

An inverse problem in coupled mode theory

Paul Sacks
Department of Mathematics
Iowa State University
Ames, IA 50011
psacks@iastate.edu

Abstract

We study an inverse problem for the Zakharov-Shabat system which is motivated by an application to the design of co-directional couplers with prescribed response properties.

An inverse problem in coupled mode theory

Paul Sacks
Department of Mathematics
Iowa State University
Ames, IA 50011

January 30, 2004

1 Statement of direct and inverse problem

In this paper we are interested in an inverse problem for a system of ordinary differential equations which arises in the design of optical fiber devices. In the direct problem, we are given a coupling coefficient $q(x)$, which is a bounded, complex valued function on $[0, X]$. For any $k \in \mathbb{C}$ and $x \in [0, X]$ let $A = A(x, k)$, $B = B(x, k)$ denote the solution of

$$\frac{\partial A}{\partial x} = ikA + q^*B \quad \frac{\partial B}{\partial x} = -ikB \pm qA \quad (1.1)$$

with initial conditions

$$A(0, k) = 1 \quad B(0, k) = 0 \quad (1.2)$$

The corresponding inverse problem is to determine $q(x)$ on $[0, X]$ given the scattering data

$$\{B(X, k) : k \in \mathbb{R}\} \quad (1.3)$$

A closely related problem is the system (1.1) with boundary conditions

$$A(0, k) = 1 \quad B(X, k) = 0 \quad (1.4)$$

in which case the data for the inverse problem is

$$\{B(0, k) : k \in \mathbb{R}\} \quad (1.5)$$

For convenience of reference, let us refer to (1.1)-(1.2)-(1.3) as $\mathbf{P1}_\pm$ ¹ and (1.1)-(1.4)-(1.5) as $\mathbf{P2}_\pm$. Below we will say more about the distinctions among these four cases.

The system (1.1) is often referred to as the Zakharov-Shabat system, following the original work of Zakharov and Shabat [23] showing that the Cauchy problem for the cubic Schrödinger equation

$$i\Psi_t = \Psi_{xx} \mp \Psi^2\Psi^* \quad (1.6)$$

can be solved by means of an inverse scattering transform based on $\mathbf{P2}_\pm$. In [23] the spatial domain $[0, X]$ is replaced by \mathbb{R} , but if q in $\mathbf{P2}_\pm$ is extended to be 0 outside of $[0, X]$ then the problem coincides with that of [23]. This theory has been elaborated in many books and papers, see e.g. [1, 5, 20, 23, 24]. See also [3] for a generalization of the $+q$ case to a $2N$ component system. The system (1.1) also arises in the description of the propagation of coupled modes of electromagnetic waves in a waveguide. When the coefficient in the second equation is $+q$ one has the so-called contra-directional case, while $-q$ is the co-directional case. We mention also that (1.1) is related by a simple change of variables to the Dirac system for which direct and inverse scattering theory is of interest, see for example [6, 12].

The main motivation for the study of $\mathbf{P1}_-$ is the works [11, 13], in which this problem arises in connection with the design of optical fiber devices with certain prescribed properties. In this paper we will allow both choices of sign in (1.1) since it seems to be conventional in the literature to do so, and is of mathematical interest, but we don't know of any specific application for $\mathbf{P1}_+$. Problem $\mathbf{P2}_\pm$ has arisen in a larger number of recent works concerned with design of fiber gratings, (e.g. [9, 15, 19, 20, 22], see also [17] for a corresponding time domain problem), the $+$ sign being usually the physically significant case here. As mentioned above, $\mathbf{P2}_\pm$ is by now fairly well understood from a theoretical point of view – these papers are more concerned with computational issues.

¹ \pm referring to the sign in the second equation of (1.1). Here and elsewhere when the symbols \pm and \mp are used it is understood that the top sign pertains to (1.1) in the $+q$ case and the lower sign to the $-q$ case.

Let us emphasize, however, that in the sort of application just mentioned, the questions of ultimate importance have to do with the *design* of the coupling coefficient q rather than *reconstruction* of q . The literature on inverse scattering is concerned almost entirely with reconstruction, so that uniqueness of solutions and, to a somewhat lesser extent, characterization of the data for which a solution exists, are the principal concerns. By contrast in a design problem we seek to choose a coefficient q so as to obtain a physical system with prescribed properties, and in many cases these are idealized properties which are not physically realizable². From the mathematical point of view it means we are given data for which no solution exists, and we attempt to find a solution anyway – or, less facetiously, obtain an ‘optimal’ solution in some sense. There seems to be very little mathematical analysis of the design problem from this point of view (see e.g. [10] for some such work), even for the simpler case of Schrödinger scattering and none will be given in this paper either. However we do believe this is an important problem for future investigation, and that the material in this paper on the reconstruction problem for $\mathbf{P1}_\pm$ will be useful for that.

