Chapter 2

Review of Vectors

2.1 Definition of a Vector

A vector is a quantity that possesses both a magnitude and a direction and obey the parallelogram law of addition.

2.2 Vector Addition

\[
\vec{C} = \vec{A} + \vec{B}
\]

![Figure 2.1: Vector addition](image)

2.3 Vector Subtraction

\[
\vec{D} = \vec{A} - \vec{B}
\]

2.4 Properties of Vectors

If \( s \) and \( t \) are two scalars and \( \vec{A} \) and \( \vec{B} \) are two vectors then:

\[
\begin{align*}
0\vec{A} & = \vec{0} ; \quad +\vec{A} = \vec{A} ; \quad (-1)\vec{A} = -\vec{A} \\
(s + t)\vec{A} & = s\vec{A} + t\vec{A} \\
s(\vec{A} + \vec{B}) & = s\vec{A} + s\vec{B} \\
st(\vec{A}) & = s(t\vec{A}) = t(s\vec{A})
\end{align*}
\]
2.4.1 Explanation

If s is a number and \( \vec{A} \) is a vector, \( s \vec{A} \) is defined to be the vector having magnitude \( s \) times that of \( \vec{A} \) and pointing in the same direction if \( s > 0 \) and in the opposite direction if \( s \) is negative. Any vector \( s \vec{A} \) is a scalar multiple of \( \vec{A} \).

2.5 Scalar Product

\[ \vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta \]

Ex.1) Work done by a force \( \vec{F} \) during an infinitesimal displacement \( \delta \vec{s} \) is

\[ W = \vec{F} \cdot \delta \vec{s} \]

Ex.2)

\[ -p \delta \vec{A} = \delta \vec{F} \]

where \( p \) is pressure at a point, \( \delta \vec{A} \) is infinitesimal area vector, \( \delta \vec{F} \) is the corresponding force

\[ \int \delta \vec{F} = \vec{F} \]

where \( \vec{F} \) is the total force on a body

\[ \vec{F} \cdot \hat{i} = Drag \]
\[ \vec{F} \cdot \hat{k} = Lift \]

2.6 Vector Product

\[ \vec{A} \times \vec{B} = |\vec{A}||\vec{B}||\sin\theta \hat{e} \]

where \( \hat{e} \) is a unit vector perpendicular to both \( \vec{A} \) and \( \vec{B} \) and \( | | \) sign denotes the magnitude of the vector.

Ex.1)

\[ \vec{r} \times \vec{F} = M_o \]

where \( M_o \) is moment about \( o \)

2.7 Triple Product

\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A}) \]
\[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \]
2.8 Unit Vector

A vector whose magnitude is 1 is called a unit vector.

\[ \hat{e}_A = \frac{\vec{A}}{|\vec{A}|} \]

where \(|\vec{A}|\) is the magnitude of vector \(\vec{A}\) and \(\hat{e}_A\) is a unit vector in the direction of \(\vec{A}\).

2.9 Vector Differentiation

If \(\vec{A}\) and \(\vec{B}\) are two vectors and \(\vec{U} = \vec{A} + \vec{B}\) then:

\[
\begin{align*}
\frac{d\vec{U}}{dt} &= \frac{d|\vec{U}|}{dt} \hat{e}_u \\
\frac{d\vec{U}}{dt} &= \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt} \\
\frac{d(n\vec{U})}{dt} &= \frac{dn\vec{U}}{dt} + n\frac{d\vec{U}}{dt}
\end{align*}
\]

2.10 Product Rules

\[
\begin{align*}
\vec{A} \cdot \vec{A} &= (|\vec{A}|)^2 \\
\hat{e}_i \cdot \hat{e}_j &= \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases} \\
\hat{e}_i \times \hat{e}_j &= \begin{cases} 
0 & \text{if } i = j, \\
1 & \text{if } i \neq j.
\end{cases}
\end{align*}
\]

