

Chapter 4

Conservation Equations

4.1 Methods of Describing Fluid Motion

Motion of fluid particles can be described by two methods. In **Lagrangian** method the motion of each particle of fixed identity is described at all times. In **Eulerian** method the motion of a fluid at every point is described at all times without considering the whereabouts of individual particles.

4.1.1 Lagrangian Approach

Lagrangian approach, an extension of particle mechanics, considers the individual molecules and obtains conservation equations (mass, momentum, energy) based on individual molecular motion.

Attention is paid to what happens to the individual fluid particle (identified usually by its position at $t=0$) in the course of time, what paths they describe, what velocities or accelerations they possess, and so on.

The temperature in Lagrangian variables is given by:

$$T = T(a, b, c, t)$$

where (a, b, c) is the position of the particle at time $t = 0$.

Also $\vec{r} = \vec{r}(a, b, c, t)$ is the position of the particle at time t tagged by (a, b, c) . $t, a, b,$ and c are the independent variables in Lagrangian frame. Since the fluid elements are continuously distributed, the values of the parameters (a, b, c) will assume for the various elements are continuous.

4.1.2 Eulerian Approach

Everything is viewed from an aggregate sense in terms of Eulerian fields.

Eulerian Field

In the Eulerian description of fluids in motion we describe the distribution of a macroscopic property as a function of space and time which we refer to as an Eulerian field of that property. Thus the complete representation of a property for a general three-dimensional, unsteady field is given by:

$$\varphi = \varphi(q_1, q_2, q_3, t)$$

where φ can be any property such as \vec{V}, p, T, ρ etc.

Examples

$T = T(\vec{r}, t)$	Temperature
$p = p(\vec{r}, t)$	Pressure
$\rho = \rho(\vec{r}, t)$	Density
$s = s(\vec{r}, t)$	Entropy
$\vec{V} = \vec{V}(\vec{r}, t)$	Velocity
$\vec{M} = \vec{M}(\vec{r}, t)$	Momentum

In Eulerian description we watch a fixed point (x, y, z) in space as time t proceeds. The independent variables are the spatial coordinates x, y, z , and time t .

- The temperature of the fluid is given by $T = T(x, y, z, t)$.
- At a given (x, y, z) $T = T(x, y, z, t)$ gives the time history of T at that point.
- At a given time t , $T = T(x, y, z, t)$ gives the spatial variation of T . In other words, for any fluid quantity Q , we can write $Q = Q(\vec{r}, t)$ - a scalar or a vector field.

4.2 Substantial/Total Derivative

(The relation between the derivatives in Lagrangian and Eulerian derivatives.)

In Eulerian viewpoint, since our attention is focused upon specific points in space at various times. The history of the individual particle is not explicit. The substantial derivative allows us to express the time rate of change of a particle property in terms of the spatial Eulerian derivatives of that property at a given point. Consider a general flow variable $\varphi = \varphi(q_1, q_2, q_3, t)$. Then:

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial\varphi}{\partial q_2} \frac{dq_2}{dt} + \frac{\partial\varphi}{\partial q_3} \frac{dq_3}{dt} + \frac{\partial\varphi}{\partial t} \frac{dt}{dt}$$

which becomes:

$$\frac{D\varphi}{Dt} = \frac{1}{h_1} \frac{\partial\varphi}{\partial q_1} v_1 + \frac{1}{h_2} \frac{\partial\varphi}{\partial q_2} v_2 + \frac{1}{h_3} \frac{\partial\varphi}{\partial q_3} v_3 + \frac{\partial\varphi}{\partial t}$$

or

$$\underbrace{\frac{D\varphi}{Dt}}_{\text{Lagrangian}} = \underbrace{\overbrace{\vec{V} \cdot \nabla \varphi}^{\text{Convective Derivative}} + \overbrace{\frac{\partial\varphi}{\partial t}}^{\text{Local Derivative}}}_{\text{Eulerian}}$$

The total derivative can be written in operator notation as:

$$\frac{D()}{Dt} = \left[\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right] ()$$

- $\frac{D}{Dt}$ - is the time rate of change of a fluid property as the given fluid element moves through space.
- $\frac{\partial}{\partial t}$ - is the time rate of change of fluid property at the given fixed point (local derivative).
- $\vec{V} \cdot \nabla$ - is the time rate of change due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different (connective derivative).