To conclude this introduction we discuss the difference in character between problems $\mathbf{P1}_\pm$ and $\mathbf{P2}_\pm$, due to the different side conditions. In certain cases (see Section 3 below) there is an exact correspondence to an inverse scattering problem for a potential $V(x)$ in the more familiar Schrödinger equation. The case of $\mathbf{P2}_\pm$ corresponds to data being $L(k)$, the left hand reflection coefficient, (see below for review of the definitions) whereas in the case of $\mathbf{P1}_\pm$ the data amounts to $L(k)/T(k)$, the ratio of reflection coefficient to transmission coefficient. The use of L/T as basic scattering data, in the case of a real V has appeared in a few previous papers ([16, 18]), but usually in a somewhat artificial way. Here it seems that there is a clear physical motivation. In the Schrödinger scattering case it is not difficult to show that L/T uniquely determines L and hence the potential, at least if there are no bound states (see [4] for a careful study of multiplicity results when bound states are allowed), and we will derive the analogous result for $\mathbf{P1}_\pm$. The proof relies on analytic continuation, thus

²For example, $B(X, k) \equiv 0$ outside of some k interval.

does not immediately show how known numerical techniques for $\mathbf{P2}_\pm$ could be adapted to $\mathbf{P1}_\pm$. From another point of view the inverse problem $\mathbf{P2}_\pm$ is ‘local in depth’, whereas $\mathbf{P1}_\pm$ is not, meaning that layer stripping type methods cannot be used, at least in any direct way. One alternative computational approach, somewhat analogous to [18] will be discussed in Section 7.

2 Conservation of energy and bound states

Clearly a solution of (1.1),(1.2) exists on $[0, X]$ for any $k \in \mathbb{C}$. Extending q by zero for $x \notin [0, X]$ we may assume when convenient that $A(x, k), B(x, k)$ are defined for all $x \in \mathbb{R}, k \in \mathbb{C}$. If we let

$$E(x, k) = (|A(x, k)|^2 \mp |B(x, k)|^2) \quad (2.1)$$

then simple calculation gives

$$\frac{\partial E}{\partial x} = i(k - k^*)(|A(x, k)|^2 \pm |B(x, k)|^2) \quad (2.2)$$

In particular for $k \in \mathbb{R}$

$$|A(X, k)|^2 \mp |B(X, k)|^2 \equiv 1 \quad (2.3)$$

A bound state of (1.1) is a nonzero solution of (1.1) for some fixed $k \in \mathbb{C}$ with $A(\cdot, k), B(\cdot, k) \in L^2(\mathbb{R})$. Such solutions are impossible in the $+q$ case of (1.1) since by (2.2) and the fact that E must be exponentially decaying at $\pm\infty$ we must have $\text{Im } k \neq 0$ and

$$0 = \int_{-\infty}^{\infty} \frac{\partial E}{\partial x}(x, k) dx = i(k - k^*) \int_{-\infty}^{\infty} (|A(x, k)|^2 + |B(x, k)|^2) dx \quad (2.4)$$

In the $-q$ case bound states may exist, although only for sufficiently large q . This is in contrast to the standard Schrödinger scattering case in which any potential $V(x) \leq 0, V(x) \not\equiv 0$ has at least one bound state. As is well known, it is the existence of such bound states which give rise to the existence of solitary wave solutions of the cubic Schrödinger equation (1.6).

It is easy to check that if $\text{Im } k < 0$ and $A(X, k) = 0$ then both components are either identically zero or exponentially decaying at both $\pm\infty$, hence $\{A(x, k), B(x, k)\}$ is a bound state.

3 Relation to other forms of scattering data

Consider the special case that $\pm q = q^*$, i.e. q is real in the $+q$ case or imaginary in the $-q$ case. If we set $\psi = A + B$ then straightforward computation gives

$$\psi'' + (k^2 - V(x))\psi = 0 \quad (3.1)$$

where $V = \pm q' + q^2$, a Schrödinger equation with (possibly complex) potential V .

