

Chapter 5

Simplifications to Conservation Equations

5.1 Steady Flow

If fluid properties at a point in a field do not change with time, then they are a function of space only. They are represented by:

$$\varphi = \varphi(q_1, q_2, q_3)$$

Therefore for a steady flow $\frac{\partial \varphi}{\partial t} = 0$.

5.2 One-, Two-, and Three-Dimensional Flows

A flow is classified as one-, two-, or three-dimensional depending on the number of space coordinates required to specify all the fluid properties and the number of components of the velocity vector. For example a steady three-dimensional flow requires three space coordinates to specify the property and the velocity vector is given by: $\vec{V} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$. Most real flows are three-dimensional in nature. On the other hand any property of a two-dimensional flow field requires only two space coordinates to describe it and its velocity has only two components along the two space coordinates that describe the field. The third component of velocity is identically zero everywhere. Steady channel flow between two parallel plates is a perfect example of two-dimensional flow if the viscous effects on the plates are neglected. The properties of the flow can be uniquely represented by $\varphi = \varphi(q_1, q_2)$ and the velocity vector can be written as $\vec{V} = v_1 \hat{e}_1 + v_2 \hat{e}_2$. In addition $\frac{\partial \varphi}{\partial q_3} = 0$. The complexity of analysis increases considerably with the number of dimensions of the flow field. In one-dimensional flow properties vary only as a function of one spatial coordinate and the velocity component in the other two directions are identically zero. All derivatives in the other directions are identically zero. In other words $\varphi = \varphi(q_1)$, $\vec{V} = v_1 \hat{e}_1$ and $\frac{\partial \varphi}{\partial q_2} = \frac{\partial \varphi}{\partial q_3} = 0$.

5.3 Axisymmetric Flow

In axisymmetric flow the variation of flow variables are zero in the direction of rotation but the velocity component in the rotation direction is not zero. For example if the flow is symmetric about the q_1 axis and the plane containing the axis q_1 and q_3 are rotated in the direction of q_2 then $\frac{\partial \varphi}{\partial q_2} = 0$ but $v_2 \neq 0$.

5.4 Ideal Fluid

Non-heat conducting, inviscid, incompressible, homogeneous fluid is defined as ideal fluid. The dependent variables of *ideal fluid* are p and \vec{V} . The equations of the Fluid flow are:

$$(1) \quad \nabla \cdot \vec{V} = 0$$

$$(2) \quad \frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{\vec{V} \cdot \vec{V}}{2} \right) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla \left(\frac{p}{\rho} \right) + \vec{f}$$

If we consider conservative body forces only ($\rightarrow \vec{f} = \nabla U$), then the above equation becomes:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{\vec{V} \cdot \vec{V}}{2} \right) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla \left(\frac{p}{\rho} \right) + \nabla U$$

Rearrange the above equation as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) - \vec{V} \times (\nabla \times \vec{V}) = \vec{0}$$

The above equation is valid at any point in an ideal fluid and can be integrated in closed form for two situations.

1. Steady flow along a streamline.
2. Unsteady irrotational flow.

5.5 Streamlines and Stream Function

5.5.1 Streamlines

A streamline is defined as an imaginary line drawn in the fluid whose tangent at any point is in the direction of the velocity vector at that point.

- By definition there is no flow across it at any point.
- Any streamline may be replaced by a solid boundary without modifying the flow.
- Any solid boundary is itself a streamline of the flow around it.

5.5.2 Pathline

This is the path traced out by any one particle of the fluid in motion.

- In unsteady flow, the two are in general different, while in steady flow both are identical.

5.5.3 Equation for A Streamline

$$d\vec{s} \times \vec{V} = 0$$

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

$$d\vec{s} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

$$d\vec{s} \times \vec{V} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ h_1 dq_1 & h_2 dq_2 & h_3 dq_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

$$\underbrace{(V_3 h_2 dq_2 - V_2 h_3 dq_3)}_{=0} \hat{e}_1 + \underbrace{(V_1 h_3 dq_3 - V_3 h_1 dq_1)}_{=0} \hat{e}_2 + \underbrace{(V_2 h_1 dq_1 - V_1 h_2 dq_2)}_{=0} \hat{e}_3 = 0$$

- Differential equations

$$V_3 h_2 dq_2 - V_2 h_3 dq_3 = 0$$

$$V_1 h_3 dq_3 - V_3 h_1 dq_1 = 0$$

$$V_2 h_1 dq_1 - V_1 h_2 dq_2 = 0$$

- Symmetric form

$$\underbrace{\frac{h_1 dq_1}{V_1} = \frac{h_2 dq_2}{V_2} = \frac{h_3 dq_3}{V_3}}_{2-D}$$

From the symmetric form in 2-D:

$$\frac{h_2 dq_2}{h_1 dq_1} = \frac{V_2}{V_1} = \frac{ds_2}{ds_1}$$

where $\frac{ds_2}{ds_1}$ is the slope for the line.

