Chapter 6

Basic flows

6.1 Uniform Flow at An Angle α

Given velocity field is:

$$\vec{V} = (V_\infty \cos\alpha, V_\infty \sin\alpha)$$

Check if conservation of mass is satisfied first to test if it is a physically possible flow?

$$\nabla \cdot \vec{V} \stackrel{?}{=} 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \stackrel{?}{=} 0$$

Since u and v are both constants, $\nabla \cdot \vec{V} = 0$
Therefore ψ exists. From conservation of mass,

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$
$$u = V_\infty \cos\alpha = \frac{\partial \psi}{\partial y}$$
$$\psi = V_\infty \cos\alpha \ \ y + f(x)$$
$$\frac{\partial \psi}{\partial x} = -v = 0 + f'(x)$$
$$f'(x) = -V_\infty \sin\alpha \quad or \quad f(x) = -V_\infty \sin\alpha \ \ x + g(y)$$
$$\psi = -V_\infty \sin\alpha \ \ x + V_\infty \cos\alpha \ \ y$$
$$\psi = const. = -V_\infty \sin\alpha \ \ x + V_\infty \cos\alpha \ \ y$$
$$\frac{\psi}{V_\infty} = -\sin\alpha \ \ x + \cos\alpha \ \ y$$
$$\frac{\psi}{V_\infty \cos\alpha} = -\tan\alpha \ \ x + y$$

Equation of streamlines:

$$y = tan\alpha \ \ x + \frac{\psi}{V_\infty \cos\alpha}$$

Check if the given flow is a potential flow?

$$\nabla \times \vec{V} \stackrel{?}{=} 0$$
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \stackrel{?}{=} 0$$

Since V_∞ and α are constant throughtout the flow, $\nabla \times \vec{V} = 0$
Therefore ϕ exists and $\vec{V} = \nabla\phi$.

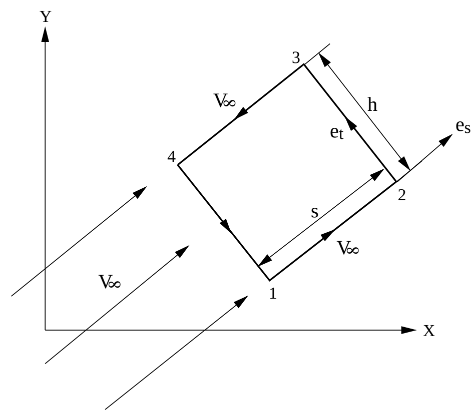
$$\vec{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath}$$
$$\frac{\partial\phi}{\partial x} = V_\infty \cos\alpha = u$$
$$\phi = V_\infty \cos\alpha \;\; x + f(y)$$
$$\frac{\partial\phi}{\partial y} = 0 + f'(y) = v$$
$$f'(y) = V_\infty \sin\alpha \quad or \quad f(y) = V_\infty \sin\alpha \;\; y + f(x)$$
$$\phi = V_\infty \cos\alpha \;\; x + V_\infty \sin\alpha \;\; y \quad \text{(uniform flow at an angle } \alpha)$$
$$\phi = const. = V_\infty \cos\alpha \;\; x + V_\infty \sin\alpha \;\; y$$
$$\frac{\phi}{V_\infty \sin\alpha} = \frac{x}{\tan\alpha} + y$$

Equation of Equipotential lines:

$$y = -\frac{1}{\tan\alpha}x + \frac{\phi}{V_\infty \sin\alpha}$$

ϕ constant lines are orthogonal to ψ constant lines.

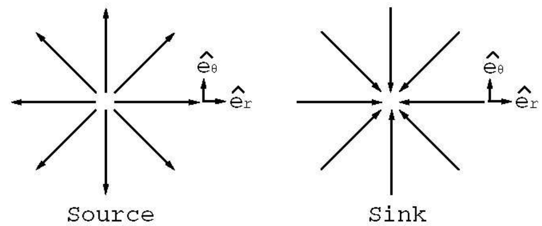
6.1.1 Γ: Contour Integral over a Close Curve$C$



$$\Gamma = -\oint \vec{V} \cdot d\vec{l}$$
$$= -\left[\int_1^2 \vec{V} \cdot d\vec{l} + \int_2^3 \vec{V} \cdot d\vec{l} + \int_3^4 \vec{V} \cdot d\vec{l} + \int_4^1 \vec{V} \cdot d\vec{l}\right]$$
$$= -\left[\int_1^2 (V_\infty \hat{e}_s) \cdot (ds\ \hat{e}_s) + \int_2^3 (V_\infty \hat{e}_s) \cdot (dh\ \hat{e}_t) + \int_3^4 (V_\infty \hat{e}_s) \cdot (-ds\ \hat{e}_s) + \int_4^1 (V_\infty \hat{e}_s) \cdot (-dh\ \hat{e}_t)\right]$$
$$= -\left[(V_\infty s) + 0 + (-V_\infty s) + 0\right] \equiv 0$$

6.2 2-D Source (Line Source)

Definition: A source is a point from which fluid issues along radial lines. Streamlines are straight lines emanating from a central point. Velocity varies inversely with distance from the origin.



