangle && \text{by Def.4.1} \\ &= \langle 1, p_0\rangle && \text{by Th.1.3(i), Ax4, normality of } \mathsf{d}, \mathsf{r} \end{aligned}$$

Similarly, $1;w = \langle 1, p_1\rangle$. Thus

$$\begin{aligned} w;1 \cdot -w &= \langle 1, p_0\rangle \cdot -\langle 1, 0\rangle \\ &= \langle 1, p_0\rangle \cdot \langle 0, 1\rangle && \text{by Def.4.1} \\ &= \langle 0, p_0\rangle && \text{by Def.4.1} \\ &= \varphi(p_0) \end{aligned}$$

Similarly, $1;w \cdot -w = \varphi(p_1)$.

To prove the uniqueness statement, we must express converse and relative product in $\mathfrak{A}$ in terms of p_0, p_1, w, and the operations of $\mathfrak{Rl}_{-w}\mathfrak{A}$. Since $w \leq 1'$, we have $w^{\smile} = w$ by Theorem 1.8(i). Also, for $x \leq -w$ we have

$$\begin{aligned} x^{\smile} \cdot w &= (x \cdot w^{\smile})^{\smile} && \text{by Ax2} \\ &= (x \cdot w)^{\smile} && \text{since } w^{\smile} = w \\ &= 0^{\smile} && \text{since } x \leq -w \\ &= 0 && \text{by Th.1.3(i)} \end{aligned}$$

Thus $x^{\smile} = x^{\smile} \cdot w + x^{\smile} \cdot -w = x^{\smile} \cdot -w = x^{\smile(-w)}$ by Definition 2.1. Thus converse in $\mathfrak{A}$ is uniquely determined. Now $w;w = w \cdot w = w$ by Th.1.8(i), and, for $x \leq -w$, we have

$$\begin{aligned} w;x &= w;1 \cdot x && \text{by Th.1.8(iii)} \\ &= w;1 \cdot -w \cdot x && \text{since } x \leq -w \\ &= p_0 \cdot x && \text{by (ii)} \end{aligned}$$

Similarly, $x;w = p_1 \cdot x$.

Finally, let $x, y \leq -w$. Then $x;y = x;y \cdot w + x;y \cdot -w = x;y \cdot w + x;^{-w}y$ by Definition 2.1.

To finish the uniqueness proof, we need only determine whether $x;y \cdot w$ is 0 or w (since w is an atom). But

$$\begin{aligned} x;y \cdot w &= x;y \cdot 1' \cdot w && \text{since } w \leq 1' \\ &= y^{\smile};x^{\smile} \cdot 1' \cdot w && \text{by Th.1.8(vi)} \\ &= y^{\smile};x^{\smile} \cdot w && \text{since } w \leq 1' \end{aligned}$$

Thus

$$\begin{aligned} x;y \cdot w = 0 \quad &\text{iff} \quad y^{\smile};x^{\smile} \cdot w = 0 \\ &\text{iff} \quad y^{\smile\smile};w \cdot x^{\smile} = 0 && \text{by Th.1.7(i)} \\ &\text{iff} \quad p_1 \cdot y^{\smile\smile} \cdot x^{\smile} = 0 && \text{by above} \\ &\text{iff} \quad p_1 \cdot y^{\smile(-w)\smile(-w)} \cdot x^{\smile(-w)} = 0 && \text{by above} \quad \square \end{aligned}$$

Theorem 4.9. *Let* $\mathfrak{B} \in \mathsf{REL}$ *and let* $p_0, p_1 \in |\mathfrak{B}|$ *satisfy* $p_0^{\smile\smile} = p_1^{\smile}$ *as well as the following equations*

$$\begin{aligned} 1';p_0 &= 0 & p_1;1' &= 0 \\ -p_0;1 &\leq -p_0 & 1;-p_1 &\leq -p_1 \\ p_0;1 &\leq p_0 & 1;p_1 &\leq p_1 \\ p_1;-p_0 &= 0 & -p_1;p_0 &= 0 \end{aligned}$$

Then there exists an $\mathfrak{A} \in \mathsf{REL}$ *(unique up to isomorphism) such that*

(i) $\mathfrak{B} = \mathfrak{Rl}_{-w}\mathfrak{A}$ *for some* $w \leq 1'$ *with* w *an atom of* $\mathfrak{A}$*,*
(ii) $w;1 \cdot -w = p_0$ *holds in* $\mathfrak{A}$*,*
(iii) $1;w \cdot -w = p_1$ *holds in* $\mathfrak{A}$*.*

