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Equations not preserved by complete extensions

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In this paper we construct a representable relation algebra $\mathfrak{B}$ and an equation which holds in $\mathfrak{B}$ but fails in every complete extension of $\mathfrak{B}$. In particular, the equation fails in every perfect extension [3] and every completion [4] of $\mathfrak{B}$.

We also outline how to carry out the same construction for cylindric algebras. For every finite $n \geq 3$ there is a representable CA_n and an equation which holds in that algebra but fails in every complete extension. This yields a negative answer to Problem 2.15 of [2].

Let $\mathcal{G} = \langle G, \circ, ^{-1}, e \rangle$ be a group. The *complex algebra of* $\mathcal{G}$ is

$$\mathcal{C}m(\mathcal{G}) = \langle Sb(G), +, \cdot, -, \varnothing, 1, ;, ^{\cup}, 1' \rangle$$

where $Sb(G)$ is the set of subsets of $G, +, \cdot,$ and $-$ are union, intersection, and complementation with respect to G. Also, $1, ;, ^{\cup}$, and 1' are defined as follows:

$$X; Y = \{x \circ y : x \in X, y \in Y\},$$
$$X^{\cup} = \{x^{-1} : x \in X\},$$
$$1' = \{e\} \quad \text{and} \quad 1 = G,$$

for all $X, Y \in Sb(G)$. For any group $\mathcal{G}$, $\mathcal{C}m(\mathcal{G})$ is a representable relation algebra [3, 5.10 and 4.31]. Furthermore, $\mathcal{C}m(\mathcal{G})$ is integral, hence also simple, and so for every $X \in Sb(G)$, $X \neq \varnothing$ iff $X; G = G$ iff $G; X = G$ iff $G; X; G = G$. All subalgebras of $\mathcal{C}m(\mathcal{G})$ have these properties as well [3, 5.10, 4.10, 4.17, 4.18]. It is easy to check that $X \in Sb(G)$ is a subgroup of $\mathcal{G}$ iff $X; X^{\cup} = X$ [1, p. 357].

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LEMMA 1. *Let $\mathcal{G}$ be any group and let $B \subseteq Sb(G)$ be the set of all finite or cofinite subsets of G. Then*

(i) *B is a subuniverse of $\mathcal{C}m(\mathcal{G})$,*

(ii) *if $X \in B$ is a subgroup of $\mathcal{G}$, then either X is finite or $X = G$.*

Proof. (i) Clearly B is closed under $+, \cdot$, and $-$. Also $\varnothing, G, 1' \in B$. B is closed under $^{\cup}$ since if $X \in B$ is finite (cofinite), then $X^{\cup} = \{x^{-1} : x \in X\}$ is also finite (cofinite). It remains to show that B is closed under ;. Let $X, Y \in B$. If X, Y are both finite, then clearly $X; Y$ is finite. So without loss of generality, assume X is cofinite. If $Y = \varnothing$, then $X; Y = X; \varnothing = \varnothing$, so we can assume that $Y \neq \varnothing$. Let $y \in Y$. Then

$$X;\, Y \supseteq X; \{y\} = \{x \circ y : x \in X\}.$$

But the right side is a "coset" of X, and hence is cofinite since X is cofinite. Thus $X; Y$ is cofinite also since it contains a cofinite set.

(ii) Suppose $X \in B$ is an infinite proper subgroup of $\mathcal{G}$. Since X is proper, it has a coset disjoint from it which is infinite since X is, contradicting the hypothesis that $X \in B$.

THEOREM. *Let $\mathcal{G}$ be an infinite group with no nontrivial finite subgroups. (For example, the integers under addition and negation with zero.) Let B be the subuniverse of $\mathcal{C}m(\mathcal{G})$ consisting of the finite and cofinite subsets of G, and let $\mathfrak{B}$ be the corresponding subalgebra with universe B. Let $\mathfrak{A}$ be any complete extension of $\mathfrak{B}$. Then the following equation holds in $\mathfrak{B}$ but not in $\mathfrak{A}$:*

$$1{:}\,(-1' \cdot X);\, 1 \cdot 1.\, -X;\, 1 \leq 1;\, (X;\, X^{\cup} \oplus X);\, 1.$$