2.11 Components of a Vector

In 3-D a vector has 3 components. These 3 components are independent of each other. Consider three vectors \(\vec{A}, \vec{B}, \vec{C}\). In component form, these vectors in general can be written as:

\[
\begin{align*}
\vec{A} &= A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \\
\vec{B} &= B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3 \\
\vec{C} &= C_1 \hat{e}_1 + C_2 \hat{e}_2 + C_3 \hat{e}_3
\end{align*}
\]

Based on the component form the following relations can be established:

\[
\begin{align*}
\vec{A} \cdot \vec{B} &= A_1 B_1 + A_2 B_2 + A_3 B_3 \\
\vec{A} \times \vec{B} &= \begin{vmatrix} 
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
A_1 & A_2 & A_3 \\
B_1 & B_2 & B_3
\end{vmatrix} \\
\vec{A} \cdot (\vec{B} \times \vec{C}) &= \begin{vmatrix} 
A_1 & A_2 & A_3 \\
B_1 & B_2 & B_3 \\
C_1 & C_2 & C_3
\end{vmatrix} \\
\vec{A} \times (\vec{B} \times \vec{C}) &= \begin{vmatrix} 
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
A_1 & A_2 & A_3 \\
(B_2 C_3 - B_3 C_2) & (B_3 C_1 - B_1 C_3) & (B_1 C_2 - B_2 C_1)
\end{vmatrix}
\]

Cartesian and cylindrical coordinate systems are two coordinate systems widely used. Their component forms are discussed next.
2.11.1 Cartesian Coordinate System

\[ \vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \]

where \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors, and \( V_x, V_y, V_z \) are components. In general the components are independent of each other.

For example:

\[ V_x \neq f(V_y, V_z) \]

The position vector in cartesian system is given as:

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

2.11.2 Cylindrical Coordinate System

\[ \vec{V} = V_r \hat{r} + V_\theta \hat{\theta} + V_z \hat{z} \]

2.12 Relation Between Coordinate Systems

2.12.1 General Transformation

\((q_1, q_2, q_3)\) are the general coordinates of a 3-D coordinate system.

\[ q_1 = q_1(x, y, z) \]
\[ q_2 = q_2(x, y, z) \]
\[ q_3 = q_3(x, y, z) \]

2.12.2 General Inverse Transformation

\[ x = x(q_1, q_2, q_3) \]
\[ y = y(q_1, q_2, q_3) \]
\[ z = z(q_1, q_2, q_3) \]
2.12.3 Transformation
\[ r = (x^2 + y^2)^{\frac{1}{2}} \quad (0 \leq r < \infty) \]
\[ \theta = \arctan(y/x) \quad (0 \leq \theta \leq 2\pi) \]
\[ z = z \quad (-\infty < z < \infty) \]

2.12.4 Inverse transformation
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

2.13 A procedure for Evaluating Scalar Factors, Unit Vectors and Their Derivatives from Inverse Transformation

2.13.1 Scale Factor
Scale factor defines the relationship between coordinates and distance along coordinates.

2.13.2 General Coordinate System
A position vector \( \mathbf{r} \) in cartesian system is given by
\[ \mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k} \]
and using the inverse transformation can also be written as:
\[ \mathbf{r} = x(q_1, q_2, q_3) \hat{i} + y(q_1, q_2, q_3) \hat{j} + z(q_1, q_2, q_3) \hat{k} \]
The variation of the position vector along the coordinate direction defines the following relations:
\[ \frac{\partial \mathbf{r}}{\partial q_1} = h_1 \hat{e}_1 \]
\[ \frac{\partial \mathbf{r}}{\partial q_2} = h_2 \hat{e}_2 \]
\[ \frac{\partial \mathbf{r}}{\partial q_3} = h_3 \hat{e}_3 \]
where \( h_1, h_2, \) and \( h_3 \) are the scale factors and \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) are the unit vectors in the \( q_1, q_2, \) and \( q_3 \) directions respectively.