From the definition of the source the velocity vector can be written as:

$$\vec{V} = v_r \hat{e}_r$$

where $v_r \propto \frac{1}{r}$ or $v_r = \frac{c}{r}$, and $v_\theta = 0$ where C is a constant.
Check if the assumed flow is physically possible.

$$\vec{V} = \frac{c}{r}\hat{e}_r + 0\hat{e}_\theta$$
$$\nabla \cdot \vec{V} \stackrel{?}{=} 0$$
$$\nabla \cdot \vec{V} = \frac{1}{r}\left[\frac{\partial(rv_r)}{\partial r} + \frac{\partial(v_\theta)}{\partial \theta}\right] = \frac{1}{r}\left[\frac{\partial(c)}{\partial r} + \frac{\partial(0)}{\partial \theta}\right] \equiv 0$$

Flow is physically possible and ψ exists.

6.2.1 Evaluation of c

From mass conservation for a steady flow we know

$$\int d\dot{m} = 0$$

From continuity the mass of fluid per unit time crossing any circle centered at the source is a constant and equal to the mass of fluid issuing per unit time from the source. Consider a cylinder centered on the source. There is mass flowing out only from the sides of the cylinder.

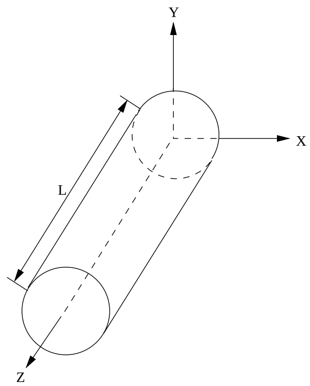
$$d\vec{A}_r = h_\theta h_z d\theta\ dz\hat{e}_r = r\ d\theta\ dz\hat{e}_r$$
$$\dot{m} = \int_0^L \int_0^{2\pi} \rho\vec{V} \cdot d\vec{A} = \int_0^L \int_0^{2\pi} \rho(V_r\ \hat{e}_r) \cdot (r\ d\theta\ dz)\hat{e}_r$$

It is a 2-D flow and hence the integral can be reduced to:

$$\dot{m} = L\int_0^{2\pi} \rho V_r\ r\ d\theta$$

V_r is not a function of θ. V_r is only a function of r.

$$\dot{m} = L\int_0^{2\pi} \rho\left(\frac{c}{r}\right) r\ d\theta = \rho\ L\ c\ 2\pi$$



Cylinder centered at a source

Volume flow per second is:

$$\frac{\dot{m}}{\rho} = c\, 2\pi\, L$$

Define K as the source strength. It is physically the rate of volume flow from the source per unit depth into the page (2-D).

$$K = 2\pi c \quad or \quad c = \frac{K}{2\pi}$$

then the velocity becomes:

$$v_r = \frac{K}{2\pi r}$$

Since $\nabla \cdot \vec{V} = 0$ is satisfied, the flow is physically possible and from the definition of ψ in polar coordinates, ψ can be found.

$$\begin{aligned}
& v_r = \frac{\partial \psi}{r \partial \theta} \quad and \quad v_\theta = -\frac{\partial \psi}{\partial r} \\
& \frac{\partial \psi}{r \partial \theta} = \frac{K}{2\pi r} \\
& \frac{\partial \psi}{\partial \theta} = \frac{K}{2\pi} \\
& \psi = \frac{K}{2\pi}\theta + f(r) \\
& \frac{\partial \psi}{\partial r} = -v_\theta = 0 + f'(r) = 0 \\
& f(r) = const. \\
& \psi = \frac{K}{2\pi}\theta
\end{aligned}$$

Since the source strength, K is a constant, ψ constant lines are radial lines.

$$\psi = \frac{K}{2\pi}\theta = const.$$

Is $\nabla \times \vec{V} \stackrel{?}{=} 0$.

$$\nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & r v_\theta & v_z \end{vmatrix} = \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial (v_r)}{\partial \theta}\right]\hat{e}_z \equiv 0$$

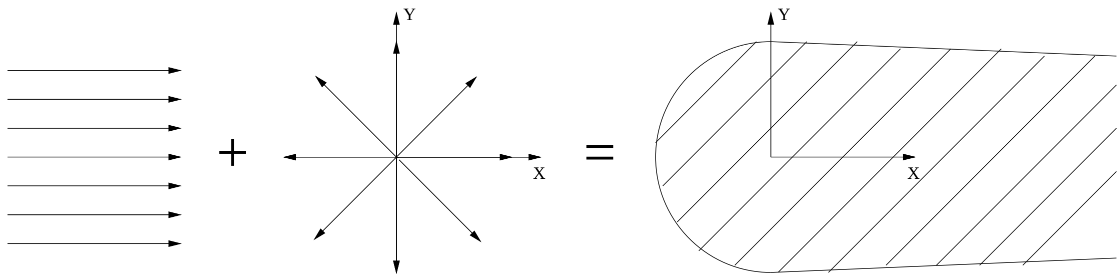
therefore ϕ exists.