4.3 Acceleration (of A Fluid Particle) at A Point

Consider a velocity field: $\vec{V} = \vec{V}(\vec{r}, t) = u\hat{i} + v\hat{j} + w\hat{k}$
 where $u = u(\vec{r}, t)$, $v = v(\vec{r}, t)$, and $w = w(\vec{r}, t)$

Acceleration $\vec{a} = \vec{a}(\vec{r}, t) = (a_x, a_y, a_z)$

where $a_x = \frac{du}{dt}$, $a_y = \frac{dv}{dt}$, and $a_z = \frac{dw}{dt}$

$$\begin{aligned} a_x = \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} + \frac{\partial u}{\partial t} \frac{dt}{dt} \\ &= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \end{aligned}$$

We know $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ therefore

$$\begin{aligned}
 a_x = \frac{du}{dt} &= (u\hat{i} + v\hat{j} + w\hat{k}) \cdot \left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k} \right) + \frac{\partial u}{\partial t} \\
 &= (\vec{V} \cdot \nabla u) + \frac{\partial u}{\partial t} \\
 \frac{d}{dt}(u) &= \left[\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right] (u) \\
 a_y = \frac{dv}{dt} &= u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\
 &= (\vec{V} \cdot \nabla v) + \frac{\partial v}{\partial t} = \left[\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right] (v) \\
 \\
 a_z = \frac{dw}{dt} &= u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \\
 &= (\vec{V} \cdot \nabla w) + \frac{\partial w}{\partial t} = \left[\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right] (w)
 \end{aligned}$$

4.4 Reynolds' Transport Theorem

If the laws of physics are written for a fixed region of space, so that different fluid particles occupy this region at different times, then the frame of reference is said to be **Eulerian**. However, if the laws govern the same fluid particles in a particular region that moves with the fluid, the laws are written in **Lagrangian** reference frame. Basic conservation laws can be applied more easily to an arbitrary collection of matter of fixed identity (a system composed of the same quantity of matter at all times) than to a volume fixed in space. Applying conservation laws to matter of fixed identity gives rise to Lagrangian reference frame and the associated substantial derivatives of volume integrals. However, control volumes of fixed shape is preferred (Eulerian reference frame) for the following reasons of difficulty with the Lagrangian reference frame.

1. The fluid media is capable of continuous distortion and deformation, since it is often extremely difficult to identify and follow the same mass of fluid at all times as must be done in Lagrangian reference frame.
2. Also, our primary interest often is not in the motion of a given mass of fluid, but rather in the effect of the overall fluid motion on some device or structure.

Thus the basic laws applied to a fixed mass in Lagrangian reference frame must be transformed to equivalent expressions in Eulerian reference frame. The theorem which permits this transformation is called Reynolds' transport theorem.

Let N be any extensive property of the identifiable fixed mass (system) such as total mass, momentum, or energy. The corresponding intensive property (extensive property per unit mass) will be designated as α . Then if :

$$N)_S = \int_{mass} \alpha dm = \int_{Volume} \alpha \rho dV$$

Reynolds' Transport theorem states:

$$\underbrace{\frac{D}{Dt} (N_S) = \frac{D}{Dt} \left(\int_{Volume} \alpha \rho dV \right)}_{Lagrangian} = \underbrace{\frac{\partial}{\partial t} \left(\int_{CV} \alpha \rho dV \right) + \int_{CS} \alpha (\rho \vec{V} \cdot d\vec{A})}_{Eulerian}$$

$\frac{D}{Dt} (N_S)$ The total rate of change of any arbitrary extensive property of the system.

$\frac{D}{Dt} \left(\int_{Volume} \alpha \rho dV \right)$ The total rate of change of any arbitrary extensive property of the system.

$\frac{\partial}{\partial t} \left(\int_{CV} \alpha \rho dV \right)$ The time rate of change of the arbitrary extensive property N within the control volume evaluated by an observer fixed in the moving control volume.

$\alpha \left(\rho \vec{V} \cdot d\vec{A} \right)$ The net rate of efflux of the extensive property N through the control surface where \vec{V} is the velocity measured relative to the control volume.

The intensive property α corresponding to N can be used for different quantities as follows:

Quantity	α
Mass	1
Linear Momentum	\vec{V}
Angular Momentum	$\vec{r} \times \vec{V}$
Energy	e
Entropy	s

4.5 Conservation of Mass

Rate of change of mass of a system is zero.

$$\frac{D}{Dt} (m_S) = 0$$

using Reynolds' Transport Theorem this can be converted to Eulerian formation as:

$$\frac{D}{Dt} (m_S) = \frac{D}{Dt} \left(\int_{Volume} \rho dV \right) = \frac{\partial}{\partial t} \left(\int_{CV} \rho dV \right) + \int_{CS} (\rho \vec{V} \cdot d\vec{A}) = 0$$

where $\alpha = 1$.