Proceeding as in the case of a real potential, (e.g. [5, 7, 8]) we introduce the standard fundamental set of solutions ψ_-, ψ_+ of (3.1) which satisfy

$$\psi_-(x, k) = \begin{cases} e^{ikx} + L(k)e^{-ikx} & x < 0 \\ T(k)e^{ikx} & x > X \end{cases} \quad (3.2)$$

$$\psi_+(x, k) = \begin{cases} T(k)e^{-ikx} & x < 0 \\ e^{-ikx} + R(k)e^{ikx} & x > X \end{cases} \quad (3.3)$$

where L, R are left and right hand reflection coefficients and T is the transmission coefficient.³

From (1.2) we clearly have

$$A(x, k) = e^{ikx} \quad B(x, k) = 0 \quad x < 0 \quad (3.4)$$

and

$$B(x, k) = B(X, k)e^{-ik(x-X)} \quad A(x, k) = A(X, k)e^{ik(x-X)} \quad x > X \quad (3.5)$$

From (3.4) it follows that

$$\psi(x, k) = \psi_-(x, k) - \frac{L(k)}{T(k)}\psi_+(x, k) \quad (3.6)$$

³It may be checked that even in the case of complex V , the left and right transmission coefficients coincide.

so that

$$\psi(x, k) = \left(T(k) - \frac{L(k)R(k)}{T(k)} \right) e^{ikx} - \frac{L(k)}{T(k)} e^{-ikx} \quad x > X \quad (3.7)$$

From this and (3.5) then

$$A(X, k) = e^{ikX} \left(T(k) - \frac{L(k)R(k)}{T(k)} \right) \quad B(X, k) = -e^{-ikX} \frac{L(k)}{T(k)} \quad (3.8)$$

At least in the case of real q , the expression for $A(X, k)$ can be further simplified, using standard identities for Schrödinger scattering, to

$$A(X, k) = \frac{e^{ikX}}{T(k)^*} \quad k \in \mathbb{R} \quad (3.9)$$

The inverse problem $\mathbf{P1}_\pm$ thus corresponds, in such a case, to Schrödinger inverse scattering in which the available data is the ratio L/T of a reflection and transmission coefficient. By comparison, the data for problem $\mathbf{P2}_\pm$ would be simply $L(k)$, by a similar calculation, and so amounts to a familiar problem. It is well known, (e.g. [5, 7, 8]) that a real V is uniquely determined by $L(k)$ if no bound states are present.

4 Uniqueness

In the Schrödinger scattering case, when V has no bound states it is not hard to see that L/T uniquely determines L and hence V . Uniqueness for $\mathbf{P1}_\pm$ amounts to essentially the same thing for the system (1.1).

Theorem 1 *There is at most one solution of $\mathbf{P1}_+$, and the same is true of $\mathbf{P1}_-$ if there are no bound states.*

Proof: First note from (2.3) that $|A(X, k)|$ is known from the data, and it is not hard to check that

$$A(X, k) = a^*(k^*) e^{ikX} \quad (4.1)$$

where $a(k)$ is defined in equation (1.3.3a) in [1] (again with the understanding that q has been extended by zero outside of $[0, X]$). The function a is entire and $a(k) \rightarrow 1$ as $|k| \rightarrow \infty$ in the upper half plane.

Recall from section 2 that there can be no bound states in the case of $\mathbf{P1}_+$, and the same is true in the case of $\mathbf{P1}_-$ by hypothesis. From the last remark in section 2 it follows that $A(X, k)$ has no zeros in the lower half of the complex plane. Thus $g(k) := \log(A(X, k)e^{-ikX})$ is analytic in the lower half plane and tends to zero as $|k| \rightarrow \infty$. It follows by the usual Hilbert transform relation that the imaginary part of g on the real axis is uniquely determined by its real part, that is to say, the phase of $A(X, k)$ may be determined from $|A(X, k)|$, and so $A(X, k)$ is itself known. Finally one may check that

$$\{A(X - x, -k)/A(X, -k), -B(X - x, -k)/A(X, -k)\} \quad (4.2)$$

is a solution pair of (1.1) with $q(x)$ replaced by $q(x - X)$, satisfying side conditions (1.4) and having known scattering data (1.5). Thus q is uniquely determined according to the standard results about problems $\mathbf{P2}_\pm$. \square