Proof. First note that

$$\begin{aligned} 1' \cdot p_0 &= 1';(1' \cdot p_0) && \text{by Th.1.8(i)} \\ &\leq 1' \cdot p_0 && \text{by Ax3} \\ &= 0 && \text{by hypothesis} \end{aligned}$$

Similarly $1' \cdot p_1 = 0$. Thus, the hypotheses of Theorem 4.8 are satisfied. To finish the proof we need only show that the algebra $\mathfrak{A}$ of Theorem 4.8 is in REL.

Let $\mathfrak{V}$, π_0, π_1, d, and r be as in the proof of Theorem 4.8. By Theorem 4.1 we need only show that following ten equations. (Here all operations are taken in $\mathfrak{B}$ or $\mathfrak{V}$.)

$$
\begin{array}{ll}
1' ; \pi_0(1) = 0 & \pi_1(1) ; 1' = 0 \\
-\pi_0(1) ; 1 \leq -\pi_0(1) & 1 ; -\pi_1(1) \leq -\pi_1(1) \\
\pi_0(a) ; 1 \leq \pi_0(a) & 1 ; \pi_1(a) \leq \pi_1(a) \\
\pi_0(a) \cdot \pi_0(-a) = 0 & \pi_1(a) \cdot \pi_1(-a) = 0 \\
\pi_1(a) ; -\pi_0(a) = 0 & -\pi_1(a) ; \pi_0(a) = 0
\end{array}
$$

We will prove the five equations on the left. The five equations on the right follow by symmetry.

$$
\begin{array}{rll}
1' ; \pi_0(1) &= 1' ; p_0 & \text{by definition} \\
&= 0 & \text{by hypothesis} \\
-\pi_0(1) ; 1 &= -p_0 ; 1 & \text{by definition} \\
&\leq -p_0 & \text{by hypothesis} \\
&= -\pi_0(1) & \text{by definition} \\
-\pi_0(0) ; 1 &= 0 ; 1 & \text{by definition} \\
&= 0 & \text{by Ax4} \\
&= \pi_0(0) & \text{by definition} \\
-\pi_0(1) ; 1 &= p_0 ; 1 & \text{by definition} \\
&\leq p_0 & \text{by hypothesis} \\
&= \pi_0(1) & \text{by definition} \\
\pi_0(0) \cdot \pi_0(1) &= 0 \cdot p_0 & \text{by definition} \\
&= 0 & \\
& - \pi_1(0) ; -\pi_0(0) & \\
&= 0 ; 1 & \text{by definition} \\
&= 0 & \text{by Ax4} \\
\pi_1(1) ; -\pi_0(1) &= p_1 ; -p_0 & \text{by definition} \\
&= 0 & \text{by hypothesis} \quad \square
\end{array}
$$

Theorem 4.10. *Let* $\mathfrak{B} \in \mathsf{NREL}$. *Then there exists an* $\mathfrak{A} \in \mathsf{NREL}$ *(unique up to isomorphism) such that*

(i) $\mathfrak{B} = \mathfrak{Rl}_{-w}\mathfrak{A}$ *for some* $w \leq 1'$ *with* w *an atom of* $\mathfrak{A}$*,*
(ii) $1' ; 1 = 1 ; 1' = 1$ *holds in* $\mathfrak{A}$*,*
(iii) $w ; 1 \cdot (1' \cdot -w) ; 1 = 0$ *holds in* $\mathfrak{A}$*.*

Proof. Let $p_0 = -(1\text{'};1)$ and $p_1 = -(1;1\text{'})$. (Here all operations and constants are taken in $\mathfrak{B}$.) Then

$$\begin{aligned}
1\text{'} \cdot p_0 &= 1\text{'} \cdot -(1\text{'};1) \\
&= 1\text{'};(1\text{'} \cdot -(1\text{'};1)) && \text{by Th.1.8(i)} \\
&\leq 1\text{'};-(1\text{'};1) && \text{by Ax3} \\
&= 0 && \text{by Th.1.8(vii)}
\end{aligned}$$

Similarly $1\text{'} \cdot p_1 = 0$.

