CE 576 Environmental Flows  
Spring 2012  
Homework 4

*20. Evaluate the Boussinesq approximation for the internal waves (see problem 10) by determining whether

\[ \frac{\rho D w}{Dt} \ll \rho' g. \]

Follow these steps:

a. Construct the profiles of temperature and density in the absence of waves. Use a buoyancy frequency \( N \) of 0.025 rad/s and note that

\[ N^2 = -\frac{g}{\rho_0} \frac{\partial \tilde{\rho}}{\partial z}. \]

Take the depth \( H \) to be 25 m, the bottom temperature to be 8°C and the equation of state to be \( \rho = \rho_0[1+\beta(T_0-T)] \), where \( \rho_0 = 999.27 \text{ kg/m}^3 \), \( \beta = 1.5 \times 10^{-4} \text{ °C}^{-1} \), and \( T_0 = 14.6°C \). Plot temperature and density profiles.

b. Discuss whether a linear equation of state works well for this case.

c. Evaluate the Boussinesq approximation by comparing the maximum values of \( \rho \partial w/\partial t \) and \( \rho' g \) and determine whether the Boussinesq approximation applies. Estimate \( \tilde{\rho} \) as half the difference between the surface and bottom densities, and estimate the density fluctuation using a maximum isotherm displacement \( \zeta_0 \) of 1 m and the temperature gradient computed from the information in part a.

21. Show that the steady equations for \( x \)-momentum, \( y \)-momentum, and temperature in terms of dimensionless variables in the problem of convection in a long box are

\[
Gr A^2 \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + A^2 \frac{\partial^2 u}{\partial x^2},
\]

\[
Gr A^4 \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \theta + A^2 \left( \frac{\partial^2 v}{\partial y^2} + A^2 \frac{\partial^2 v}{\partial x^2} \right).
\]

\[
Gr Pr A^2 \left( u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right) = \frac{\partial^2 \theta}{\partial y^2} + A^2 \frac{\partial^2 \theta}{\partial x^2}.
\]

22. For the long box problem we showed that

\[ u = \frac{y^3}{6} + c_1 \frac{y^2}{2} + c_2 y + c_3. \]

By applying the boundary conditions and the condition of no net flow, show that

\[ u = \frac{y^3}{6} - \frac{y^2}{4} + \frac{y}{12} \]

and

\[ u' = \frac{g \beta \Delta T H^3}{12 \nu L} \left( \frac{y'}{H} \right) \left[ 1 - 3 \left( \frac{y'}{H} \right) + 2 \left( \frac{y'}{H} \right)^2 \right]. \]
23. Estimate the Grashof number and the coefficients on the left sides of the equations in problem 21 in the following situations. Are those terms negligible?

a. A laboratory tank of length 5 m and depth 0.3 m with a temperature difference of $5^\circ$C. Take the thermal expansion coefficient to be $1.5 \times 10^{-4}$ $^\circ$C$^{-1}$.

b. An estuary of length 10 km and depth 3 m with a salinity difference of 35 psu (practical salinity units). The haline contraction coefficient is $7.6 \times 10^{-4}$ psu$^{-1}$.