In the Schrödinger scattering case it is known (e.g. Theorem 2.3 of [18]) that L/T does not uniquely determine the potential in the presence of bound states, hence we expect that the ‘no bound state’ hypothesis in the case of $\mathbf{P1}_-$ cannot be dispensed with. Numerical examples (see section 7 below) also indicate this quite clearly. On the other hand in the more conventional case of Schrödinger scattering with data L , even though bound state data is needed in general to uniquely determine a potential, this is not the case if V is known to be supported in a half line ([2, 14]), i.e. $L(k)$ alone suffices to determine V in such a case. We conjecture that the same is true for $\mathbf{P2}_-$.

5 Time domain problem

It is well known that problem $\mathbf{P2}_\pm$ can be analyzed in terms of an associated hyperbolic ‘time domain’ problem, which is recalled in (6.2) below, see e.g. [9, 17, 19, 20]. Here we derive a corresponding hyperbolic problem (5.15) for $\mathbf{P1}_\pm$. We remark that in the case of $\mathbf{P2}_\pm$ the derivation of (6.2) requires an assumption that there be no bound states, but this hypothesis is not needed in the case of (5.15).

Let $\{a(x, t), b(x, t)\}$ denote solutions of the hyperbolic problem in \mathbb{R}^2

$$a_t + a_x = q^*(x)b \quad b_t - b_x = \mp q(x)a \quad (5.1)$$

$$a(0, t) = \delta(t) \quad b(0, t) = 0 \quad (5.2)$$

Regarding this as an evolution equation in the space variable x we see by standard domain of dependence considerations that the support of the solution is in the triangle $\{(x, t) : |t| \leq x < X\}$. Taking the Fourier transform in t , using the convention

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(t)e^{ikt} dt \quad (5.3)$$

we find that

$$-ik\hat{a} + \hat{a}_x = q^*\hat{b} \quad -ik\hat{b} - \hat{b}_x = \mp q\hat{a} \quad (5.4)$$

and

$$\hat{a}(0, k) = 1 \quad \hat{b}(0, k) = 0 \quad (5.5)$$

That is to say $A(x, k) = \hat{a}(x, k), B(x, k) = \hat{b}(x, k)$ is the solution of (1.1),(1.2), since the solution is unique.

By a standard propagation of singularities (geometric optics) argument, we may obtain equivalent characteristic boundary conditions for a, b . The solution with $q \equiv 0$ is $a(x, t) = \delta(x - t), b(x, t) = 0$, thus near the characteristic $t = x$ we should have

$$a(x, t) = \delta(x - t) + \alpha(x)H(x - t) + \text{smoother terms} \quad (5.6)$$

$$b(x, t) = \beta(x)H(x - t) + \text{smoother terms} \quad (5.7)$$

where H denotes the unit step function and α, β are transport coefficients to be determined. Inserting these expansions into (5.1) and matching the coefficients of the most singular terms we get

$$\alpha(x) = \pm \frac{1}{2} \int_0^x |q(s)|^2 ds \quad \beta(x) = \pm \frac{1}{2} q(x) \quad (5.8)$$

When we carry out a similar calculation on the lower characteristic $t = -x$, substituting

$$a(x, t) = \alpha(x)H(x + t) + \text{smoother terms} \quad (5.9)$$

$$b(x, t) = \beta(x)H(x + t) + \text{smoother terms} \quad (5.10)$$

the relations

$$\alpha(x) = 0 \quad -\beta'(x) = \mp q(x)\alpha(x) \quad (5.11)$$

are obtained. Thus

$$\alpha(x) = 0 \quad \beta(x) = \beta(0) \quad (5.12)$$

and continuity of a, b from inside the cone $|t| < x$ then gives

$$\beta(0) = \pm \frac{1}{2}q(0) \quad (5.13)$$

Note that

$$A(X, k) = \int_{-X}^X a(X, t)e^{ikt} dt \quad B(X, k) = \int_{-X}^X b(X, t)e^{ikt} dt \quad (5.14)$$

from which it follows that $A(X, \cdot), B(X, \cdot)$ are bandlimited functions, hence uniquely determined by appropriate sampling.