$$
\begin{aligned}
&v_r = \frac{\partial \phi}{\partial r} = \frac{K}{2\pi r} \quad and \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \\
&\phi = \frac{K}{2\pi}\ln r + g(\theta) \\
&\frac{\partial \phi}{\partial \theta} = 0 + g'(\theta) = 0 \\
&g(\theta) = const. \\
&\therefore \phi = \frac{K}{2\pi}\ln r
\end{aligned}
$$

6.3 Combination of Potential Flows

Uniform flow and source/sink satisfy Laplace equation and therefore superposition is possible.

6.3.1 Combination of a Uniform Flow to The Right ($\alpha = 0$) and A Source at The Origin



Quantity	Uniform flow	Source/Sink	Combination
$\vec{V}$	$V_\infty \hat{\imath}$	$\pm\frac{K}{2\pi r}\hat{e}_r$	$V_\infty \hat{\imath} \pm \frac{K}{2\pi r}\hat{e}_r$
ϕ	$V_\infty x$	$\pm\frac{K}{2\pi}\ln r$	$V_\infty x \pm \frac{K}{2\pi}\ln r$
ψ	$V_\infty y$	$\pm\frac{K}{2\pi}\theta$	$V_\infty y \pm \frac{K}{2\pi}\theta$

Stagnation Point:

At the stagnation point, $\vec{V} \equiv 0$

$$
\begin{aligned}
v_r &= V_\infty \cos\theta + \frac{K}{2\pi r} = 0 \\
v_\theta &= -V_\infty \sin\theta = 0
\end{aligned}
$$

Solve for θ and r at the stagnation point to get $(r_{stag}, \theta_{stag})$. Proceed to find ψ_{stag} to get the shape of the body. From $v_\theta = 0$:

$$
\begin{aligned}
\sin\theta &= 0 \\
\theta &= 0 \;\; or \;\; \pm\pi
\end{aligned}
$$

Case 1: $\theta_s = 0$. Sove for r_s from v_r=0

$$\begin{aligned}
v_r &= V_\infty \cos\theta + \frac{K}{2\pi r} = 0 \\
\text{if} \quad \theta &= \theta_s = 0 \\
\cos\theta_s &= 1 \\
v_r &= V_\infty + \frac{K}{2\pi r_s} = 0 \\
\text{or} \quad r_s &= -\frac{K}{2\pi V_\infty}
\end{aligned}$$

Impossible solution as $r_s < 0$; ($\because K$ and V_∞ are positive)
Case 2: $\theta_s = \pm\pi$.

$$\begin{aligned}
v_r &= -V_\infty + \frac{K}{2\pi r_s} = 0 \\
r_s &= \frac{K}{2\pi V_\infty}
\end{aligned}$$

$\theta_s = +\pi$ for the upper half of the body
$\theta_s = -\pi$ for the lower half of the body
Coordinates of the stagnation point:

$$(r_s, \theta_s) = \left(\frac{K}{2\pi V_\infty}, \pm\pi\right)$$

Body Shape (Stagnation Streamline):

A general expression for the streamfunction for the combined flow is:

$$\psi = V_\infty r \sin\theta + \frac{K}{2\pi}\theta$$

Find ψ_s (= the body shape) by substituting (r_s, θ_s).

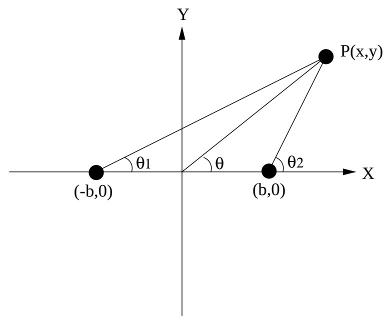
$$\psi_s = V_\infty r \sin(\pm\pi) + \frac{K}{2\pi}(\pm\pi) = \pm\frac{K}{2} = const.$$

In Cartesian coordinate, the general expression for the body streamfunction becomes:

$$\begin{aligned}
\pm\frac{K}{2} &= V_\infty y + \frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \\
\frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) &= \left(\pm\frac{K}{2} - V_\infty y\right) \\
\tan^{-1}\left(\frac{y}{x}\right) &= \left(\pm\pi - \frac{2\pi V_\infty y}{K}\right) \\
\frac{y}{x} &= \tan\left(\pm\pi - \frac{2\pi V_\infty y}{K}\right) \\
x &= \frac{y}{\tan\left(\pm\pi - \frac{2\pi V_\infty y}{K}\right)}
\end{aligned}$$