2.14 Relation Between Unit Vectors and Their Derivatives

2.14.1 Cartesian System
Cartesian unit vectors are fixed in magnitude and directions and hence are constant vectors.
\[ \frac{\partial \hat{i}}{\partial x} = \frac{\partial \hat{i}}{\partial y} = \frac{\partial \hat{i}}{\partial z} = 0 \]
\[ \frac{\partial \hat{j}}{\partial x} = \frac{\partial \hat{j}}{\partial y} = \frac{\partial \hat{j}}{\partial z} = 0 \]
\[ \frac{\partial \hat{k}}{\partial x} = \frac{\partial \hat{k}}{\partial y} = \frac{\partial \hat{k}}{\partial z} = 0 \]
2.14.2 Cylindrical System

In general the unit vectors are not constant. For example, in the cylindrical coordinate system the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ vary with the coordinate $\theta$. In contrast, the $\hat{e}_z$ vector is a fixed vector like the cartesian vectors.

For the cylindrical coordinate system:
\[
\begin{align*}
\frac{\partial \mathbf{r}}{\partial r} &= h_r \mathbf{e}_r \\
\frac{\partial \mathbf{r}}{\partial \theta} &= h_\theta \mathbf{e}_\theta \\
\frac{\partial \mathbf{r}}{\partial z} &= h_z \mathbf{e}_z
\end{align*}
\]

From the inverse transformation for the cylindrical system, the position vector in the cylindrical coordinate system can be written as:
\[
\begin{align*}
\mathbf{r} &= r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + z \mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial r} &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} = h_r \mathbf{e}_r \\
\frac{\partial \mathbf{r}}{\partial \theta} &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\
\frac{\partial ^2 \mathbf{r}}{\partial \theta^2} &= r^2 (\sin^2 \theta + \cos^2 \theta) = h_\theta^2 \\
\end{align*}
\]

Similarly $h_z = 1$.
\[
\begin{align*}
\frac{\partial \mathbf{r}}{\partial r} &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} = h_r \mathbf{e}_r = \hat{e}_r \\
\frac{\partial \mathbf{r}}{\partial \theta} &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} = h_\theta \mathbf{e}_\theta = r \hat{e}_\theta
\end{align*}
\]
From the previous equations the unit vectors in cylindrical coordinate system can be written as:

\[
\begin{align*}
\hat{e}_r &= \cos \theta \hat{i} + \sin \theta \hat{j} \\
\hat{e}_\theta &= -\sin \theta \hat{i} + \cos \theta \hat{j} \\
\hat{e}_z &= \hat{k}
\end{align*}
\]

The unit vectors in the cylindrical system can be related to the cartesian unit vectors \( \hat{i}, \hat{j}, \hat{k} \) in matrix form as:

\[
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\theta \\
\hat{e}_z
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} = [A] \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

Matrices relating unit vectors of orthogonal coordinate systems are orthogonal matrices.

The inverse of an orthogonal matrix is its transpose. Hence the following relations can be established.

\[
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\theta \\
\hat{e}_z
\end{bmatrix} = [A]^T \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

In other words the cartesian unit vectors can be written in terms of the cylindrical unit vectors using the relations.

\[
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
\hat{e}_r \\
\hat{e}_\theta \\
\hat{e}_z
\end{bmatrix}
\]

The derivatives of the unit vectors in cylindrical system are:

\[
\frac{\partial \hat{e}_r}{\partial \theta} = 0 \\
\frac{\partial \hat{e}_r}{\partial \theta} = -\sin \theta \hat{i} + \cos \theta \hat{j} = \hat{e}_\theta \\
\frac{\partial \hat{e}_r}{\partial z} = 0 \\
\frac{\partial \hat{e}_\theta}{\partial \theta} = 0 \\
\frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos \theta \hat{i} - \sin \theta \hat{j} = -\hat{e}_r \\
\frac{\partial \hat{e}_\theta}{\partial z} = 0 \\
\frac{\partial \hat{e}_z}{\partial \theta} = 0 \\
\frac{\partial \hat{e}_z}{\partial \theta} = 0 \\
\frac{\partial \hat{e}_z}{\partial z} = 0
\]