We may now reformulate the inverse problem $\mathbf{P1}_{\pm}$ as an overdetermined boundary value problem: If $B(X, k)$ is the data corresponding to q then there exist $\{a(x, t), b(x, t)\}$ such that

$$a_t + a_x = q^*(x)b \quad b_t - b_x = \mp q(x)a \quad |t| < x < X \quad (5.15a)$$

$$b(x, x) = \pm \frac{1}{2}q(x) \quad 0 < x < X \quad (5.15b)$$

$$a(x, -x) = 0 \quad 0 < x < X \quad (5.15c)$$

$$b(X, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(X, k)e^{-ikt} dk \quad |t| < X \quad (5.15d)$$

Note that the additional characteristic boundary conditions

$$a(x, x) = \pm \frac{1}{2} \int_0^x |q(s)|^2 ds \quad b(x, -x) = \pm \frac{1}{2}q(0) \quad 0 < x < X \quad (5.16)$$

follow from (5.15), consistently with the expressions for $\alpha(x)$ in (5.8) and $\beta(x)$ in (5.12).

6 Nonlinear Plancherel identity

In the context of inverse scattering for the Helmholtz equation, Sylvester, Winebrenner, and Gylys-Colwell [21] discovered an interesting identity relating the L^2 norm of the coefficient to a certain nonlinear functional of the scattering data. It was referred to as a nonlinear Plancherel identity because in the weak scattering limit, the coefficient to data mapping becomes the Fourier transform, and so the identity in question goes over to the classical Plancherel equality. In this section we'll state and prove an analogous property for the inverse scattering problem $\mathbf{P1}_\pm$. For the proof we'll need a corresponding result for $\mathbf{P2}_\pm$, which as far as we know is also new, although the proof follows closely the pattern of [21].

Before proceeding let us note that if we set $R(x, k) = B(x, k)/A(x, k)$ then it is easy to check that R satisfies the Riccati type equation

$$\frac{\partial R}{\partial x} = -2ikR(x, k) - q^*(x)R(x, k)^2 \pm q(x) \quad (6.1)$$

at least provided that $A(x, k) \neq 0$. Numerical methods for $\mathbf{P2}_\pm$ have been developed which exploit (6.1) and the fact that $R(0, k), R(X, k)$ are known from the data and side conditions ([9, 19]. For $\mathbf{P1}_\pm$ however, we only know that $R(0, k) = 0$ and have no explicit way to obtain $R(X, k)$.

We also will make use of the analog of (5.15) for $\mathbf{P2}_\pm$ which is the following: If $B(0, k)$ is the data corresponding to q there exists $\{a(x, t), b(x, t)\}$ such that

$$a_t + a_x = q^*(x)b \quad b_t - b_x = \mp q(x)a \quad 0 < x < t \quad (6.2a)$$

$$b(x, x) = \mp \frac{1}{2}q(x) \quad x > 0 \quad (6.2b)$$

$$a(0, t) = 0 \quad t > 0 \quad (6.2c)$$

$$b(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(0, k)e^{-ikt} dk \quad t > 0 \quad (6.2d)$$

Here we are again regarding $q(x)$ as defined to be zero outside of $[0, X]$. In the case of $\mathbf{P2}_-$ we must assume in addition that no bound states exist, so that $b(0, t)$ is Fourier transformable with respect to time. This solution

corresponds to (5.1) with side conditions

$$a(x, t) = \delta(x - t) \quad b(x, t) = 0 \quad t < 0 \quad (6.3)$$

and may be derived in a manner analogous to (5.15). See also [17, 20] for more about (6.2). In particular $\{a, b\}$ may be viewed as a solution of (6.2a) for all t , which is zero for $t < x$. Since $b(0, t) = 0$ for $t < 0$ it follows that $B(0, k)$ is analytic in the upper half of the complex plane.

Theorem 2 *In the case of problem $\mathbf{P2}_+$ we have*

$$-\int_{-\infty}^{\infty} \log(1 - |B(0, k)|^2) dk = \pi \int_0^X |q(x)|^2 dx \quad (6.4)$$

and in the case of problem $\mathbf{P1}_+$ we have

$$\int_{-\infty}^{\infty} \log(1 + |B(X, k)|^2) dk = \pi \int_0^X |q(x)|^2 dx \quad (6.5)$$