To find maximum y value, consider $\psi = K/2$ (upper half of the body).

$$\begin{aligned}
\frac{K}{2} &= V_\infty y + \frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \\
y_{max} &= y_{@x=\infty} = \frac{K}{2V_\infty}
\end{aligned}$$



6.3.2 Combined Flow of a Source at $(-b, 0)$ and a Sink at $(b, 0)$

$$\psi = \frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2$$

where θ_1 and θ_2 are measured from the center of the source and sink respectively.

$$\theta_1 = \tan^{-1}\left(\frac{y}{x+b}\right), \quad \theta_2 = \tan^{-1}\left(\frac{y}{x-b}\right)$$

$$\theta_2 - \theta_1 = \tan^{-1}\left(\frac{y}{x-b}\right) - \tan^{-1}\left(\frac{y}{x+b}\right)$$

$$\theta_2 - \theta_1 = \tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\psi = \psi_1 + \psi_2 = \frac{K}{2\pi}(\theta_1 - \theta_2)$$

$$\theta_1 - \theta_2 = -\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\psi_{source+sink} = -\frac{K}{2\pi}\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\frac{2\pi\psi}{K} = -\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\tan\left(\frac{2\pi\psi}{K}\right) = -\frac{2by}{x^2+y^2-b^2}$$

$$x^2 + y^2 + 2by\cot\left(\frac{2\pi\psi}{K}\right) = b^2$$

$$(x-0)^2 + \left(y + b\cot\left[\frac{2\pi\psi}{K}\right]\right)^2 = b^2\left(1 + \cot^2\left[\frac{2\pi\psi}{K}\right]\right)$$

$$(x-0)^2 + \left(y + b\cot\left[\frac{2\pi\psi}{K}\right]\right)^2 = b^2\csc^2\left[\frac{2\pi\psi}{K}\right]$$

Equation of a circle with center at $\left(0, \pm b\cot\frac{2\pi\psi}{K}\right)$ and radius of $\left(b\csc\frac{2\pi\psi}{K}\right)$. When $y = 0$, $x = \pm b$. All streamlines go through $\pm b$.

6.3.3 Uniform Flow to The Right+Source $(-b, 0)$+ Sink $(b, 0)$ (Rankine oval)

- Source of strength K placed at $(-b, 0)$
- Sink of strength K placed at $(b, 0)$
- Uniform flow to the right $(\alpha = 0)$

$$\psi = V_\infty r \sin\theta + \frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2$$

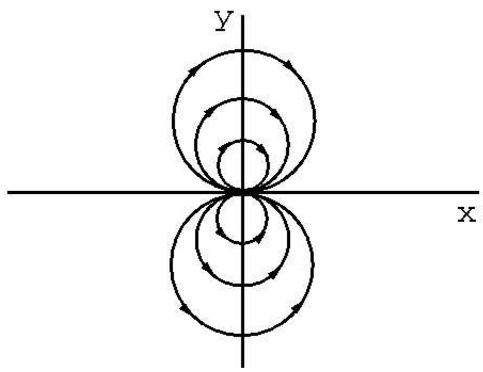
Problem:

Analyze Rankine oval.

6.3.4 2-D Doublet

Definition: A doublet is obtained when a source and sink of equal strength approach each other so that the product of their strength and the distance apart remains a constant.

$$B = K(2b) = constant$$



Doublet

$$\psi_{source+sink} = -\frac{K}{2\pi}\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$= -\frac{2bK}{4\pi}\frac{\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)}{b}$$

$$\lim_{2bK\to\mu}\psi_{source+sink} = \psi_{doublet} = -\frac{\mu}{4\pi}\lim_{b\to 0}\left[\frac{\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)}{b}\right]$$

Using L'Hospital's rule:

$$\psi_{doublet} = -\frac{\mu}{4\pi}\lim_{b\to 0}\left[\frac{\frac{\frac{d}{db}\frac{2by}{x^2+y^2-b^2}}{1+\left(\frac{2by}{x^2+y^2-b^2}\right)^2}}{\frac{db}{db}}\right] = -\frac{\mu}{4\pi}\lim_{b\to 0}\left[\frac{\frac{(x^2+y^2-b^2)(2y)-(2by)(-2b)}{(x^2+y^2-b^2)^2}}{1+\left(\frac{2by}{x^2+y^2-b^2}\right)^2}\right] = -\frac{\mu}{4\pi}\left(\frac{2y}{x^2+y^2}\right)$$

$$\psi_{doublet} = -\frac{\mu}{2\pi}\frac{\sin\theta}{r}$$

Streamlines

$$x^2 + y^2 + \frac{\mu y}{2\pi\psi} = 0$$

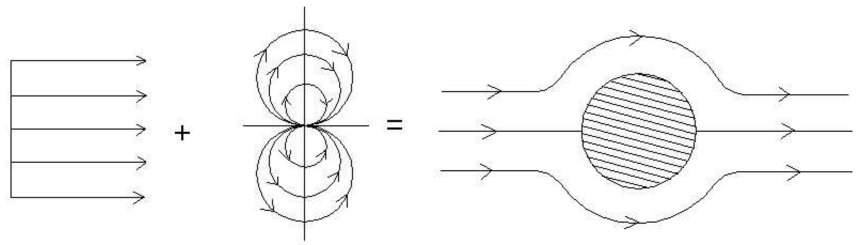
$$(x-0)^2 + \left(y + \frac{\mu}{4\pi\psi}\right)^2 = \left(\frac{\mu}{4\pi\psi}\right)^2$$