Physically:

$$\left[\begin{array}{l} \text{Rate of change of mass inside} \\ \text{the control volume} \end{array} \right] + \left[\begin{array}{l} \text{Net rate of mass efflux (outflow)} \\ \text{through the control surface} \end{array} \right] = 0$$

Since the control volume is fixed with respect to a coordinate system attached to it, the limits of integration are also fixed. Hence, the time derivative can be placed inside the volume integral, and the equation can be written as:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} (\rho \vec{V} \cdot d\vec{A}) = 0$$

which is a conservation law in a finite space.

Applying Gauss divergence Theorem we convert the surface integral to volume integral to obtain:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CV} (\nabla \cdot \rho \vec{V}) dV = 0$$

or

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} \right] dV = 0$$

But since the control volume, V was arbitrarily chosen, the only way this equation can be satisfied is for the integrand to be zero at all points within the control volume. Thus by setting the integrand to zero we have the partial equation of conservation law:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

If the integrand in the above equation was not equal to zero, it would be possible to redefine the control volume, V in such a way that the integrand was not equal to zero.

A mathematical variation of the above law:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} &= \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho \\ &= \frac{D}{Dt}(\rho) + \rho \nabla \cdot \vec{V} = 0\end{aligned}$$

This mixed form of the continuity equation in which one term is Lagrangian and the other is Eulerian derivative is not useful for flow solution. However, it is frequently used in manipulation.

Simplifications:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho = 0$$

Incompressible:

$$\begin{aligned}\rho &= \text{constant} \\ \frac{\partial \rho}{\partial t} &= 0; \nabla \rho = 0\end{aligned}$$

$$\nabla \cdot \vec{V} = 0$$

4.6 Conservation of Momentum

(Newton's second law for fluids.)

Let \vec{H} be the momentum of a system. From Newton's law:

Rate of change of momentum \equiv Sum of all external forces

$$\underbrace{\left[\frac{d(\vec{H})}{dt} \right]_S}_{\text{R.T. Theorem}} = \underbrace{\sum \vec{F}_{ext}}_{\text{expand to include all types of forces acting on the control volume}}$$

$$\vec{H}_S = \iiint_V \rho \vec{V} dV$$

$$\frac{d}{dt} \left[\iiint_V \vec{V} \overbrace{(\rho dV)}^{dm} \right]_S = \sum \vec{F}_{ext}$$

using Reynolds' Transport theorem the left hand side can be written for a control volume as:

$$\frac{\partial}{\partial t} \iiint_V \vec{V} \rho dV + \oint_S \vec{V} (\rho \vec{V} \cdot d\vec{A}) \equiv \sum \vec{F}_{ext}$$

$\sum \vec{F}_{ext}$ is all the forces exerted on the fluid as it flows through the control volume.

$$\vec{F}_{ext} = \vec{F}_{Body} + \vec{F}_{Surface} + \underbrace{\vec{R}}_{\text{Reaction}}$$

- \vec{F}_{Body} - gravitational force, electromagnetic forces, etc.
- $\vec{F}_{Surface}$ - surface forces such as normal pressure and tangential shear (due to viscous phenomena).

Tangential shear stress can be predicted only using viscous theory. Normal pressure can be predicted using inviscid theory also. The body force term can be easily expressed as:

$$\vec{F}_{Body} \equiv \iiint_V \rho \vec{f} dV$$

where \vec{f} is a body force per unit mass.

The surface forces can be expressed in terms of the stress tensor as follows:

$$\vec{F}_{Surface} = \oint_S \tilde{p} \cdot d\vec{S}$$

where \tilde{p} is the stress tensor exerted by the surrounding on the surface.

$$\tilde{p} = -p\vec{I} + \tilde{\tau}$$

where \vec{I} is an identity tensor and $\tilde{\tau}$ is a shear stress tensor. $\tilde{\tau}$ becomes zero when viscosity is neglected. After neglecting viscosity $\vec{F}_{Surface}$ becomes:

$$\vec{F}_{Surface} = \oint_S p d\vec{S}$$

Collecting all the terms the conservation of momentum reduces to:

$$\frac{\partial}{\partial t} \iiint_V \rho \vec{V} dV + \underbrace{\oint_S \vec{V} (\rho \vec{V} \cdot d\vec{S})}_{\text{term A}} = - \underbrace{\oint_S p d\vec{S}}_{\text{term B}} + \iiint_V \rho \vec{f} dV + \vec{R}$$

Note: Use the \vec{R} term only when solving control volume type problems.

