THE PERIODIC HOPF RING OF CONNECTIVE MORAVA K-THEORY

J. MICHAEL BOARDMAN, RICHARD L. KRAMER, AND W. STEPHEN WILSON

Abstract. Let $K(n)_*(-)$ denote the n-th periodic Morava K-theory for any fixed odd prime p. Let $\underline{k(n)}_*$ denote the Ω-spectrum of the n-th connective Morava K-theory. We give a calculation of the Hopf ring $K(n)_*\underline{k(n)}_*$, the main result of the second author's thesis. This is a new, shorter, easier proof.

1. Introduction

Let p be any odd prime, fixed for the duration of this article. Let $K(n)$ be the spectrum of the periodic (of period $2(p^n-1)$) n-th Morava K-theory and let $k(n)$ be the spectrum of the connective n-th Morava K-theory. Morava K-theory was first defined and studied by Jack Morava in numerous unpublished preprints. Both $K(n)$ and $k(n)$ are commutative, associative, ring spectra with unit. The coefficient rings $K(n)_*$ and $k(n)_*$ are $\mathbf{F}_p[v_n, v_n^{-1}]$ and $\mathbf{F}_p[v_n]$, respectively, where the degree $|v_n|$ of v_n is $2(p^n-1)$ and $\mathbf{F}_p$ is the p-element field. See [JW75] for an early published account of some of Morava's work.

There is a canonical map of ring spectra $k(n) \to K(n)$. Recall that $K(n)_*X = v_n^{-1}k(n)_*X = k(n)_*X \otimes_{k(n)_*} K(n)_*$. The map $k(n)_*X \to K(n)_*X$ induced by the above map of spectra $k(n) \to K(n)$ is the obvious one. Specializing to the case $X = pt$, we get the localization $\mathbf{F}_p[v_n] \to \mathbf{F}_p[v_n, v_n^{-1}]$. Since $K(n)_*$ is a graded field (every homogeneous nonzero element has a multiplicative inverse), we have a Künneth isomorphism,

$$K(n)_*X \otimes_{K(n)_*} K(n)_*Y \cong K(n)_*(X \times Y),$$

for spaces X and Y. Let $\underline{K(n)}_*$ and $\underline{k(n)}_*$ be the Ω-spectra for the spectra $K(n)$ and $k(n)$, respectively. Since $K(n)_*(-)$ has Künneth isomorphisms, both $K(n)_*\underline{k(n)}_r$ and $K(n)_*\underline{K(n)}_r$ have coproducts, and thus are coalgebras. This puts us in the setting described in [RW77]. Namely, both $k(n)^*(-)$ and $K(n)^*(-)$ are graded rings. This means their classifying spaces are graded ring objects in the homotopy category. Continuing as in [RW77], our Künneth isomorphism puts the Morava K-theory of these classifying spaces in the category of coalgebras and thus both $K(n)_*\underline{k(n)}_*$ and $K(n)_*\underline{K(n)}_*$ are graded ring objects in the category of coalgebras, i.e. they are *Hopf rings* or *coalgebraic rings.* In particular, each $K(n)_*\underline{k(n)}_r$ and $K(n)_*\underline{K(n)}_r$ is a group object in this category, i.e. a Hopf algebra or *coalgebraic group.* The canonical map of ring spectra $k(n) \to K(n)$ induces a Hopf ring homomorphism $K(n)_*\underline{k(n)}_* \longrightarrow K(n)_*\underline{K(n)}_*$.

$K(n)_*\underline{K(n)}_*$ was computed in [Wil84]. It is clear that $K(n)_*\underline{k(n)}_r \cong K(n)_*\underline{K(n)}_r$ for $r < 2(p^n-1)$, since $\underline{k(n)}_r \simeq \underline{K(n)}_r$ for such r. This grounds our inductive calculation.