Streamlines are circles centerd on the y-axis a distance $-\frac{\mu}{4\pi\psi}$ from the x-axis with a radius of $\left|\frac{\mu}{4\pi\psi}\right|$. All circles pass through the origin.

Problem:

Show that $\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r}$ for a 2-D doublet.

6.3.5 Uniform Flow to The Right + A 2-D Doublet



Quantity	Uniform flow	2-D doublet	Combination
$\vec{V}$	$V_\infty \hat{\imath}$		
ϕ	$V_\infty x$	$\frac{\mu}{2\pi}\frac{\cos\theta}{r}$	$V_\infty x + \frac{\mu}{2\pi}\frac{\cos\theta}{r}$
ψ	$V_\infty y$	$-\frac{\mu}{2\pi}\frac{\sin\theta}{r}$	$V_\infty y - \frac{\mu}{2\pi}\frac{\sin\theta}{r}$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\left[V_\infty r\cos\theta - \frac{\mu}{2\pi}\frac{\cos\theta}{r}\right]$$

$$= V_\infty\cos\theta\left[1 - \underbrace{\frac{\mu}{2\pi V_\infty}}_{1/R^2}\frac{1}{r^2}\right] = V_\infty\cos\theta\left[1 - \left(\frac{R}{r}\right)^2\right]$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = -\left[V_\infty + \frac{\mu}{2\pi r^2}\right]\sin\theta = -V_\infty\sin\theta\left[1 + \left(\frac{R}{r}\right)^2\right]$$

Where $R^2 = \frac{\mu}{2\pi V_\infty}$.

Stagnation Points ($\vec{V} = 0$)

Set $V_\theta = 0$.

$$0 = -V_\infty\sin\theta\left[1 + \left(\frac{R}{r}\right)^2\right]$$

$$\sin\theta = 0 \quad \text{or} \quad \theta_s = (0 \ \text{ or } \ \pi)$$

Now set $V_r = 0$.

$$0 = V_\infty\sin\theta\left[1 - \left(\frac{R}{r}\right)^2\right]$$

$$\text{For } \theta = 0 \ \text{ or } \ \pi, \quad \cos\theta \neq 0$$

$$\therefore \left[1 - \left(\frac{R}{r}\right)^2\right] \equiv 0 \ \text{ or } \ r^2 = R^2 = \frac{\mu}{2\pi V_\infty}$$

The stagnation points are located at

$$(r_s, \theta_s) \equiv (R, 0) \text{ and } (R, \pi)$$

For Cylindrical System

$$\left.\begin{aligned} \phi &= V_\infty r \cos\theta \left(1 + \frac{R^2}{r^2}\right) \\ \psi &= V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) \\ V_r &= V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) \\ V_\theta &= -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) \end{aligned}\right\} (r \geq R)$$

Where $R^2 = \dfrac{\mu}{2\pi V_\infty}$
Substitute $(r_s, \theta_s) = (R, 0)$ or (R, π) in the expression for ψ.

$$\begin{aligned} &\psi_s = 0 \\ &\text{at } r = R \text{ (surface of the cylinder)} \\ &V_r = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right) = 0 \text{ (no flow out of the cylinder)} \\ &V_\theta = -2V_\infty \sin\theta \\ &C_p|_{r=R} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty} = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \frac{V_r^2 + V_\theta^2}{V\infty^2} = 1 - \left(\frac{V_\theta}{V_\infty}\right)^2 = 1 - 4\sin^2\theta \\ &C_p \text{ (2-D cylinder) } = 1 - 4\sin^2\theta \end{aligned}$$

6.4 2-D Vortex Flow (Potential Vortex)

A 2-D point vortex is a mathematical concept that induces a velocity field given by

$$V_r = 0, \quad V_\theta = \frac{const.}{r} = \frac{C}{r}$$

1. Check if the flow satisfies conservation of mass (Is it a physically possible flow?)

$$\begin{aligned} &\nabla \cdot \vec{V} \stackrel{?}{=} 0 \\ &\nabla \cdot \vec{V} = \frac{1}{r}\left[\frac{\partial(V_r r)}{\partial r} + \frac{\partial V_\theta}{\partial\theta}\right] = 0 \rightarrow \psi \text{ exist.} \\ &V_r = \frac{\partial\psi}{r\partial\theta} = 0 \rightarrow \psi = g(r) \\ &V_\theta = -\frac{\partial\psi}{\partial r} = \frac{C}{r} \rightarrow \psi = -C\ln r + f(\theta) \\ &\frac{\partial\psi}{\partial\theta} = f'(\theta) = 0 \\ &f(\theta) = const. \\ &\therefore \psi = -C\ln r + const. \end{aligned}$$