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>General</th>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
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</thead>
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<td>((q_1, q_2, q_3))</td>
<td>((x,y,z))</td>
<td>((r, \theta, z))</td>
<td>((R, \varphi, \theta))</td>
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<tr>
<td>((e_1, e_2, e_3))</td>
<td>((i, j, k))</td>
<td>((\hat{e}<em>r, \hat{e}</em>\theta, \hat{e}_z))</td>
<td>((\hat{e}<em>R, \hat{e}</em>\varphi, \hat{e}_\theta))</td>
<td></td>
</tr>
<tr>
<td>((\delta s_1, \delta s_2, \delta s_3))</td>
<td>((\delta x, \delta y, \delta z))</td>
<td>((\delta r, r \delta \theta, \delta z))</td>
<td>((\delta R, R \delta \varphi, R \sin \varphi \delta \theta))</td>
<td></td>
</tr>
<tr>
<td>((h_1, h_2, h_3))</td>
<td>((1, 1, 1))</td>
<td>((1, r, 1))</td>
<td>((1, R, R \sin \varphi))</td>
<td></td>
</tr>
</tbody>
</table>

Relationship between coordinate systems and the coordinate space is unique. For example the relation between cartesian and cylindrical coordinate system is given by:

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

### 2.15 Vector Calculus

#### 2.15.1 Del, The Vector Differential Operator: \( \nabla \)

\[
\nabla = \hat{e}_1 \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{\partial}{\partial q_2} + \hat{e}_3 \frac{\partial}{\partial q_3}
\]

or

\[
\nabla = \frac{\partial}{h_1 \partial q_1} + \frac{\partial}{h_2 \partial q_2} + \frac{\partial}{h_3 \partial q_3}
\]
where \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) are three mutually orthogonal unit vectors. \( \delta s_1, \delta s_2, \) and \( \delta s_3 \) denote infinitesimal distances along the coordinate axes. Thus, knowing the scale factors the vector differential operator \( \nabla \) can be written for other coordinate systems.

### 2.15.2 Cartesian

\[
\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}
\]

### 2.15.3 Cylindrical

\[
\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{\theta} \frac{\partial}{r \partial \theta} + \hat{e}_z \frac{\partial}{\partial z}
\]

### 2.15.4 General Rules of Differentiation

\[
(e_1 \frac{\partial}{\partial q_1}) \cdot (\phi_2 \hat{e}_2) = (\hat{e}_1) \cdot \left( \frac{\partial \phi_2}{\partial q_1} \hat{e}_2 \right) + (\hat{e}_1) \cdot \left( \phi_2 \frac{\partial \hat{e}_2}{\partial q_1} \right)
\]

where \( q_1, q_2, \) and \( q_3 \) are general coordinates. \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) are general unit vectors.

Note: The order of the dot (cross) product should be preserved.

### 2.16 Scalar and Vector Field

A scalar (vector) quantity given as a function of coordinate space and time is called a scalar (vector) field.

Pressure \( p = p(\vec{r}, t) \)

- general = \( p(q_1, q_2, q_3, t) \)
- cartesian = \( p(x, y, z, t) \)
- cylindrical = \( p(r, \theta, z, t) \)

### 2.16.1 Examples

Scalar fields

- \( T = T(\vec{r}, t) \) (temperature)
- \( \rho = \rho(\vec{r}, t) \) (density)
- \( \epsilon = \epsilon(\vec{r}, t) \) (internal energy)

Velocity vector field

- \( \vec{V} = \vec{V}(\vec{r}, t) \) (velocity vector)
- \( \vec{V} = (V_1, V_2, V_3) \)
- \( V_1 = V_1(\vec{r}, t), V_2 = V_2(\vec{r}, t), \) and \( V_3 = V_3(\vec{r}, t) \).

In general a field denotes a region throughout which a quantity is defined as a function of location within the region and time.

If the quantity is independent of time, the field is steady or stationary.

### 2.17 Concept of Gradient

Let \( \phi = \phi(\vec{r}) \) be a scalar function of position \( \vec{r} \).

Find the spatial variation of \( \phi \).