Also

$$\begin{aligned}
p_1^{\smile\smile} &= (-(1;1\text{'}))^{\smile\smile} \\
&= (-(1;1\text{'})^{\smile} \cdot 1^{\smile})^{\smile} && \text{by Th.1.3(viii)} \\
&= (-(1;1\text{'})^{\smile})^{\smile} \cdot 1^{\smile\smile} && \text{by Th.1.3(vi)} \\
&= (-(1;1\text{'})^{\smile})^{\smile} && \text{by Th.1.3(iii)(iv)} \\
&= -(1;1\text{'})^{\smile\smile} \cdot 1^{\smile} && \text{by Th.1.3(viii)} \\
&= -(1\text{'};1)^{\smile} \cdot 1^{\smile} && \text{by Th.1.8(ix)} \\
&= (-(1\text{'};1))^{\smile} && \text{by Th.1.3(viii)} \\
&= p_0^{\smile}
\end{aligned}$$

Thus, the hypotheses of Theorem 4.8 are satisfied. Now, we calculate in $\mathfrak{A}$ and use Theorem 4.8.

$$\begin{aligned}
1\text{'};1 \cdot w &= 1\text{'};w && \text{by Th.1.8(iii)} \\
&= w;w + (1\text{'} \cdot -w);w && \text{by Ax3} \\
&= w \cdot w + 1\text{'} \cdot -w \cdot w && \text{Th.1.8(i)} \\
&= w \\
1\text{'};1 \cdot w &= w;1 \cdot -w + (1\text{'} \cdot -w);1 \cdot -w && \text{by Ax3} \\
&= w;1 \cdot -w + (1\text{'} \cdot -w);w \cdot -w + \\
&\quad + (1\text{'} \cdot -w);-w \cdot -w && \text{by Ax3} \\
&= p_0 + 1\text{'} \cdot -w \cdot w \cdot -w + 1\text{'}_{-w};^{-w} - w && \text{by Th.4.8(ii), Th.1.8(i), Def.2.1} \\
&= p_0 + 0 + ({}_{(-w)} - p_0) && \text{by definition of } p_0 \\
&= p_0 + -p_0 \cdot -w && \text{by Def.2.1} \\
&= -w && \text{since } p_0 \leq -w
\end{aligned}$$

Thus $1\text{'};1 = 1$. Similarly $1;1\text{'} = 1$. Next, we note that

$$\begin{aligned}
w;1 \cdot (1\text{'} \cdot -w);1 = 0 \quad &\text{iff} \quad 1 \cdot (1\text{'} \cdot -w);(w;1) = 0 && \text{by Th.1.7(i), Th.1.8(ii)} \\
&\text{iff} \quad (1\text{'} \cdot -w);(w;1) = 0
\end{aligned}$$

But

$$\begin{aligned}(1\text{'} \cdot -w);(w;1) &= (1\text{'} \cdot -w);(w;w) + (1\text{'} \cdot -w);(w;-w) && \text{by Ax3}\\ &= (1\text{'} \cdot -w) \cdot w \cdot w + (1\text{'} \cdot -w);(w;-w) && \text{by Th.1.8(i)}\\ &= (1\text{'} \cdot -w);(w;-w)\\ &= (1\text{'} \cdot -w);(w;1 \cdot -w) && \text{by Th.1.8(iii)}\\ &= (1\text{'} \cdot -w);(w;1 \cdot -w) \cdot -w && \text{by Ax3, Ax6}\\ &= 1\text{'}_{-w};^{-w} p_0 && \text{by Def.2.1, Th.4.8(ii)}\end{aligned}$$

Now, if we express this in $\mathfrak{B}$, we get

$$1\text{'};p_0 = 1\text{'};-(1\text{'};1) = 0$$

by Theorem 1.8(vii). Thus, returning to $\mathfrak{A}$, we have

$$w;1 \cdot (1\text{'} \cdot -w);1 = 0$$

as desired.

For the uniqueness statement it suffices to assume (ii) and (iii) of Theorem 4.10 and prove (ii) and (iii) of Theorem 4.8. Then we can use the uniqueness of $\mathfrak{A}$ in Theorem 4.8.

