Lec. 39: 4-20-09

1. Boundary layers
2. Flat plate BL
3. Momentum integral, eqn

Boundary layers

Need for BL

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{2} \rho \frac{\partial u}{\partial t} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

Baseball

\[ U_0 = \text{25 m/s} \]
\[ D = \text{4 cm} \]

\[ \frac{V^2}{k} \frac{\partial^2 \nu/\partial x^2}{\nu} \approx \frac{V}{U_0} \frac{\nu}{k} \approx \frac{1}{\text{Re}} \approx \frac{10^{-5} \text{ m}^2/\text{s}}{(25 \text{ m/s}) \times (0.4 \text{ m})} \]

\[ = \frac{10^{-5} \text{ m}^2/\text{s}}{1 \text{ m}^2/\text{s}} \approx 10^{-5} \]

\[ \therefore \text{Neglect viscous terms} \]

\[ \Rightarrow \text{Inertia-pressure balance} \]
1800s: Perfect (or ideal) fluid theory - mathematicians

Early 1900s: Engineer (Prandtl) recognized a problem

with no viscous effects, momentum balance includes
inertia, pressure, & gravity (Bernoulli)

Problems: Slip velocity (i.e. no-slip is violated)
No drag

How do you fix this problem?

Patch the solution with a
boundary layer so that
viscosity is important
near wall.

We want to choose \( \delta \) so that
viscosity is important

\[
\frac{\nu \frac{\partial^2 y}{\partial x^2}}{\nu \frac{\partial^2 y}{\partial y^2}} \sim \frac{\frac{U_0}{L^2}}{\frac{U_0}{\delta^2}} = \left( \frac{\delta}{L} \right)^2
\]

Away from the leading edge, \( \delta \ll L \)

\( \Rightarrow \) Expect \( \frac{\partial^2 y}{\partial y^2} \) to be
the most important
viscosity term
To ensure that viscosity is important in the BL

\[ l \sim \frac{v^{\frac{3}{2}}}{\frac{\nu^2}{\delta x}} = \frac{v L v}{u_0 \delta x} = \frac{v L}{u_0 \delta x} \Rightarrow \delta \sim \left( \frac{v L}{u_0} \right)^{\frac{1}{2}} \]

Impulsively started plate: \( \delta \sim (vt)^{\frac{1}{2}} \)

Flat plate BL: \( \delta \sim \left( \frac{v L}{u_0} \right)^{\frac{1}{2}} \)

Check that BL is thin

\[ \delta l \sim \frac{1}{l} \left( \frac{v L}{u_0} \right)^{\frac{1}{2}} = \left( \frac{u}{u_0 L} \right)^{\frac{1}{2}} \approx 3 \times 10^{-3} \text{ for baseball} \]

Conclusion:

Away from the plate: Inertia \& pressure

Near the plate: Inertia \& pressure \& viscous
Analysis of the flat plate BL

To get the details of the velocity profile, use a similarity solution. Cast the PDE as an ODE in

\[ \eta = \frac{y}{(\nu y u_0)^{1/2}} \]

We will use an approx. approach that gives bulk measures (BL thickness, shear stress)
BL thickness

Where is \( \delta ? \)

1. \( \delta_0 \)
   
   You could say that \( u(\delta) = U_0 \), but you never get there, so define
   
   \[
   \frac{U(\delta_0)}{U_0} = 0.39 \quad \text{Arbitrary}
   \]

2. Displacement thickness \( \delta^+ \)
   
   Amount that streamlines are displaced by the BL
   
   \[
   \int_0^h U_0 \, dy = \int_0^{\delta^+} u \, dy
   \]
   
   \[
   = \int_0^h u \, dy + \int_{\delta^+}^{h} u \, dy
   \]
   
   \[
   = \int_0^h u \, dy + U_0 \delta^+
   \]
   
   \[
   \delta^+ = \int_0^h \frac{1}{U_0} (U_0 - u) \, dy
   \]
   
   Let \( h \to \infty \) (i.e., outside BL)
   
   \[
   \delta^+ = \int_0^\infty (1 - \frac{u}{U_0}) \, dy \quad \text{Based on volume flux}
   \]

3. Momentum thickness
   
   \[
   \Theta = \int_0^\infty \frac{U_0}{U_0} (1 - \frac{u}{U_0}) \, dy \quad \text{Based on momentum flux}
   \]