Theorem 1.1. *For p an odd prime, the fibration sequence*

$$(1.2)\quad \underline{k(n)}_{i-1} \longrightarrow K(\mathbf{Z}/(p), i-1) \longrightarrow \underline{k(n)}_{i+2(p^n-1)} \xrightarrow{v_n} \underline{k(n)}_i \longrightarrow K(\mathbf{Z}/(p), i)$$

induces a short exact sequence of Hopf algebras

(1.3)

$$K(n)_* \to K(n)_*\underline{k(n)}_{i+2(p^n-1)} \xrightarrow{[v_n]\circ} K(n)_*\underline{k(n)}_i \to K(n)_*K(\mathbf{Z}/(p), i) \to K(n)_*$$

or, all together,

(1.4)

$$K(n)_* \to K(n)_*\underline{k(n)}_{*+2(p^n-1)} \xrightarrow{[v_n]\circ} K(n)_*\underline{k(n)}_* \to K(n)_*K(\mathbf{Z}/(p), *) \to K(n)_*.$$

By [RW80], we know that $K(n)_*K(\mathbf{Z}/(p), i)$ is trivial for $i > n$, so this short exact sequence reduces to an isomorphism

$$K(n)_*\underline{k(n)}_{i+2(p^n-1)} \simeq K(n)_*\underline{k(n)}_i$$

for $i > n$. Since by induction we know the right two terms in the short exact sequence, it is enough for us to describe the kernel of the map between them.

The elements which generate $K(n)_*\underline{K(n)}_*$ are $e_1 \in K(n)_1\underline{K(n)}_1$, $a_{(i)} \in K(n)_{2p^i}\underline{K(n)}_1$, for $i < n$, $b_{(i)} \in K(n)_{2p^i}\underline{K(n)}_2$, $[v_n] \in K(n)_0\underline{K(n)}_{-2(p^n-1)}$, and $[v_n^{-1}] \in K(n)_0\underline{K(n)}_{2(p^n-1)}$. All but $[v_n^{-1}]$ are defined in $K(n)_*\underline{k(n)}_*$. The relations among these elements in $K(n)_*\underline{K(n)}_*$ also hold in $K(n)_*\underline{k(n)}_*$. Our theorem gives us a sub-Hopf ring

$$K(n)_*\underline{k(n)}_* \subset K(n)_*\underline{K(n)}_*.$$

In Section 2, we establish some notation and state our main theorem. In Section 3 we prove our results.

The main theorem in this article is from the 1990 dissertation of Richard Kramer at the Johns Hopkins University, [Kra90]. Normally, he would publish this without his advisors' names on the paper. However, by the time the paper was accepted for publication, modulo various revisions suggested by the referee, he was without a research academic job and unable to pursue the rewriting. The paper languished. The result is the first known instance of what is now more commonplace. Consequently, this result is regularly referred to and his advisors felt it should be in the literature. After discussions with Richard Kramer, the outcome was this paper which has been significantly revised by the advisors. The main result, as stated, is due to Richard Kramer. However, in the process of rewriting the paper, with the advantage of hindsight, his proof was scrapped in favor of a much shorter (and easier) new proof. Regretably, a number of very nice formulas about iterated p-th powers from Kramer's thesis fell by the wayside in this process. The second author's original proof was a difficult calculation comparing the bar spectral sequence for $K(n)_*\underline{k(n)}_*$ with that for $K(n)_*\underline{K(n)}_*$. The main ingredient for the new proof is that we already know the answer. This allows us to see a relatively easy direct proof. The second author wishes to thank Jack Morava and Jeff Smith for many interesting conversations regarding algebraic topology and homotopy theory and acknowledge the support of his advisors, now his coauthors. The third author wishes to thank Takuji Kashiwabara and Douglas Ravenel for various interactions over the years which helped give insight into a new short proof of this result and the Centre de Recerca Matemàtica, Institut d'Estudis Catalans, Barcelona, Spain, for their ideal work environment during the revising of this paper. We all wish to thank the editors for their patience and support in this matter.