Proof. Let $X \in B$. If $X = G$ then $1; -X;\ 1 = 1;\ \varnothing;\ 1 = \varnothing$ and the equation is satisfied, so assume $X \neq G$. If $X \leq 1'$ then $1; (-1' \cdot X); 1 = 1; \varnothing; 1 = \varnothing$, so we also assume $X \nleq 1'$. Thus we have $-1' \cdot X \neq \varnothing \neq -X$, so $1; (-1' \cdot X); 1 \cdot 1; -X; 1 = 1$ since $\mathfrak{B}$ is simple. The only way the right side can fail to be 1 is if $X; X^{\cup} \oplus X = \varnothing$, i.e. $X; X^{\cup} = X$ and X is a subgroup. By Lemma 1 (ii) this can happen only if $X = G$ or X is finite. But $X \neq G, X$ is nontrivial ($X \nleq 1'$), and $\mathcal{G}$ has no nontrivial finite subgroups. Hence the equation holds in $\mathfrak{B}$.

Now we show the equation fails in $\mathfrak{A}$. Let $H \subseteq G$ be a proper infinite subgroup of $\mathcal{G}$. (For example, the infinite cyclic group generated by a non-generator of $\mathcal{G}$.) Let $X \in A$ be the element $X = \sum_{h \in H}^{(\mathfrak{A})} \{h\}$ which exists since $\mathfrak{A}$ is complete. Choose some $h \in H$ with $h \neq e$. Then $-1' \cdot \{h\} = \{h\} \leq X$, so $1; (-1' \cdot X); 1 \geq 1; (-1' \cdot \{h\}); 1 = 1; \{h\}; 1 = 1$ since $\mathfrak{B}$ is simple. Choose some $g \in G$ with $g \notin H$.

Then $\{g\}\cdot X=\sum_{h\in H}^{(\mathfrak{A})}(\{g\}\cdot\{h\})=\varnothing$, so $\{g\}\leq -X$, and hence $1;-X;1\geq 1;\{g\};1=1$. Thus the left side of the equation is 1. To show the right side is $\varnothing$, it suffices to show $X;X^{\cup}=X$. Using the complete additivity of ; and $^{\cup}$ we have:

$$\begin{aligned} X;X^{\cup} &= \Big(\sum_{h\in H}\{h\}\Big);\Big(\sum_{g\in H}\{g\}\Big)^{\cup} \\ &= \sum_{h,g\in H}(\{h\};\{g\}^{\cup}) \\ &= \sum_{h,g\in H}\{h\circ g^{-1}\} \\ &= \sum_{k\in H}\{k\}=X. \end{aligned}$$

Now we turn to cylindric algebras. For notation and terminology see [2]. Let $\mathcal{G}=\langle G,\circ,{}^{-1},e\rangle$ be a group and let $n\geq 3$. Set

$$G_n=\{g\in {}^{n\times n}G\colon \text{for all } i,j,k<n,\ g_{ii}=e,\ g_{ij}^{-1}=g_{ji},\text{ and } g_{ij}\circ g_{jk}=g_{ik}\}.$$

For all $X\subseteq G_n$ and all $i,j<n$ let

$$\begin{aligned} &c_iX=\{g\in G_n\colon \text{for some } x\in X,\ x_{jk}=g_{jk} \text{ whenever } i\neq j,\ k<n\},\\ &d_{ij}=\{g\in G_n : g_{ij}=e\},\\ &\mathcal{G}_n=\langle Sb(G_n),+,\cdot,-,\varnothing,G_n,c_i,d_{ij}\rangle_{i,j<n}. \end{aligned}$$

Then $\mathcal{G}_n$ is a simple representable CA_n. Let B be the set of all subsets X of G_n such that for all $i,j<n$, $\{x_{ij} : x\in X\}$ is either finite of cofinite. then B is a subuniverse of $\mathcal{G}_n$. Let $\mathfrak{B}$ be the subalgebra of $\mathcal{G}_n$ with universe B. $\mathfrak{B}$ is also a simple representable CA_n, and the following equation holds in $\mathfrak{B}$, but fails in every complete extension of $\mathfrak{B}$:

$$c_{(n)}(-d_{01}\cdot c_{(n-2)}X)\cdot c_{(n)}(-c_{(n-2)}X)\leq c_{(n)}[c_2(s_2^1c_{(n-2)}X\cdot s_1^0s_2^1c_{(n-2)}X)\oplus c_{(n-2)}X].$$

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