To prove that $w;1 \cdot -w = p_0$ it suffices to show that $w;1 \cdot -w \cdot -p_0 = 0$ and $(w;1 \cdot -w) + -p_0 = 1$. Recall that

$$\begin{aligned}p_0 &= {}_{(-w)} - (1\text{'}_{-w};^{-w} - w)\\ &= -((1\text{'} \cdot -w);-w \cdot -w) \cdot -w && \text{by Def.2.1}\\ &= -((1\text{'} \cdot -w);-w \cdot -w + w)\end{aligned}$$

Thus

$$\begin{aligned}-p_0 &= (1\text{'} \cdot -w);-w \cdot -w + -w\\ &= (1\text{'} \cdot -w);-w + w && \text{by Ax3 and Ax6}\end{aligned}$$

Now we calculate

$$\begin{aligned}w;1 \cdot -w \cdot -p_0 &= w;1 \cdot -w \cdot ((1\text{'} \cdot -w);-w + w)\\ &= w;1 \cdot -w \cdot (1\text{'} \cdot -w);-w\\ &\leq w;1 \cdot (1\text{'} \cdot -w);1 && \text{by Ax3}\\ &= 0 && \text{by (iii)}\end{aligned}$$

Also,

$$\begin{aligned}w;1 \cdot -w + -p_0 &= w;1 \cdot -w + (1\text{'} \cdot -w);-w + w\\ &= w;-w + (1\text{'} \cdot -w);-w + w && \text{by Th.1.8(iii)}\\ &= 1\text{'};-w + w\\ &= 1\text{'};-w + 1\text{'};w && \text{Th1.8(i)}\\ &= 1\text{'};1 && \text{by Ax3}\\ &= 1 && \text{by (ii)}\end{aligned}$$

Thus $w;1 \cdot -w = p_0$. Likewise $1;w \cdot -w = p_1$. Thus $\mathfrak{A}$ is uniquely determined up to isomorphism. □

Theorem 4.11. *Let* $\mathfrak{B} \in \mathsf{REL}$. *Then there exists an* $\mathfrak{A} \in \mathsf{REL}$ *(unique up to isomorphism) such that*

(i) $\mathfrak{B} \in \mathfrak{Rl}_{-w}\mathfrak{A}$ *for some* $w \leq 1'$ *with* w *an atom of* $\mathfrak{A}$*,*
(ii) $1';1 = 1;1' = 1$ *in* $\mathfrak{A}$*.*

Proof. Let $p_0 = -(1';1)$ and $p_1 = -(1;1')$. (Here all operations and constants are taken in $\mathfrak{B}$.) As in the proof of Theorem 4.10, we have $p_0^{\smile\smile} = p_1^{\smile}$. In order to apply Theorem 4.9, we must prove the following eight equations:

$$
\begin{array}{ll}
1';-(1';1) = 0 & -(1;1');1' = 0 \\
1';1;1 \leq 1';1 & 1;(1;1') \leq 1;1' \\
-(1';1);1 \leq -(1';1) & 1;-(1;1') \leq -(1;1') \\
-(1;1');(1';1) = 0 & 1;1';-(1';1) = 0
\end{array}
$$

We will prove the four equations on the left. The four equations on the right will follow by symmetry.

First note that $1';-(1';1) = 0$ follows from Theorem 1.8(vii), and $1';1;1 \leq 1';1$ follows from Theorem 1.9(i). Next,

$$
\begin{array}{lll}
-(1';1);1 \leq -(1';1) & \text{iff} \quad 1';1 \cdot -(1';1);1 = 0 & \\
 & \text{iff} \quad 1';(-(1';1);1) \cdot 1 = 0 & \text{by Th.1.7(i), Th.1.8(ii)} \\
 & \text{iff} \quad 1';(-(1';1);1) = 0 &
\end{array}
$$

which follows from Th.1.9(ii). $-(1;1');(1';1) = 0$ follows from Theorem 1.9(iv). Thus, Theorem 4.9 applies. As in the proof of Theoem 4.10, we see that Theorem 4.8 applies also. Theorem 4.9 guarantees the existence of an $\mathfrak{A} \in \mathsf{REL}$ satisfying (i), (ii), and (iii) of Theorem 4.9 (and thus of Theorem 4.8). This existence coupled with the uniqueness of the $\mathfrak{A} \in \mathsf{NREL}$ satisfying (i), (ii), and (iii) of Theorem 4.8 says that the $\mathfrak{A} \in \mathsf{NREL}$ in Theorem 4.10 is actually in REL, since it comes from Theorem 4.8. To prove uniqueness, we need only show that $\mathfrak{A}$ satisfies (iii) of Theorem 4.10.