In differential form or while formulating other forms the reaction term \vec{R} can be conveniently left out.

4.6.1 Gradient Theorem

$$\oint_S \phi d\vec{S} = \iiint_V \nabla \phi dV$$

Using Gauss divergence theorem on term A and Gradient theorem on term B we can write the conservation of momentum after dropping the \vec{R} term as:

$$\frac{\partial}{\partial t} \iiint_V \rho \vec{V} dV + \iiint_V \nabla \cdot (\rho \vec{V} \vec{V}) dV = - \iiint_V \nabla p dV + \iiint_V \rho \vec{f} dV$$

Consider a fixed control volume

$$\begin{aligned} \iiint_V \frac{\partial}{\partial t} (\rho \vec{V}) dV + \iiint_V \nabla \cdot (\rho \vec{V} \vec{V}) dV &= - \iiint_V \nabla p dV + \iiint_V \rho \vec{f} dV \\ \iiint_V \left[\frac{\partial}{\partial t} (\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) + \nabla p - \rho \vec{f} \right] dV &\equiv 0 \end{aligned}$$

For any arbitrary control volume the above equation is satisfied if and only if $[] = 0$. Hence, the differential form of the conservation of an inviscid fluid is:

$$\underbrace{\frac{\partial}{\partial t} (\rho \vec{V})}_{\text{Unsteady}} + \underbrace{\nabla \cdot (\rho \vec{V} \vec{V})}_{\text{Convection}} + \underbrace{\nabla p}_{\text{Normal pressure}} - \underbrace{\rho \vec{f}}_{\text{Body force}} \equiv 0$$

4.6.2 Substantial Derivative Form

The following vector identity is useful for better manipulation.

$$\nabla \cdot (\phi \vec{A}) = (\vec{A} \cdot \nabla) \phi + \phi \nabla \cdot \vec{A}$$

where ϕ is any vector or scalar and \vec{A} is a general vector.

Consider the differential form:

$$\begin{aligned}
& \frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot \underbrace{(\rho \vec{V})}_{\vec{A}} \underbrace{\vec{V}}_{\phi} + \nabla p - \rho \vec{f} \equiv 0 \\
& \frac{\partial}{\partial t}(\rho \vec{V}) + \vec{V} \nabla \cdot (\rho \vec{V}) + (\rho \vec{V} \cdot \nabla \vec{V}) + \nabla p - \rho \vec{f} \equiv 0 \\
& \rho \frac{\partial \vec{V}}{\partial t} + \vec{V} \frac{\partial \rho}{\partial t} + \vec{V} \nabla \cdot (\rho \vec{V}) + \rho \vec{V} \cdot \nabla \vec{V} + \nabla p - \rho \vec{f} \equiv 0 \\
& \rho \frac{\partial \vec{V}}{\partial t} + \vec{V} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} \right]}_{\text{Conservation of mass} \equiv 0} + \rho \vec{V} \cdot \nabla \vec{V} + \nabla p - \rho \vec{f} \equiv 0 \\
& \rho \left[\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right] + \nabla p - \rho \vec{f} \equiv 0 \\
& \rho \frac{D\vec{V}}{Dt} + \nabla p - \rho \vec{f} \equiv 0 \text{ [Inviscid momentum equation]} \\
& \rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{f} \\
& \frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} + \vec{f} \text{ [Euler equation]}
\end{aligned}$$

4.7 Discussion of Flow Equations

The properties and the flow pattern of a moving fluid are governed by the fundamental laws of physics expressed as:

1. Conservation of mass
2. Conservation of momentum
3. Conservation of energy
4. Equation of state

when the mathematical equations expressing these laws are solved satisfying the appropriate initial and boundary conditions, the fluid properties, and the flow pattern result.

These conservation equations involve three scalar fields ρ , p , T and one vector field \vec{V} as the unknown functions.

- Independent variables: (q_1, q_2, q_3, t)
- Dependent variables: $(v_1, v_2, v_3, \rho, p, T)$
- Prescribed quantities: $\vec{f}, \mu(T), C_p(T), R$, etc.