Spatial derivative of \( \phi \) at a point is expressed as derivatives of \( \phi \) in three independent directions. Gradient of a scalar is a vector.
2.17.1 Concept of Gradient
At any point, the gradient of a scalar function $\phi$ of position is equal in magnitude and direction to the greatest derivative of $\phi$ with respect to distance at that point. Rate of change of the scalar $\phi$ along two paths are of special importance.

1. Path along which the scalar is constant (Isolines)
2. Path along which the rate of change of the scalar is the maximum (Gradient line)

2.17.2 General

$$\nabla \phi = \hat{e}_1 \frac{\partial \phi}{\partial s_1} + \hat{e}_2 \frac{\partial \phi}{\partial s_2} + \hat{e}_3 \frac{\partial \phi}{\partial s_3}$$

$$\nabla \phi = \hat{e}_1 \frac{\partial \phi}{h_1 \partial q_1} + \hat{e}_2 \frac{\partial \phi}{h_2 \partial q_2} + \hat{e}_3 \frac{\partial \phi}{h_3 \partial q_3}$$

$$\nabla \phi = \left( \frac{\partial \phi}{\partial s_1}, \frac{\partial \phi}{\partial s_2}, \frac{\partial \phi}{\partial s_3} \right) = \left( \frac{\partial \phi}{h_1 \partial q_1}, \frac{\partial \phi}{h_2 \partial q_2}, \frac{\partial \phi}{h_3 \partial q_3} \right)$$

2.17.3 Cartesian

$$\nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

2.17.4 Cylindrical

$$\nabla \phi = \hat{r} \frac{\partial \phi}{\partial r} + \hat{\theta} \frac{\partial \phi}{\partial \theta} + \hat{z} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \left( \frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

2.18 Concept of Directional Derivative

Let $d\vec{r}$ be a small increment in $\vec{r}$ in some direction $\hat{\epsilon}$. If $\phi = \phi(q_1, q_2, q_3)$ using chain rule we can write:

$$d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3$$

By multiplying the numerator and denominator by the respective scale factors it can also be written as:

$$d\phi = \frac{\partial \phi}{h_1 \partial q_1} h_1 dq_1 + \frac{\partial \phi}{h_2 \partial q_2} h_2 dq_2 + \frac{\partial \phi}{h_3 \partial q_3} h_3 dq_3$$

$$= \frac{\partial \phi}{\partial s_1} ds_1 + \frac{\partial \phi}{\partial s_2} ds_2 + \frac{\partial \phi}{\partial s_3} ds_3$$

$$= \left( \frac{\partial \phi}{\partial s_1}, \frac{\partial \phi}{\partial s_2}, \frac{\partial \phi}{\partial s_3} \right) \cdot (ds_1, ds_2, ds_3)$$

where $\left( \frac{\partial \phi}{\partial s_1}, \frac{\partial \phi}{\partial s_2}, \frac{\partial \phi}{\partial s_3} \right) = \nabla \phi$ and $(ds_1, ds_2, ds_3) = d\vec{r}$.

Therefore

$$d\phi \equiv (\nabla \phi \cdot d\vec{r})$$

Change in a general direction $d\phi$ is simply the scalar product of the gradient at that point and $d\vec{r}$ in that general direction. That is,

$$d\phi = \nabla \phi \cdot d\vec{r}$$

$$d\phi = \nabla \phi \cdot |d\vec{r}| \hat{\epsilon} = \nabla \phi \cdot (d\vec{r} \hat{\epsilon})$$
Therefore
\[ \frac{d\phi}{dr} = \nabla \phi \cdot \hat{e} \]

- Directional derivative of \( \phi(\vec{r}) \) in any chosen direction is equal to the component of the gradient vector in that direction.
- The greatest rate of change of \( \phi \) with respect to coordinate space at a point takes place in the direction of \( \nabla \phi \) and has the magnitude of the vector \( \nabla \phi \).