2. Preliminaries and the main theorem

Throughout, we assume that $I = (i_0, \ldots, i_{n-1})$, where $i_k \in \{0,1\}$. We also assume that $J = (j_0, j_1, \ldots)$ where $0 \leq j_k$ and $j_k = 0$ for all but finitely many $k \geq 0$. After Corollary 2.2, we also insist $j_k < p^n$. We also define $I(1)$ to be the sequence I with $i_0 = \ldots = i_{n-1} = 1$. Let Δ_k be the sequence I (or J) with $i_k = 1$ (or $j_k = 1$) and zero elsewhere. Given I, let $\rho(I)$ be defined by

$$\rho(I) = \begin{cases} \infty & \text{if } I = I(1) \\ \min\{k \mid i_{n-k} = 0\} & \text{if } I \neq I(1)\ (n \geq k > 0) \end{cases} .$$

In [Wil84], certain elements, a_i and b_i were defined in $P(n)_*\underline{P(n)}_*$. These elements could be pushed down non-trivially to $H_*\underline{H}_*$ or anywhere in between. In particular, they are defined in $k(n)_*\underline{k(n)}_*$, $K(n)_*\underline{k(n)}_*$, and $K(n)_*\underline{k(n)}_*$. On the occasion we need these elements in more than one place we will abuse notation and use the same name. The elements are easily defined. The b_i come from CP^∞ and the fact that all of our theories are complex oriented. The a_i are low degree elements which are in the range where they are just standard homology elements. These a_i survive non-trivially to $K(n)_*K(\mathbf{Z}/(p), 1)$. They have the following properties in our setting.

Proposition 2.1. *Let p be an odd prime and $n \geq 1$. Then there exist elements $e_1 \in K(n)_1\underline{k(n)}_1$, $a_i \in K(n)_{2i}\underline{k(n)}_1$ for $i < p^n$, and $b_i \in K(n)_{2i}\underline{k(n)}_2$. Let $b_{(i)} = b_{p^i}$ and $a_{(i)} = a_{p^i}$. Let ψ be the coproduct. We have:*

- *(i) $e_1 \circ (-)$ is the homology suspension map.*
- *(ii) $\psi(a_i) = \sum_{j=0}^{i} a_{i-j} \otimes a_j$ and $\psi(b_i) = \sum_{j=0}^{i} b_{i-j} \otimes b_j$.*
- *(iii) The standard mod p homology a_i and b_i are all permanent cycles in the Atiyah-Hirzebruch spectral sequence for $K(n)_*\underline{k(n)}_*$ and represent the corresponding a_i and b_i.*
- *(iv) $e_1 \circ e_1 = -b_1 = -b_{(0)}$.*
- *(v) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$.*
- *(vi) $b_{(i)}^{*p} = 0$.*
- *(vii) $a_{(i)}^{*p} = 0$ for $i < n-1$.*
- *(viii) $a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ(p^n-1)} \circ [v_n]$.*
- *(ix) $v_n e_1 = b_{(0)}^{\circ(p^n-1)} \circ e_1 \circ [v_n]$.*
- *(x) $b_{(k)}^{\circ p^n} \circ [v_n] = v_n^{p^k} b_{(k)}$ for $k \geq 0$.*

Proof. See Proposition 1.1 of [Wil84] with the correction of (*iv*) from [BJW95]. □

For any I and J, we define

$$a^I b^J = a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(n-1)}^{\circ i_{n-1}} \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots ,$$

with the convention that $a^0 b^0 = [1] - [0]$. Furthermore, we define $a^I = a^I b^0$ and $b^J = a^0 b^J$.

Corollary 2.2. *In $K(n)_*\underline{k(n)}_*$, if $k \geq \rho(I)$, then $(a^I b^J \circ [v_n^s])^{*p^k} = 0$.*

Proof. This follows from Proposition 2.1 (*vii*) and (*viii*). □

Let $E(x)$ be the exterior algebra with generator x and let $P(x)$ be the polynomial algebra generated by x. Let $TP_k(x)$ be the truncated polynomial algebra $P(x)/(x^{p^k})$ generated by x for $k > 0$. We use the convention that $TP_\infty(x) = P(x)$. Finally, if $J \neq 0$, we define $m(J) = \min\{k \mid j_k \neq 0\}$. The main theorem of this article is the following:

Theorem 2.3 (Kramer)**.** *Let p be an odd prime. Then* $K(n)_*\underline{k(n)}_* \simeq$

$$\bigotimes_{\substack{j_0 < p^n - 1 \\ s \geq 0}} E\left(a^I b^J \circ e_1 \circ [v_n^s]\right) \tag{2.4}$$

$$\bigotimes_{\substack{j_0 < p^n - 1 \\ s > 0}} E\left(a^I b^J \circ e_1 \circ b_{(0)}^{\circ(p^n-1)s}\right) \tag{2.5}$$

$$\bigotimes_{\substack{(a)\quad \{s \geq 0,\ \text{and } i_0 = 0 \text{ or } j_0 < p^n - 1\} \text{ or} \\ (b)\quad \{s = 0,\quad J = (p^n-1)\Delta_0,\quad i_0 = 1\}}} TP_{\rho(I)}\left(a^I b^J \circ [v_n^s]\right) \tag{2.6}$$

$$\bigotimes_{\substack{s > 0,\quad J \neq 0 \text{ and either} \\ (a)\quad \{i_0 = 0 \text{ or } j_0 < p^n - 1\} \text{ or} \\ (b)\quad \{J = (p^n-1)\Delta_0,\quad i_0 = 1\}}} TP_{\rho(I)}\left(a^I b^J \circ b_{(m(J))}^{\circ(p^n-1)s}\right) \tag{2.7}$$

where I and J are taken as above.

We remark that it is easy to see the correct stable result in here by suspending an infinite number of times.

We recall:

Theorem 2.8 ([Wil84])**.** *For p an odd prime, $s \in \mathbf{Z}$,*

$$K(n)_*\underline{K(n)}_* = \bigotimes_{j_0<p^n-1} E\left(a^I b^J \circ e_1 \circ [v_n^s]\right) \bigotimes_{i_0=0 \text{ or } j_0<p^n-1} TP_{\rho(I)}\left(a^I b^J \circ [v_n^s]\right)$$

3. Proof

We proceed by proving Theorem 1.1 and then identifying the kernel by induction.

Lemma 3.1. *For p an odd prime, we have a surjection of Hopf rings:*

$$K(n)_*\underline{k(n)}_* \longrightarrow K(n)_*K(\mathbf{Z}/(p), *).$$

Proof. As in the discussion of the definition of the elements $a_{(i)}$, they are defined in $K(n)_*\underline{k(n)}_1$ and map non-trivially to elements with the same name in $K(n)_*K(\mathbf{Z}/(p), 1)$. These elements generate the Hopf ring $K(n)_*K(\mathbf{Z}/(p), *)$, [RW80]. Our surjection follows. □

We need to look at the sequence of fibrations (1.2).

Lemma 3.2. *For p an odd prime, the map $K(\mathbf{Z}/(p), i-1) \to \underline{k(n)}_{i+2(p^n-1)}$ induces the trivial map on $K(n)_*(-)$.*

Proof. The composition of the first two maps of (1.2) is trivial. The first one is surjective on $K(n)_*(-)$ by Lemma 3.1, so the second one must be trivial on $K(n)_*(-)$. □

Proof of Theorem 1.1. We use the bar spectral sequence for the fibration

$$K(\mathbf{Z}/(p), i-1) \longrightarrow \underline{k(n)}_{i+2(p^n-1)} \xrightarrow{v_n} \underline{k(n)}_i. \tag{3.3}$$

We have $Tor_{*,*}^{K(n)_*K(\mathbf{Z}/(p),i-1)}(K(n)_*\underline{k(n)}_{i+2(p^n-1)}, K(n)_*) \Rightarrow K(n)_*\underline{k(n)}_i$. By Lemma 3.2, this reduces to

$$Tor_{*,*}^{K(n)_*K(\mathbf{Z}/(p),i-1)}(K(n)_*, K(n)_*) \bigotimes K(n)_*\underline{k(n)}_{i+2(p^n-1)}.$$

We map this to the bar spectral sequence for the fibration

$$K(\mathbf{Z}/(p), i-1) \longrightarrow * \longrightarrow K(\mathbf{Z}/(p), i).$$

This map allows us to compute all of the differentials in the spectral sequence from the one for the Eilenberg–Mac Lane spaces using [RW80]. When finished with the differentials forced on us we have $K(n)_*\underline{k(n)}_{i+2(p^n-1)}$ in the zeroth filtration tensored with an associated graded object for $K(n)_*K(\mathbf{Z}/(p), i)$, all in higher filtrations. Any more differentials would have to have their source in the $K(n)_*K(\mathbf{Z}/(p), i)$ part which would contradict Lemma 3.1. The desired short exact sequence now follows from the spectral sequence. □