Also if $\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2$ then

$$\frac{V_2}{V_1} = \tan \theta \text{ which is the angle of the velocity vector}$$

The equation of the streamline $d\vec{s} \times \vec{V} = 0$ implies that the slope of the streamline is equal to the angle of the velocity vector at that point. Hence, the velocity vector at any point on the streamline is a tangent to the streamline.

5.5.4 Stream Function

From the symmetric form in 2-D:

$$\frac{h_2 dq_2}{h_1 dq_1} = \frac{V_2}{V_1}$$

Integration yields:

$$q_2 = f(q_1) \text{ or } F(q_1, q_2) = C$$

$$\text{because } V_1 = V_1(q_1, q_2) \text{ and } V_2 = V_2(q_1, q_2).$$

Let us say that F is called a stream function $\bar{\psi}$, or $\bar{\psi} = \bar{\psi}(q_1, q_2) = C$ - a stream function for compressible flows.

Different constants of integration yield different streamlines.

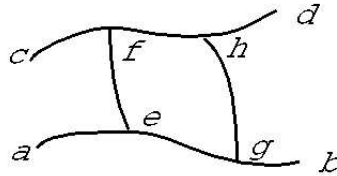


Figure 4.1: Stream lines

Let ab , cd represent two streamlines. No fluid passes ab or cd . Therefore the same mass of fluid must cross gh and ef .

If the streamline ab is arbitrarily chosen as a base, every other streamline in the field can be identified by assigning to it a number $\Delta\bar{\psi}$ equal to the mass of fluid passing, per second per unit depth perpendicular to the plane containing the base streamline and the streamline in question.

$$\Delta\bar{\psi} = C_2 - C_1 = \bar{\psi}_2 - \bar{\psi}_1 = \int_f^e \rho \vec{V} \cdot \hat{e}_n dl = \rho V_n \Delta l$$

where V_n is the normal component of velocity and Δl is the normal distance between streamlines.

$$\text{or } \Delta\bar{\psi} = \rho V_n \Delta l$$

$$\text{or } \frac{\Delta\bar{\psi}}{\Delta l} = \rho V_n$$

and in the limit $\Delta l \rightarrow 0$

$$\frac{\Delta\bar{\psi}}{\Delta l} = \frac{\partial\bar{\psi}}{\partial l} = \rho V_n$$

Thus the velocity component in any direction is obtained by differentiating $\bar{\psi}$ at right angles to that direction.

- This stream function is defined for two-dimensional flow only. In general, it is not possible to define a stream function for three dimensional flow, though there is a special form, for axi-symmetric flows known as the *Stokes stream function*.

5.6 Relation Between $\bar{\psi}$ and \vec{V}

5.6.1 Derivation from The Physical Meaning

Conventions:

- Direction of integration for the chosen coordinate system is ACW.
- Do all derivations in the first quadrant with Δx , Δy and all velocity components (u, v) or (v_r, v_θ) being positive.
- The sign convention yields positive for flow going out and negative for flow going in.
- In line integrals the integral is positive if the flow is left to right if you look in the direction of integration.

5.6.2 Cartesian Coordinate System

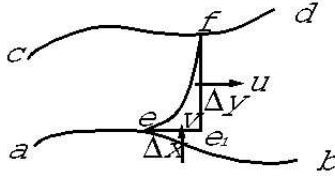


Figure 4.2: Velocity components between stream lines

Mass flow across ef :

$$ef = \Delta\bar{\psi} = - \int_e^{e_1} \rho v dx + \int_{e_1}^f \rho u dy$$

$$\text{or } \Delta\bar{\psi} = -\rho v \Delta x + \rho u \Delta y$$

$$\lim_{\Delta\bar{\psi} \rightarrow 0} d\bar{\psi} = -\rho v dx + \rho u dy \quad [1]$$

$$\text{Since } \bar{\psi} = \bar{\psi}(x, y)$$

$$d\bar{\psi} = \frac{\partial \bar{\psi}}{\partial x} dx + \frac{\partial \bar{\psi}}{\partial y} dy \quad [2]$$