2.18.1 Cartesian
\[ d\vec{r} = (dx, dy, dz) \]

Cylindrical
\[ d\vec{r} = (dr, r d\theta, dz) = |dr|\hat{e} \]

In component form the position vector can also be written as:
\[ d\vec{r} = (dr, rd\theta, dz) = ds_1 \hat{e}_1 + ds_2 \hat{e}_2 + ds_3 \hat{e}_3 \]

2.19 Divergence of a Vector Field \( (\nabla \cdot \vec{V} = 0) \)

Divergence of a vector field can be computed using vector algebra. In a general coordinate system this is tedious and yet possible as is illustrated here.

\[
\nabla = \hat{e}_1 \frac{\partial}{\partial s_1} + \hat{e}_2 \frac{\partial}{\partial s_2} + \hat{e}_3 \frac{\partial}{\partial s_3} \\
\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3
\]

\[
\nabla \cdot \vec{V} = \left( \hat{e}_1 \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{\partial}{\partial q_2} + \hat{e}_3 \frac{\partial}{\partial q_3} \right) \cdot (V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3)
\]

All the terms have to be expanded using chain rule differentiation. For example, consider \( \left( \hat{e}_1 \frac{\partial}{\partial h_1} \right) \cdot (V_2 \hat{e}_1) \):

\[
\left( \hat{e}_1 \frac{\partial}{\partial h_1} \right) \cdot (V_1 \hat{e}_1) = \hat{e}_1 \cdot \frac{\partial}{\partial h_1} (V_1 \hat{e}_1) = \hat{e}_1 \cdot \left[ \frac{\partial V_1}{\partial h_1} \hat{e}_1 \right]
\]

In specific coordinate systems the simplifications are easier if the derivatives of the unit vectors are known. The divergence of a vector in cartesian and cylindrical systems are illustrated next.

2.19.1 Cartesian
\[
\nabla \cdot \vec{A} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot \left( A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k} \right)
\]

\[
= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}
\]
2.19.2 Cylindrical

\[ \nabla \cdot \vec{A} = \left( \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z) \]

\[ \nabla \cdot \vec{A} = \begin{cases} \hat{e}_r \cdot \left( \frac{\partial A_r}{\partial r} \hat{e}_r + A_r \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial A_\theta}{\partial r} \hat{e}_\theta + \frac{\partial A_z}{\partial z} \hat{e}_z \right) = \frac{\partial A_r}{\partial r} & (1) \\ \hat{e}_\theta \cdot \left( \frac{\partial A_r}{\partial \theta} \hat{e}_r + A_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial A_\theta}{\partial \theta} \hat{e}_\theta + \frac{\partial A_z}{\partial z} \hat{e}_z \right) = \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} & (2) \\ \hat{e}_z \cdot \left[ \frac{\partial A_r}{\partial z} \hat{e}_r + A_r \frac{\partial \hat{e}_r}{\partial z} + \frac{\partial A_\theta}{\partial \theta} \hat{e}_\theta + \frac{\partial A_z}{\partial z} \hat{e}_z \right] = \frac{\partial A_z}{\partial z} & (3) \end{cases} \]

Term (1)

\[ \frac{\partial}{\partial r} A_r \hat{e}_r + A_r \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial A_\theta}{\partial r} \hat{e}_\theta + \frac{\partial A_z}{\partial z} \hat{e}_z \]

term (2)

\[ \frac{1}{r} \left( \frac{\partial A_r}{\partial \theta} + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \right) \]

term (3)

\[ \frac{\partial A_z}{\partial z} \]

therefore

\[ \nabla \cdot \vec{A} = \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \]

\[ = \frac{1}{r} \left( \frac{\partial}{\partial r} (A_r r) + \frac{\partial}{\partial \theta} (A_\theta r) + \frac{\partial}{\partial z} (r A_z) \right) \]

For any orthogonal coordinate system the conservation form of divergence of a vector \( \vec{A} \) is written as:

\[ \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_1 h_3 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right] \]

2.20 Concept of Divergence

2.20.1 Physical Meaning of Divergence of a Vector Field

The divergence of a vector at a point is the net outflow of the vector per unit volume enclosing the point.

Consider \( \vec{A} \) with components \( A_x, A_y, \) and \( A_z \) at a point in the vector field surrounded by an elemental control volume \( \Delta V \) with an elemental surface \( \Delta S \).