When $r \to 0$, $V_\theta = \infty$ and $\psi \to \infty$. To eliminate the infinite velocity it is arbitrary assumed that $\psi = 0$ at $r = R$

$$\therefore \psi = -C \ln R + const. = 0$$
$$const. = C \ln R$$
$$\psi = -C \ln\left(\frac{r}{R}\right) \text{ for } (r \geq R)$$

2. Check if the flow is irrotational

$$\nabla \times \vec{V} \stackrel{?}{=} 0$$

$$\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

Vorticity or $(\nabla \times \vec{V})$ in the $r - \theta$ plane

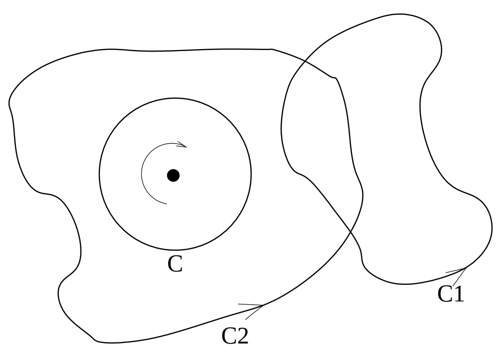
$$\frac{1}{r}\left(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right) = \left(\frac{\partial C}{\partial r} - \frac{\partial 0}{\partial \theta}\right) = 0 \to \phi \text{ exist.}$$

Problem:

Show that $\phi = \quad C\theta$.

Evaluate the Constant C

Evaluate the circulation Γ around the point vortex.



1. Around closed curve $C1$ that does not include the point vortex

$$\Gamma_{C1} = -\oint_{C1} \vec{V} \cdot d\vec{l} = \iint_{S_1} \underbrace{(\nabla \times \vec{V})}_{0} \cdot d\vec{A} = 0$$

2. Around $C2$ that includes the point vortex.

$$\begin{aligned}
\Gamma_{C2} &= -\left[\oint_{C2}(V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right] \\
&= -\left[\oint_{C}(V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right] + \left[\oint_{C2-C}(V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right] \\
&= -\left[\oint_{C}(V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta) + 0\right] \\
&= -\left[\oint_{C} V_r dr + \oint_{C} V_\theta r\ d\theta\right] = -\left[0 + \int_0^{2\pi}\left(\frac{C}{r}\right) r\ d\theta\right] = -2\pi C \\
\Gamma_{C2} &= -2\pi C \ \text{ or } \ -\frac{\Gamma}{2\pi} = C
\end{aligned}$$

This implies that the circulation evaluated for a curve enclosing the 2-D vortex is a constant and not equal to zero.

For a potential vortex, $V_\theta = -\frac{\Gamma}{2\pi r}$ and $\psi = -C \ln\frac{r}{R}$.

$$\therefore \psi = \frac{\Gamma}{2\pi}\ln\frac{r}{R} \ \text{ for } \ r \geq R$$

$\psi = const$, then $\ln\frac{r}{R} = \frac{2\pi\psi}{\Gamma}$, $\frac{r}{R} = e^{2\pi\psi/\Gamma}$, $r = Re^{2\pi\psi/\Gamma}$

Streamlines are concentric circles with center at the 2D point vortex.

$V_r = \frac{\partial\phi}{\partial r}$, $V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\Gamma}{2\pi r}$

$\phi = -\frac{\Gamma}{2\pi}\theta + C$ or $\phi = -\frac{\Gamma}{2\pi}\theta$ (straight lines form the origin).

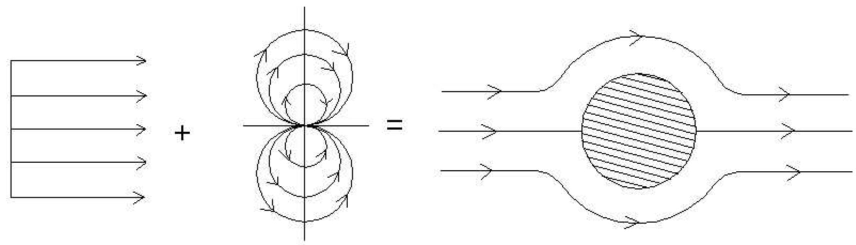
A line vortex can be described as a string of rotating particles. A chain of fluid particles are spinning on their common axis and carrying around with them a swirl of particles which flow around in circles.

A cross-section of such a string of particles and its associated flow shows a spinning point 'outside' of which is streamline flow in concentric circles.