The equations describing the fluid flow are:

Conservation laws for	Equations	Number of Eqns.	Order of Eqns.	Total order
Mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$	1	1	1
Momentum	$\frac{\partial}{\partial t} (\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V})$ $= -\nabla p + \nabla \cdot \vec{\tau} + \rho \vec{f}$	3	2	6
Energy	-	1	2	2
Equation of state	$p = f(\rho, T)$	1	0	0
	Total	6	9	9

- There are six equations and six dependent variables \rightarrow Equations can be solved.
- The sum of the order of the differential equations is equal to nine and we need nine boundary conditions.

- The conservation equations are nonlinear, that is coefficients of some of the derivatives are dependent variables. Need an interactive solution.
- All equations are coupled and hence must be solved for simultaneously.

In general, exact solutions to the conservation equations are unknown because they are nonlinear and no general method is presently known for solving nonlinear differential equations. However when restrictions are placed on the flow geometry and the fluid properties, several exact solution to the conservation equations are possible.

4.8 The Navier-Stokes Equations

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) - \nabla \tilde{p}_{ext} - \vec{f} \rho = 0$$

where

$$\tilde{p}_{ext} = -p\vec{I} + \tilde{\tau}$$

Rewriting the equation after substitution leads to:

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V} + p\vec{I} - \tilde{\tau}) - \rho \vec{f} = 0$$

$$\nabla \cdot p\vec{I} = (\vec{I} \cdot \nabla)p + p(\nabla \cdot \vec{I})$$

$$\text{similar to } (\nabla \cdot \phi \vec{A}) = (\vec{A} \cdot \nabla)\phi + \phi(\nabla \cdot \vec{A})$$

$$\nabla \cdot \vec{I} = 0 \text{ for orthogonal coordinate system and } (\vec{I} \cdot \nabla) = \nabla$$

$$\text{Therefore } \frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

$$\text{where } \tilde{\tau} = \mu[\nabla \vec{V} + (\nabla \vec{V})^T - \frac{2}{3}(\nabla \cdot \vec{V})\vec{I}]$$

4.9 Another Form of N-S Equations

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

$$\vec{V} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \vec{V}}{\partial t} + (\rho \vec{V} \cdot \nabla) \vec{V} + \vec{V}(\nabla \cdot \rho \vec{V}) + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

$$\vec{V} \frac{\partial \rho}{\partial t} + \vec{V}(\nabla \cdot \rho \vec{V}) + \rho \frac{\partial \vec{V}}{\partial t} + (\rho \vec{V} \cdot \nabla) \vec{V} + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

$$\vec{V} \left[\underbrace{\text{Continuity}}_{=0} \right] + \rho \left[\frac{D\vec{V}}{Dt} \right] + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

$$\rho \left[\frac{D\vec{V}}{Dt} \right] + \nabla p - \nabla \cdot \tilde{\tau} - \rho \vec{f} = 0$$

Expansion of Navier-Stokes Equations in cylindrical coordinates

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) - \nabla \cdot \vec{p}_{ext} - \vec{f} \rho = 0$$

$$\vec{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z$$

For inviscid and orthogonal coordinates the above equation becomes:

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \vec{V}) - \nabla p - \vec{f} \rho = 0$$

Unit vectors do not change with respect to time.