Comparing equation [1] and [2] we get:

$$\rho u = \frac{\partial \bar{\psi}}{\partial y} ; \rho v = -\frac{\partial \bar{\psi}}{\partial x} \quad (\text{compressible flow})$$

For incompressible flow:

$$u = \frac{\partial \bar{\psi} / \rho}{\partial y} = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \bar{\psi} / \rho}{\partial x} = -\frac{\partial \psi}{\partial x}$$

5.7 Stream Function

5.7.1 Ex

Given: 2-D incompressible flow

$$\begin{cases} u = 2x \\ v = -6x - 2y \end{cases}$$

$$\begin{aligned} \frac{dx}{u} &= \frac{dy}{v} \\ \frac{dx}{2x} &= \frac{dy}{-6x - 2y} \quad \text{not a variable separable} \\ \frac{\partial\psi}{\partial y} &= u = 2x, \quad \psi = 2xy + f(x) + C_1 \\ \frac{\partial\psi}{\partial x} &= -v = 6x + 2y = 2y + f'(x) + 0 \\ f'(x) &= 6x, \quad f(x) = 3x^2 + C_2 \\ \psi &= 2xy + 3x^2 + C \end{aligned}$$

5.8 Vorticity, Circulation & Stokes Theorem

5.8.1 Vorticity

Vorticity is defined as twice the angular velocity.

$$\vec{\xi} = 2\vec{\omega} = \nabla \times \vec{V}$$

In 3-D Cartesian coordinates

$$\begin{aligned} \vec{\omega} &= \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \\ \vec{\omega} &= \frac{1}{2} \left\{ \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \right\} \end{aligned}$$

5.8.2 Irrotational Flow

The flow is defined irrotational if $\nabla \times \vec{V} = 0$.

1. $\nabla \times \vec{V} = 0$ at every point in the flow then the flow is irrotational.
2. $\nabla \times \vec{V} \neq 0$ at any point the flow is rotational.

General

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

5.8.3 Circulation

Circulation is defined as the line integral of the velocity around any closed curve.

$$\Gamma = - \oint_C \vec{V} \cdot d\vec{l}$$

Circulation is a kinematic property that depends only on the velocity field and the choice of the curve C . When circulation exists in a flow it simply means that the line integral $\Gamma = - \oint_C \vec{V} \cdot d\vec{l}$ is finite.

5.8.4 Stokes Theorem

The line integral of a vector \vec{V} over C is equal to the surface integral of the normal component of the curl of \vec{V} over S .

$$\oint_C \vec{V} \cdot d\vec{l} = \iint_S (\nabla \times \vec{V}) \cdot d\vec{s}$$

$$\text{or } \Gamma = - \oint_C \vec{V} \cdot d\vec{l} = - \iint_S (\nabla \times \vec{V}) \cdot d\vec{s}$$

1. ϕ exists if and only if

$$\oint_C \vec{V} \cdot d\vec{l} = 0$$

2. If $\oint_C \vec{V} \cdot d\vec{l} = 0$, it does not imply ϕ exists.

$$\vec{V} = \nabla\phi \quad \text{if} \quad \nabla \times \vec{V} = 0$$

5.9 Bernoulli's Equation for A Steady Flow Along A Streamline

For a conservative body force field the equation of motion for an ideal fluid flow is:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) - \vec{V} \times (\nabla \times \vec{V}) = \vec{0}$$

For a steady flow the above equation becomes:

$$\nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) - \vec{V} \times (\nabla \times \vec{V}) = \vec{0}$$

If we scalar multiply both sides of the above equation by $d\vec{S}$ we get:

$$\nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) \cdot (d\vec{S}) - \vec{V} \times (\nabla \times \vec{V}) \cdot (d\vec{S}) = \vec{0} \cdot (d\vec{S})$$

Using the definition of the streamline ($(d\vec{S} \times \vec{V} = \vec{0})$) the second term on the left hand side of the above equation goes to zero reducing to:

$$\nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) \cdot (d\vec{S}) = \vec{0}$$