Proof: First consider the case of $\mathbf{P2}_+$. Note from (2.1), (2.2) that $|A(x, k)|^2 - |B(x, k)|^2$ is equal to the constant $|A(X, k)|^2$ for real k , hence $A(x, k) = 0$ is impossible for $k \in \mathbb{R}$ or $x \in [0, X]$. Now multiply (6.1) by $R(x, k)^*$ to get

$$R(x, k)^* \frac{\partial R}{\partial x}(x, k) = -2ik|R(x, k)|^2 - q^*(x)R(x, k)|R(x, k)|^2 + q(x)R(x, k)^* \quad (6.6)$$

Taking the real part of this identity gives

$$\frac{\partial}{\partial x} |R(x, k)|^2 = 2(1 - |R(x, k)|^2) \operatorname{Re}(q^*(x)R(x, k)) \quad (6.7)$$

or

$$-\frac{\partial}{\partial x} \log(1 - |R(x, k)|^2) = 2 \operatorname{Re}(q^*(x)R(x, k)) \quad (6.8)$$

We next claim that

$$\int_{-\infty}^{\infty} R(x, k) dk = -\frac{\pi}{2}q(x) \quad 0 \leq x \leq X \quad (6.9)$$

Given this, we integrate both sides of (6.8) with respect to k to get

$$\frac{\partial}{\partial x} \int_{-\infty}^{\infty} \log(1 - |R(x, k)|^2) dk = \pi |q(x)|^2 \quad (6.10)$$

Finally integrate with respect to x from 0 to X , using the fact that $R(0, k) = B(0, k)$, $R(X, k) = 0$ to obtain (6.4).

To verify the claim (6.9), consider first the case $x = 0$. in which case $R(0, k) = B(0, k)$, the Fourier transform of $b(0, t)$. Since $b(0, 0+) = \lim_{t \rightarrow 0+} b(0, t) = -\frac{q(0)}{2}$ and $b(0, 0-) = \lim_{t \rightarrow 0-} b(0, t) = 0$, it follows that $\frac{1}{2\pi} \int_{-\infty}^{\infty} B(0, k)$, the inverse Fourier transform of $B(0, k)$ evaluated at $t = 0$ is the average of the one sided limits of $b(0, t)$, i.e.

$$\int_{-\infty}^{\infty} R(0, k) dk = \int_{-\infty}^{\infty} B(0, k) dk = 2\pi \left(-\frac{q(0)}{4} \right) \quad (6.11)$$

which is the required result.

Now for any fixed $x_0 \in [0, X]$ it is not hard to check that $R(x_0, k) = \tilde{B}(x_0, k)$ where \tilde{A}, \tilde{B} solves

$$\tilde{A}' = ik\tilde{A} + q^*\tilde{B} \quad \tilde{B}' = -ik\tilde{B} + q\tilde{A} \quad x_0 < x < X \quad (6.12)$$

$$\tilde{A}(x_0, k) = 1 \quad \tilde{B}(X, k) = 0 \quad (6.13)$$

The argument in the previous paragraph shows that $\int_{-\infty}^{\infty} \tilde{B}(x_0, k) dk = -\frac{\pi}{2}q(x_0)$ and so (6.9) follows for all $x \in [0, X]$.

In the case of problem **P1**₊, if $A(x, k), B(X, k)$ is the solution of (1.1),(1.2), and

$$A_1(x, k) = \frac{A(X - x, -k)}{A(X, -k)} \quad B_1(x, k) = -\frac{B(X - x, -k)}{A(X, -k)} \quad (6.14)$$

then one easily checks that $A_1(x, k), B_1(x, k)$ satisfies (1.1),(1.4) with coefficient $q(x)$ replaced by $q_1(x) = q(X - x)$. From the first part of the proof it then follows that

$$\begin{aligned} \pi \int_0^X |q(x)|^2 dx &= \pi \int_0^X |q_1(x)|^2 dx = - \int_{-\infty}^{\infty} \log(1 - |B_1(0, k)|^2) dk \\ &= - \int_{-\infty}^{\infty} \log(1 - |R(X, k)|^2) dk = - \int_{-\infty}^{\infty} \log \left(\frac{1}{1 + |B(X, k)|^2} \right) dk \end{aligned} \quad (6.15)$$

where in the last equality we used the conservation property (2.3). The conclusion (6.5) follows. \square

Using the elementary inequalities $\log(1+r) \leq r, r \geq 0$ and $r \leq -\log(1-r)$ for $0 \leq r \leq 1$ and the fact that $|B(0, k)| \leq 1$ in $\mathbf{P2}_+$ we obtain the immediate corollaries

$$\int_{-\infty}^{\infty} |B(X, k)|^2 dk \geq \pi \int_0^X |q(x)|^2 dx \quad (6.16)$$

in the case of $\mathbf{P2}_+$ and

$$\int_{-\infty}^{\infty} |B(0, k)|^2 dk \leq \pi \int_0^X |q(x)|^2 dx \quad (6.17)$$

in the case of $\mathbf{P1}_+$.