$$
\begin{array}{lll}
w;1 \cdot (1' \cdot -w);1 = 0 & \text{iff} \quad w;((1' \cdot -w);1) \cdot 1 = 0 & \text{by Th1.7(i), Th.1.8(ii)} \\
 & \text{iff} \quad w;(1' \cdot -w);1 = 0 & \text{by Ax9, since } \mathfrak{A} \in \mathsf{REL} \\
 & \text{iff} \quad (w \cdot 1' \cdot -w);1 = 0 & \text{by Th.1.8(i)} \\
 & \text{iff} \quad 0;1 = 0 &
\end{array}
$$

which holds by Ax4. The Theorem follows. □

§5. The REL Representation Theorem

Definition 5.1. *We define* NA *to be the class of all* $\mathfrak{A} \in \mathsf{NREL}$ *such that* $1';1^{\smile} = 1$ *in* $\mathfrak{A}$*. We define* WA *to be the class of all* $\mathfrak{A} \in \mathsf{REL}$ *such that* $1';1^{\smile} = 1$ *in* $\mathfrak{A}$*.*

Remark 5.2. The definitions given here for NA and WA are easily seen to be equivalent to the definitions given in Maddux [2].

Theorem 5.3.

(i) *Let* $\mathfrak{B} \in \mathsf{NREL}$ *be complete. Then there exists a complete* $\mathfrak{A} \in \mathsf{NA}$ *and* $r \in |\mathfrak{A}|$ *such that* $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{A}$.

(ii) *Let* $\mathfrak{B} \in \mathsf{REL}$ *be complete. Then there exists a complete* $\mathfrak{A} \in \mathsf{WA}$ *and* $r \in |\mathfrak{A}|$ *such that* $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{A}$.

Proof. (i). By Theorem 3.3, condition $(*)$ of Remark 3.2 is satisfied for $s = 1$. Thus, by Theorem 3.12 there exists a $\mathfrak{C} \in \mathsf{NREL}$ such that $\mathfrak{C}$ satisfies $1^{\smile} = 1$ (this follows from Theorem 3.12(i) since $s = 1$) and $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{C}$ for some $r \in |\mathfrak{C}|$. By Theorem 4.10 there exists an $\mathfrak{A} \in \mathsf{NREL}$ such that $\mathfrak{A}$ satisfies $1' ; 1 = 1 ; 1' = 1$ and $\mathfrak{C} = \mathfrak{Rl}_{-w}\mathfrak{A}$ for some atom $w \in |\mathfrak{A}|$ with $w \leq 1'$.

Thus $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{C} = \mathfrak{Rl}_r\mathfrak{Rl}_{-w}\mathfrak{A} = \mathfrak{Rl}_r\mathfrak{A}$. (Note that $r \leq -w$ in $\mathfrak{A}$.) It only remains to show that $\mathfrak{A}$ satisfies $1' ; 1^{\smile} = 1$. Since $\mathfrak{C}$ satisfies $1^{\smile} = 1$, we have $(-w)^{\smile(-w)} = -w$ in $\mathfrak{A}$. This is equivalent, by Definition 2.1, to $(-w)^{\smile} \cdot -w = -w$, which in turn is equivalent to $-w \leq (-w)^{\smile}$. But

$$
\begin{aligned}
(-w)^{\smile} \cdot w &= (-w \cdot w^{\smile})^{\smile} && \text{by Ax2}\\
&= (-w \cdot w)^{\smile} && \text{by Th.1.8(ii)}\\
&= 0^{\smile} \\
&= 0 && \text{by Th.1.3(i)}
\end{aligned}
$$

Thus $(-w)^{\smile} = -w$. Now, we calculate

$$
\begin{aligned}
1' ; 1^{\smile} &= 1' ; (w + -w)^{\smile} \\
&= 1' ; (w^{\smile} + (-w)^{\smile}) && \text{by Th.1.3(iii)}\\
&= 1' ; (w + -w) && \text{by Th.1.8(ii)}\\
&= 1' ; 1 \\
&= 1
\end{aligned}
$$

(ii). The proof proceeds as in the proof of part (i). The only changes are: the use of Theorem 3.14 to insure that $\mathfrak{C} \in \mathsf{REL}$ and the use of Theorem 4.11 to insure that $\mathfrak{A} \in \mathsf{REL}$ and thus $\mathfrak{A} \in \mathsf{WA}$. $\square$

Theorem 5.4. (The REL Representation Theorem.) *Let* $\mathfrak{B} \in \mathsf{REL}$.