From the definition of directional derivative the above equation becomes:

$$d \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) = 0$$

which upon integration yields the Bernoulli's equation along a streamline:

$$\left(\frac{p}{\rho} + \frac{V^2}{2} - U \right) = \text{constant}$$

If the body force \vec{f} is $(0, 0, -g)$ then $U = -gZ$ in Cartesian coordinates and the Bernoulli equation becomes:

$$\left(\frac{p}{\rho} + \frac{V^2}{2} + gZ \right) = \text{constant}$$

5.10 Bernoulli's Equation for Irrotational Flow

For irrotational flow ($\rightarrow \nabla \times \vec{V} = 0$) equation of motion becomes:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{\vec{V} \cdot \vec{V}}{2} \right) - \underbrace{\vec{V} \times (\nabla \times \vec{V})}_{=0} = -\nabla \left(\frac{p}{\rho} \right) + \vec{f}$$

For steady flow ($\rightarrow \frac{\partial}{\partial t} = 0$), the above equation becomes:

$$\nabla \left(\frac{V^2}{2} \right) = -\nabla \left(\frac{p}{\rho} \right) + \vec{f}$$

If we consider conservative body forces only ($\rightarrow \vec{f} = \nabla U$), then:

$$\nabla \left(\frac{p}{\rho} + \frac{V^2}{2} - U \right) = \vec{0}$$

Take a dot product with $d\vec{l}$, an elemental length along any arbitrary path:

$$\begin{aligned} & \left[\nabla \left(\frac{p}{\rho} + \frac{V^2}{2} - U \right) = \vec{0} \right] \cdot d\vec{l} \\ & \nabla() \cdot d\vec{l} = d() \\ & d \left[\frac{p}{\rho} + \frac{V^2}{2} - U \right] = 0 \\ & \frac{p}{\rho} + \frac{V^2}{2} - U = \text{constant} \end{aligned}$$

For gravitational body force ($\rightarrow U = -gz$):

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Bernoulli's eqn. valid for ideal, irrotational, steady flow

5.11 Potential Flow

Non-heat conducting, inviscid, incompressible, and irrotational flow of a homogeneous fluid is defined as potential flow.

The dependent variables of *ideal fluid* are p and \vec{V} . The equations of the Fluid flow are:

$$(1) \quad \nabla \cdot \vec{V} = 0$$

$$(2) \quad \frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{\vec{V} \cdot \vec{V}}{2} \right) = -\nabla \left(\frac{p}{\rho} \right) + \vec{f}$$

If we consider conservative body forces only ($\rightarrow \vec{f} = \nabla U$), then the above equation becomes:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{\vec{V} \cdot \vec{V}}{2} \right) = -\nabla \left(\frac{p}{\rho} \right) + \nabla U$$

Rearrange the above equation as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{p}{\rho} + \frac{\vec{V} \cdot \vec{V}}{2} - U \right) = \vec{0}$$

5.12 Velocity Potential (ϕ)

Velocity potential is defined only for ideal irrotational flow for steady or unsteady flow as:

$$\begin{aligned}\vec{V} &= \nabla\phi \\ \vec{V} &= \frac{1}{h_1} \frac{\partial\phi}{\partial q_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial\phi}{\partial q_2} \hat{e}_2 = v_1 \hat{e}_1 + v_2 \hat{e}_2 \\ v_1 &= \frac{1}{h_1} \frac{\partial\phi}{\partial q_1} \quad \text{and} \quad v_2 = \frac{1}{h_2} \frac{\partial\phi}{\partial q_2}\end{aligned}$$

ϕ is defined for 2-D or 3-D and for unsteady flow. ψ , stream function is defined only for steady 2-D or axisymmetric flows as long as the flow is physically possible.

5.13 Laplace Equation

Irrotational and incompressible flow.

From the mass conservation equation

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \rho\vec{V} = 0$$

Since ρ is constant $\rightarrow \frac{\partial\rho}{\partial t} = 0$

$$\begin{aligned}\nabla \cdot \rho\vec{V} &= \rho\nabla \cdot \vec{V} = 0 \\ \text{or } \nabla \cdot \vec{V} &= 0\end{aligned}$$

If the flow is irrotational $\vec{V} = \nabla\phi$.