For convenience the elemental control volume with its center having a vector with components \( A_x, A_y, \) and \( A_z \) is oriented with edges parallel to x, y, and z axes respectively.

Outflow of \( \vec{A} \) through any side = Component of \( \vec{A} \) in the direction normal to side \( \times \) Area of the surface.

Net outflow of \( \vec{A} \) in x-direction(Outflow \( \vec{A} \) from the x-faces):

\[ = \left[ A_x + \frac{\partial A_x}{\partial x} \frac{\Delta x}{2} \right] - \left( A_x - \frac{\partial A_x}{\partial x} \frac{\Delta x}{2} \right) \Delta y \Delta z = \left( \frac{\partial A_x}{\partial x} \right) (\Delta x \Delta y \Delta z) = \left( \frac{\partial A_x}{\partial x} \right) \Delta V \]
Similarly, net outflow of $\vec{A}$ in y-direction $= \frac{\partial A_y}{\partial y} \Delta V$ and
net outflow of $\vec{A}$ in z-direction $= \frac{\partial A_z}{\partial z} \Delta V$

Outflow of $\vec{A}$ from a point is by definition:

$$\nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \left[ \frac{\text{Outflow of } \vec{A} \text{ in all directions}}{\Delta V} \right] = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

2.20.2 Example

If $\vec{A} = \vec{V}$, the velocity vector then $\nabla \cdot \vec{V}$ is the volume flux from a point.
It is a scalar and its magnitude is the rate at which fluid volume is leaving a point per unit volume.
Flux of volume from a point is by definition:

$$\nabla \cdot (\rho \vec{V}) = \text{mass flux.}$$
where $\rho$ is density.

2.20.3 Problem

Show from the physical meaning of divergence that in polar coordinates:

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

2.21 Gauss Divergence Theorem

Consider a finite control volume $V$ in space as shown in Figure 2.6.

$$d\vec{s} = \hat{e}_n ds$$
where $d\vec{s}$ is the elemental vector area with $\hat{e}_n$ as the unit vector normal to $ds$ being counted positive when directed outward.

If $\vec{A}$ points out $\vec{A} \cdot d\vec{s}$ is positive.
If $\vec{A}$ points in $\vec{A} \cdot d\vec{s}$ is negative.

The region $V$ is divided into elementary cubes and $\text{div} \vec{A}$ is found for each cube at its center. The Flow of
Figure 2.6: A finite control volume

$\vec{A}$ through common faces of adjacent cubes cancel because the inflow through one face equals the outflow through the other. If we now sum the net outflow of $\vec{A}$ of all the cubes, only faces on the surface enclosing the region will contribute to the summation.

In the limit, therefore as the cubes approach zero volume the integral of $\text{div}\vec{A}$ over $V$ is equal to the net outflow through $ds$, the surface enclosing the region.

Stated in integral notation:

$$\oiint_S \vec{A} \cdot d\vec{s} = \iiint_V (\nabla \cdot \vec{A}) dV$$

**Vector Identities Useful for the Manipulation of Conservation Equations:**

$$\nabla \cdot (\varphi \vec{A}) = (\vec{A} \nabla) \varphi + \varphi (\nabla \cdot \vec{A})$$

$$\nabla \cdot (\varphi \vec{A} \vec{A}) = (\varphi \vec{A} \nabla) \vec{A} + \vec{A} (\nabla \cdot \vec{A})$$

$$\nabla \cdot (\varphi \vec{B}) = (\vec{B} \nabla) \varphi + \varphi (\nabla \cdot \vec{B})$$

$$\nabla \cdot (\vec{B} \vec{A}) = (\vec{B} \nabla) \vec{A} + \vec{A} (\nabla \cdot \vec{B})$$

**Gauss Divergence Theorem:**

$$\int_{CS} (\vec{F} \cdot d\vec{A}) = \int_{CV} (\nabla \cdot \vec{F}) dV$$

**Gradient Theorem:**

$$\int_{CS} (\varphi d\vec{A}) = \int_{CV} \nabla \varphi dV$$