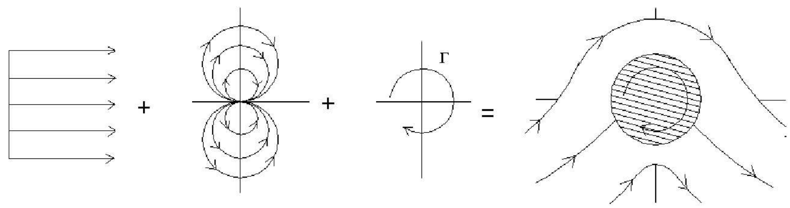
Vortices are common in nature, the difference between a real vortex as opposed to a theoretical line vortex is that the former has a core of fluid which is rotating as a 'solid', although the associated 'swirl' outside is the same as the flow 'outside' the point vortex.

6.4.1 Uniform Flow to The Right ($\alpha = 0$) + A 2-D Doublet + A 2-D Point Vortex

- As we all know, uniform flow to the right + 2-D Doublet = non-lifting over a cylinder



- Uniform flow to the right + 2-D Doublet + 2-D Point Vortex = Lifting flow over a cylinder



The parameters for lifting flow over a cylinder are as follow (spinning cylinder):

Quantity	Non-lifting flow over a cylinder	Vortex of Strength Γ	Combination
ψ	$V_\infty r \sin\theta(1-\frac{R^2}{r^2})$	$\frac{\Gamma}{2\pi}\ln\frac{r}{R}$	$V_\infty r \sin\theta(1-\frac{R^2}{r^2}) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$
ϕ	$V_\infty r \cos\theta(1+\frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi}\theta$	$V_\infty r \cos\theta(1+\frac{R^2}{r^2}) - \frac{\Gamma}{2\pi}\theta$
V_r	$V_\infty \cos\theta(1-\frac{R^2}{r^2})$	0	$V_\infty \cos\theta(1-\frac{R^2}{r^2})$
V_θ	$-V_\infty \sin\theta(1+\frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi r}$	$-V_\infty \sin\theta(1+\frac{R^2}{r^2}) - \frac{\Gamma}{2\pi r}$

- Flow satisfies continuity at every point $r \geq R$.
  $\therefore \nabla \cdot \vec{V} = 0.$
- Flow satisfies irrotationality at every point $r \geq R$.
  $\therefore \nabla \times \vec{V} = 0.$

Determine the stagnation points for the combined flow

At the stagnation points, $\vec{V} = 0$,$V_r = 0 = V_\theta$. If we set $V_r = 0$, we get $r_s = R$ *or* $\theta_s = \pm\frac{\pi}{2}$,

$\underline{Case(1)}$: $r = R_s = R$

$$V_\theta = -V_\infty \sin\theta_s(1+1) - \frac{\Gamma}{2\pi R} = 0$$

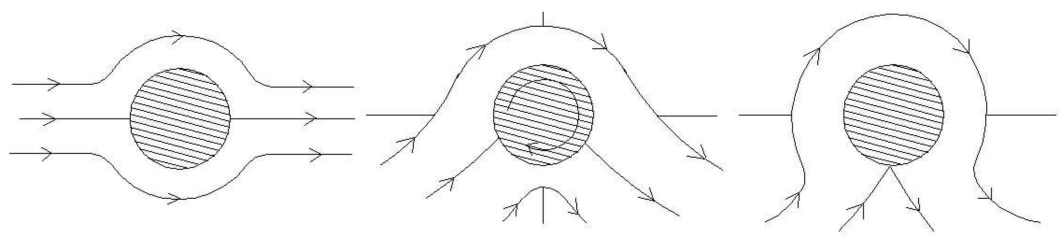
$$sin\theta_s = -\frac{\Gamma}{4\pi R V_\infty} \leq 0$$

Because $\Gamma > 0$, $4\pi R V_\infty > 0$, $\frac{\Gamma}{4\pi V_\infty} > 0$. When $\frac{\Gamma}{4\pi V_\infty} < R$, θ_s has one value in the third quadrant and one in the fourth quadrant that will satisfy the above relation.
The coordinates of the stagnation point are:

$$y_s = Rsin\theta_s = -\frac{\Gamma}{4\pi V_\infty}$$

$$x_s = \pm\sqrt{R^2 - y_s^2} = \pm\sqrt{R^2 - (\frac{\Gamma}{4\pi V_\infty})^2}$$

When $\frac{\Gamma}{4\pi V_\infty} = R$, there is only one solution. However, the method fails when $\frac{\Gamma}{4\pi V_\infty} > R$.