$$\begin{aligned} \frac{\partial \rho \vec{V}}{\partial t} &= \frac{\partial \rho v_r}{\partial t} \hat{e}_r + \frac{\partial v_\theta}{\partial t} \hat{e}_\theta + \frac{\partial \rho v_z}{\partial t} \hat{e}_z \\ \nabla \vec{p}_{ext} &= \frac{\partial p}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{e}_\theta + \frac{\partial p}{\partial z} \hat{e}_z \\ \vec{f} \rho &= \rho f_r \hat{e}_r + \rho f_\theta \hat{e}_\theta + \rho f_z \hat{e}_z \\ \nabla &= \underbrace{\hat{e}_r \frac{\partial}{\partial r}}_{A_1} + \underbrace{\hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}}_{A_2} + \underbrace{\hat{e}_z \frac{\partial}{\partial z}}_{A_3} \\ \rho \vec{V} \vec{V} &= \rho (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \\ &= \underbrace{\rho v_r v_r \hat{e}_r \hat{e}_r}_{B_1} + \underbrace{\rho v_r v_\theta \hat{e}_r \hat{e}_\theta}_{B_2} + \underbrace{\rho v_r v_z \hat{e}_r \hat{e}_z}_{B_3} \\ &+ \underbrace{\rho v_\theta v_r \hat{e}_\theta \hat{e}_r}_{B_4} + \underbrace{\rho v_\theta v_\theta \hat{e}_\theta \hat{e}_\theta}_{B_5} + \underbrace{\rho v_\theta v_z \hat{e}_\theta \hat{e}_z}_{B_6} \\ &+ \underbrace{\rho v_z v_r \hat{e}_z \hat{e}_r}_{B_7} + \underbrace{\rho v_z v_\theta \hat{e}_z \hat{e}_\theta}_{B_8} + \underbrace{\rho v_z v_z \hat{e}_z \hat{e}_z}_{B_9} \\ \nabla \cdot (\rho \vec{V} \vec{V}) &= (A_1 + A_2 + A_3) \cdot (B_1 + B_2 + \dots + B_9) \\ &= \hat{e}_r \left[\frac{\partial}{\partial r} \underbrace{(\rho v_r v_r)}_{A_1 B_1} + \frac{\partial}{r \partial \theta} \underbrace{(\rho v_\theta v_r)}_{A_2 B_4} + \frac{\partial}{\partial z} \underbrace{(\rho v_z v_r)}_{A_3 B_7} + \rho \frac{v_r v_r}{r} - \rho \frac{v_\theta v_\theta}{r} \right] \\ &+ \hat{e}_\theta \left[\frac{\partial}{\partial r} \underbrace{(\rho v_r v_\theta)}_{A_1 B_2} + \frac{\partial}{r \partial \theta} \underbrace{(\rho v_\theta v_\theta)}_{A_2 B_5} + \frac{\partial}{\partial z} \underbrace{(\rho v_z v_\theta)}_{A_3 B_8} + \rho \frac{v_r v_\theta}{r} + \rho \frac{v_\theta v_r}{r} \right] \\ &+ \hat{e}_z \left[\frac{\partial}{\partial r} \underbrace{(\rho v_r v_z)}_{A_1 B_3} + \frac{\partial}{r \partial \theta} \underbrace{(\rho v_\theta v_z)}_{A_2 B_6} + \frac{\partial}{\partial z} \underbrace{(\rho v_z v_z)}_{A_3 B_9} + \rho \frac{v_r v_z}{r} \right] \end{aligned}$$

For example

$$\begin{aligned} A_2 \cdot B_2 &= \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} \cdot (\rho v_r v_\theta \hat{e}_r \hat{e}_\theta) = \frac{\hat{e}_\theta}{r} \cdot \frac{\partial}{\partial \theta} (\rho v_r v_\theta \hat{e}_r \hat{e}_\theta) \\ &= \frac{\hat{e}_\theta}{r} \cdot \frac{\partial}{\partial \theta} (\rho v_r v_\theta) \hat{e}_r \hat{e}_\theta + (\rho v_r v_\theta) \frac{\partial \hat{e}_r}{\partial \theta} \hat{e}_\theta + (\rho v_r v_\theta) \hat{e}_r \frac{\partial \hat{e}_\theta}{\partial \theta} \\ &= \frac{\partial}{\partial \theta} (\rho v_r v_\theta) \left[\underbrace{\frac{\hat{e}_\theta}{r} \cdot \hat{e}_r \hat{e}_\theta}_{=0} \right] + (\rho v_r v_\theta) \left[\underbrace{\frac{\hat{e}_\theta}{r} \cdot \frac{\partial \hat{e}_r}{\partial \theta}}_{=\hat{e}_\theta} \hat{e}_\theta \right] + (\rho v_r v_\theta) \left[\underbrace{\frac{\hat{e}_\theta}{r} \cdot \hat{e}_r \hat{e}_\theta}_{=0} \right] \\ &= (\rho v_r v_\theta) \frac{\hat{e}_\theta}{r} \cdot \hat{e}_\theta \hat{e}_\theta = \frac{\rho v_r v_\theta}{r} \hat{e}_\theta \end{aligned}$$

Inviscid N-S equations in cylindrical coordinates

r-direction

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta v_r) + \frac{\partial}{\partial z}(\rho v_z v_r) - \rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \rho f_r$$

θ -direction

$$\frac{\partial}{\partial t}(\rho v_\theta) + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r v_\theta) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta v_\theta) + \frac{\partial}{\partial z}(\rho v_z v_\theta) + \rho \frac{v_r v_\theta}{r} = -\frac{\partial p}{r \partial \theta} + \rho f_\theta$$

z-direction

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r v_z) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta v_z) + \frac{\partial}{\partial z}(\rho v_z v_z) = -\frac{\partial p}{\partial z} + \rho f_z$$