(i) *There exists a complete atomic* $\mathfrak{A} \in \mathsf{RRA}$ *and* $r \in |\mathfrak{A}|$ *such that* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{Rl}_r\mathfrak{A}$. *Furthermore, if* $\mathfrak{B}$ *is complete and atomic then* $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{A}$.

(ii) *There exists a set* U *and a relation* $R \subseteq U \times U$ *such that* $\mathfrak{B}$ *is isomorphic to a subalgebra of* $\mathfrak{Rl}_R\mathfrak{Re}U$, *where* $\mathfrak{Re}U$ *is the relation algebra of all binary relations on* U.

(iii) $\mathsf{REL} = \mathbf{IS}\{\mathfrak{Rl}_R\mathfrak{Re}U : R \subseteq U \times U\}$.

Proof. (i). Let $\mathfrak{B} \in \mathsf{REL}$. Denote by $\mathfrak{B}^+$ the perfect extension of $\mathfrak{B}$ as a Boolean algebra with operators. (See Jónsson-Tarski [1].) Clearly, $\mathfrak{B}^+ \in \mathsf{REL}$ since negation occurs only in a constant axiom and in the Boolean axioms (see Jónsson-Tarski [1], Theorem 2.18). Since $\mathfrak{B}^+$ is complete, by Theorem 5.3(ii) there exists a complete $\mathfrak{C} \in \mathsf{WA}$ such that $\mathfrak{B}^+ = \mathfrak{Rl}_r\mathfrak{C}$ for some $r \in |\mathfrak{C}|$.

Note that the process of adjoining converses preserves atomicity of algebras. Also, we can assume the adjoining of identity involved adding a single atom. Thus, since $\mathfrak{B}^+$ is atomic, we can choose $\mathfrak{C}$ atomic also. Similarly, completeness can be preserved. Thus, we can assume $\mathfrak{C}$ is complete and atomic. Since $\mathfrak{C} \in \mathsf{WA}$ we can use part (1) of the WA Representation Theorem (Theorem 5.20 of Maddux [2]) to find a complete atomic $\mathfrak{A} \in \mathsf{RRA}$ and $s \in |\mathfrak{A}|$ such that $\mathfrak{C} = \mathfrak{Rl}_s\mathfrak{A}$. Then $\mathfrak{B}^+ = \mathfrak{Rl}_r\mathfrak{C} = \mathfrak{Rl}_r\mathfrak{Rl}_s\mathfrak{A} = \mathfrak{Rl}_r\mathfrak{A}$. But $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}^+ = \mathfrak{Rl}_r\mathfrak{A}$. In case $\mathfrak{B}$ is already complete and atomic, we replace $\mathfrak{B}^+$ with $\mathfrak{B}$, and the argument above shows $\mathfrak{B} = \mathfrak{Rl}_r\mathfrak{A}$.

(ii). By part (i), $\mathfrak{B}$ is a subalgebra of a relativization of some $\mathfrak{A} \in \mathsf{RRA}$. By Definition 1.8 of Maddux [2], $\mathfrak{A}$ is isomorphic to a subalgebra of a relativization of some $\mathfrak{Re}\,U$. Thus $\mathfrak{B}$ is isomorphic to a subalgebra of a relativization of a subalgebra of a relativization of some $\mathfrak{Re}\,U$. Theorem 5.7 of Maddux [2] is easily seen to hold with NA replaced by NREL. Part (5) of that theorem says that a relativization of a subalgebra is a subalgebra of a relativization, so $\mathfrak{B}$ is isomorphic to a subalgebra of a subalgebra of a relativization of a relativization of some $\mathfrak{Re}\,U$. Thus $\mathfrak{B}$ is isomorphic to a subalgebra of a relativization of some $\mathfrak{Re}\,U$ as desired.

(iii). The inclusion from left to right is part (ii). For the inclusion from right to left, note that $\mathfrak{Rl}_R\mathfrak{Re}\,U \in \mathsf{REL}$ whenever $R \subseteq U \times U$, and REL is closed under the formation of isomorphic images and subalgebras. $\square$

References

1. B. Jónsson and A. Tarski, *Boolean algebras with operators, Part I*, American Journal of Mathematics **73** (1951), 891–939.

2. R. Maddux, *Some varieties containing relation algebras*, Transactions of the American Mathematical Society **272** (1982), 501–526.