$\Gamma = 0$ $0 < \frac{\Gamma}{4\pi V_\infty} < R$ $\frac{\Gamma}{4\pi V_\infty} = R$

$\underline{Case(2)}$: $\theta = \pm\frac{\pi}{2}$

$\underline{Case(2a)}$: $\theta = \frac{\pi}{2}$, $r = r_s$

$$V_r = V_\infty \cos(\frac{\pi}{2})(1 - \frac{R^2}{r^2}) = 0$$

$$V_\theta = -V_\infty \sin(\frac{\pi}{2})(1 + \frac{R^2}{r^2}) - \frac{\Gamma}{2\pi r} = 0$$

$${r_s}^2 + \frac{\Gamma}{2\pi V_\infty} r_s + R^2 = 0$$

$$r_s = -\frac{\Gamma}{4\pi V_\infty} \pm \sqrt{(\frac{\Gamma}{4\pi V_\infty})^2 - R^2}$$

When $\frac{\Gamma}{4\pi V_\infty} > R$, r_s results in negative number for all cases. Because both roots are negative, the solution is impossible.

$\underline{Case(2b)}$: $\theta = -\frac{\pi}{2}$, $r = r_s$

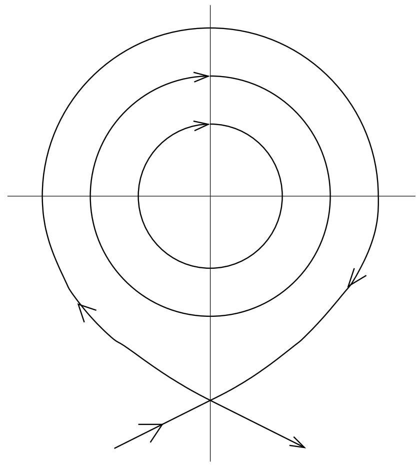
$$V_r = V_\infty \cos(-\frac{\pi}{2})(1 - \frac{R^2}{r^2}) = 0$$

$$V_\theta = -V_\infty \sin(-\frac{\pi}{2})(1 + \frac{R^2}{r^2}) - \frac{\Gamma}{2\pi r} = 0$$

$${r_s}^2 - \frac{\Gamma}{2\pi V_\infty} r_s + R^2 = 0$$

$$r_s = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{(\frac{\Gamma}{4\pi V_\infty})^2 - R^2}$$

When $\frac{\Gamma}{4\pi V_\infty} > R$, we get $r_s = \frac{\Gamma}{4\pi V_\infty} - \sqrt{(\frac{\Gamma}{4\pi V_\infty})^2 - R^2} < R$. So we can't use this solution.
However, $\theta_s = -\frac{\pi}{2}$ and $\frac{\Gamma}{4\pi V_\infty} > R$ is an acceptable solution when $r_s = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{(\frac{\Gamma}{4\pi V_\infty})^2 - R^2} > R$



Force on a Cylinder with Circulation in a Uniform Steady Flow

Force on an elemental distance on the surface of the cylinder:

$$d\vec{F} = -p_b R d\theta \hat{e_r}$$

$$d\vec{F} = -p_b R d\theta(\cos\theta\hat{i} + \sin\theta\hat{j})$$

$$\vec{F} = \int_0^{2\pi} -p_b R d\theta(\cos\theta\hat{i} + \sin\theta\hat{j})$$

The drag per unit span is

$$D' = \vec{F}\cdot\hat{j} = \int_0^{2\pi} -p_b cos\theta R d\theta$$

The lift per unit span is

$$L' = \vec{F}\cdot\hat{i} = \int_0^{2\pi} -p_b sin\theta R d\theta$$

As we know, in incompressible flow the total pressure $p_o = p + \frac{\rho V^2}{2}$, which is a constant throughout the flow. $p_b = p_o - \frac{\rho(V_r^2+V_\theta^2)}{2}$. Besides, there is no flow normal to the surface, $V_r = 0$.

$$p_b = p_o - \frac{\rho}{2}V_\theta^2$$

$$p_b = p_o - \frac{\rho}{2}(-2V_\infty\sin\theta - \frac{\Gamma}{2\pi R})^2$$

$$p_b = p_o - 2\rho {V_\infty}^2(\sin\theta)^2 - \rho V_\infty sin\theta\frac{\Gamma}{R\pi} - \frac{\rho\Gamma^2}{8\pi^2 R^2}$$

$$\therefore D' = R\int_0^{2\pi} -(p_o - 2\rho {V_\infty}^2(\sin\theta)^2 - \rho V_\infty sin\theta\frac{\Gamma}{R\pi} - \frac{\rho\Gamma^2}{8\pi^2R^2})cos\theta d\theta = 0$$

which means that d'Alembert's paradox still prevails.

$$L' = R\int_0^{2\pi} -(p_o - 2\rho {V_\infty}^2(\sin\theta)^2 - \rho V_\infty sin\theta\frac{\Gamma}{R\pi} - \frac{\rho\Gamma^2}{8\pi^2R^2})sin\theta d\theta = \rho V_\infty\Gamma$$

which is the Kutta-Joukowski theorem.

In inviscid, incompressible flow, the resultant force per unit span acting on a 2-D body of any cross section is equal to $\rho V_\infty \Gamma$ and acts perpendicular to V_∞.