Proof of Theorem 2.3. First we must ground our induction. The spaces $\underline{k(n)}_r$ are the same as $\underline{K(n)}_r$ for $r < 2(p^n - 1)$. For these spaces, we must compare the answers in Theorem 2.3 and Theorem 2.8. The element $a^I b^J[v_n^s]$ is in $K(n)_*\underline{k(n)}_r$ where $r = \sum i_k + 2\sum j_k - 2(p^n - 1)s$. All elements with s negative in Theorem 2.8 lie outside our range, i.e. have $r \geq 2(p^n - 1)$. The first tensor product in Theorem 2.8 corresponds to (2.4). The second tensor product in Theorem 2.8 corresponds to (2.6) (a). All of the other elements in Theorem 2.3 fall outside our range.

We can now make our inductive assumption that Theorem 2.3 is correct for $K(n)_*\underline{k(n)}_r$ for r less some $i + 2(p^n - 1)$. We then use the known short exact sequence, (1.3), to prove the result for $r = i + 2(p^n - 1)$. Actually, we do them all at once with (1.4). All of the elements defined in Theorem 2.3 exist in the asserted spaces. All of our asserted generators certainly have the appropriate iterated p-th powers trivial by Corollary 2.2. Our only remaining question: is our asserted answer the kernel of (1.4)? (It obviously maps to the kernel because of the fibration.)

There are two parts to this last question. We must show that our asserted algebra injects and that its cokernel is the Morava K-theory of Eilenberg–Mac Lane spaces. Since our generators will map to distinct generators of the same height, injection will not be a problem. To this end we must apply $[v_n]\circ$ to $K(n)_*\underline{k(n)}_{*+2(p^n-1)}$. First we look at (2.4). The generators of (2.4) map to generators of (2.4). They inject and hit all generators except those with $s = 0$. A similar thing happens with all of the generators of (2.6) (a). The height of source and target generators is the same so we have no kernel on this part. Looking at (2.5), we multiply by $[v_n]\circ$, use Proposition 2.1 (ix) to surject the generators back on to the generators of (2.5). We must determine what happens to the generators with $s = 1$ though. They map to the $s = 0$ generators we missed before in (2.4). When $s > 1$ in (2.7) we can use Proposition 2.1 (x) to see that this maps isomorphically to the part of (2.7) with $s > 0$.

All we have remaining is to show that (2.6) (b) and the $s = 1$ part of (2.7) maps injectively to the $s = 0$ part of (2.6) with cokernel $K(n)_*K(\mathbf{Z}/(p), *)$. For the (2.7), $s = 1$ part, because $J \neq 0$ we can still use Proposition 2.1 (x). When we apply

$[v_n]\circ$ we get $a^I b^J$, $J \neq 0$, and either of the other two conditions on (2.7). The first of these conditions, (2.7) (a), gives all $a^I b^J$, $i_0 = 0$ or $j_0 < p^n - 1$, $J \neq 0$, which is all of the $J \neq 0$ part of the $s = 0$ part of (2.6) (a). The second condition, (2.7) (b), just gives (2.6) (b).

The remaining terms, (2.6) (b), are just $a^I b^{(p^n-1)\Delta_0}$ with $i_0 = 1$. We have not yet hit the a^I part of (2.6). Now, we must circle multiply Propostion 2.1 $(viii)$ by $a^{(0,i_1,\dots,i_{n-1})}$ to see that when $i_0 = 1$,

$$a^I b^{(p^n-1)\Delta_0}[v_n] = v_n a^I \pm (a^{(i_1,i_2,\dots,i_{n-1},1)})^{*p}.$$

$K(n)_*K(\mathbf{Z}/(p),*)$, has all a^I non zero, but if $i_0 = 1$, it is a p-th power. Thus, we see this last part of the map misses all a^I, $i_0 = 0$ and forces the a^I, $i_0 = 1$, to be p-th powers in the cokernel. □

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E-mail address: `jmb@math.jhu.edu, wsw@math.jhu.edu`

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218