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla\phi = 0 = \nabla^2\phi \quad \leftarrow \text{Laplace Equation}$$

5.13.1 Cartesian

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

5.13.2 Cylindrical

$$\begin{aligned}\nabla \cdot \nabla\phi &= \nabla \cdot \left(\frac{\partial\phi}{\partial r} \hat{e}_r + \frac{\partial\phi}{r\partial\theta} \hat{e}_\theta + \frac{\partial\phi}{\partial z} \hat{e}_z \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0\end{aligned}$$

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$

$$\begin{aligned}\frac{\partial\hat{e}_r}{\partial\theta} &= \hat{e}_\theta \\ \frac{\partial\hat{e}_\theta}{\partial\theta} &= -\hat{e}_r\end{aligned}$$

5.13.3 Irrotational 2-D

$$\begin{aligned}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial}{\partial x} \left(-\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) &= 0 \\ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= \nabla^2 \psi = 0\end{aligned}$$

Laplace equation has solutions which are called as harmonic functions.

For 2-D flow

1. Any irrotational and incompressible flow has a velocity potential ϕ and stream function ψ that both satisfy Laplace equation.
2. Conversely any solution represents the velocity potential ϕ or stream function ψ for an irrotational and incompressible flow.

A powerful procedure for solving irrotational flow problems is to represent ϕ and ψ by linear combinations of known solutions of Laplace equation.

$$\phi = \sum C_i \phi_i, \quad \psi = \sum C_i \psi_i$$

Finding the coefficients C_i so that the boundary conditions are satisfied both far from the body and the body surface.

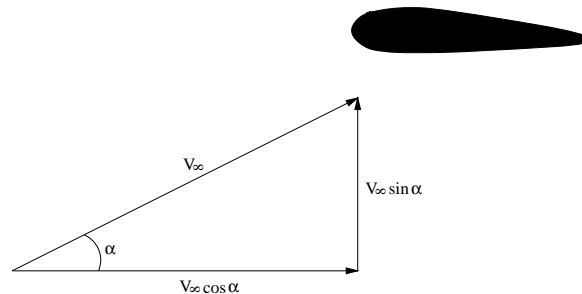
Say ϕ_1 and ϕ_2 are solutions of $\nabla^2 \phi = 0$, therefore

$$\begin{aligned}\nabla^2(\phi_1) &= 0; & \nabla^2(\phi_2) &= 0 \\ A_1 \nabla^2(\phi_1) &= 0 & \text{or} & \nabla^2(A_1 \phi_1) = 0 \\ \text{Similarly} & & \nabla^2(A_2 \phi_2) &= 0 \\ \text{Therefore} & & \nabla^2(A_1 \phi_1 + A_2 \phi_2) &= 0 \\ \phi &= A_1 \phi_1 + A_2 \phi_2 & \text{is also a solution.}\end{aligned}$$

A complicated flow pattern for an irrotational and incompressible flow can be synthesized by adding together a number of elementary flows which are also irrotational and incompressible.

5.14 Boundary Conditions

5.14.1 Infinity Boundary Conditions



$$\vec{V}_\infty = V_\infty \cos \alpha \hat{i} + V_\infty \sin \alpha \hat{j}$$

$$\begin{aligned}
 u &= \frac{\partial \phi}{\partial x} = V_{\infty} \cos \alpha = \frac{\partial \psi}{\partial y} \\
 v &= \frac{\partial \phi}{\partial y} = V_{\infty} \sin \alpha = -\frac{\partial \psi}{\partial x}
 \end{aligned}$$

The coordinate axes are attached to the body.

5.14.2 Wall Boundary Conditions

At the body, the velocity must be tangential to the surface, that is, a streamline must conform to the contour of the body.

$$\begin{aligned}
 \psi_{surface} &= \text{constant} \\
 \text{or } \frac{\partial \psi}{\partial s} &= 0
 \end{aligned}$$

where s is the distance measured along the body surface.

5.14.3 Streamline

$$\begin{aligned}
 \vec{V} \times d\vec{s} &= 0 \\
 u \, dy - v \, dx &= 0 \\
 \left(\frac{dy}{dx} \right)_{surface} &= \left(\frac{v}{u} \right)_{surface}
 \end{aligned}$$

5.14.4 Solid Body

Component of velocity normal to the surface is zero.

$$\begin{aligned}
 \vec{V} \cdot \hat{n} &= 0 \\
 \nabla \phi \cdot \hat{n} &= 0 \\
 \text{or } \left(\frac{\partial \phi}{\partial n} \right)_{surface} &= 0
 \end{aligned}$$