Similar properties can be proved for problems $\mathbf{P1}_-, \mathbf{P2}_-$ with further restrictions.

Theorem 3 *Assume that there are no bound states and that $A(x, k) \neq 0$ for $x \in [0, X]$ and $k \in \mathbb{R}$. Then in the case of problem $\mathbf{P2}_-$ we have*

$$\int_{-\infty}^{\infty} |B(0, k)|^2 dk \geq \int_{-\infty}^{\infty} \log(1 + |B(0, k)|^2) dk = \pi \int_0^X |q(x)|^2 dx \quad (6.18)$$

and in the case of problem $\mathbf{P1}_-$ we have

$$\int_{-\infty}^{\infty} |B(X, k)|^2 dk \leq - \int_{-\infty}^{\infty} \log(1 - |B(X, k)|^2) dk = \pi \int_0^X |q(x)|^2 dx \quad (6.19)$$

7 Constructive methods

A simple approximate solution for $\mathbf{P2}_\pm$ which may nevertheless be quite accurate for sufficiently small q 's is the Born approximation. If we replace q by $q_0 + \epsilon q_1$ with corresponding solutions $A_0 + \epsilon A_1, B_0 + \epsilon B_1$ and match powers of ϵ we get $A_0(x, k) = e^{ikx}, B_0(x, k) = 0$ and

$$B_1' + ikB_1 = \pm q(x)e^{ikx} \quad (7.1)$$

Solving with $B_1(0, k) = 0$ leads to

$$B_1(X, k) = \pm e^{-ikX} \int_0^X q(s) e^{2iks} ds \quad (7.2)$$

and so by Fourier inversion

$$q(x) \approx \pm \frac{1}{\pi} \int_{-\infty}^{\infty} B(X, k) e^{-ik(2x-X)} dk \quad (7.3)$$

The same result may be obtained by ignoring the quadratic term in (6.1), or from (5.15) as explained below. This formula is exploited in [11] (also [22] in the case of $\mathbf{P2}_{\pm}$). An improved method based on the calculation of transfer matrices for piecewise constant q is presented in [13].

Several types of exact methods may be derived by first transforming the data for $\mathbf{P1}_{\pm}$ to that of $\mathbf{P2}_{\pm}$, in the manner indicated by the proof of the uniqueness result Theorem 1. From the data and (2.3) one may obtain $|A(X, k)|$, a Hilbert transform calculation of $\log |A(X, k)|$ then yields $A(X, k)$ from which we obtain data for a reflected version of $\mathbf{P2}_{\pm}$, according to (4.2). One can then proceed to use one of the methods which have been developed for numerical solution of $\mathbf{P2}_{\pm}$, e.g. [9, 15, 19, 20]. For this transformation it is necessary to assume that there are no bound states. Furthermore the Hilbert transform step may present numerical difficulties if $B(X, k)$ is known only on a coarse grid of k values, whereas coarse sampling need not in itself be a problem due to the bandlimitation property of $B(X, k)$ discussed above.

A final possibility for an exact method which we discuss in more detail is based on the equivalent time domain problem (5.15). Let us define the mapping $q \mapsto \Lambda(q)$ by $\Lambda(q)(t) = b(X, t)$, $|t| < 1$ where $\{a(x, t), b(x, t)\}$ is the solution pair of the well-posed characteristic boundary value problem (5.15 a–c). We may then attempt to obtain q as the solution of the nonlinear operator equation

$$\Lambda(q) = g \quad (7.4)$$

where $g(t) = b(X, t)$ is known from the given data. Recalling that $b(X, t)$ has support in $[-X, X]$ it follows from (5.14) that we may represent g as

a Fourier series

$$b(X, t) = \frac{1}{2X} \sum_{n=-\infty}^{\infty} B(X, \frac{n\pi}{X}) e^{-\frac{in\pi t}{X}} \quad (7.5)$$

involving only sampled values of $B(X, k)$. Of course in theory $b(X, t)$ is uniquely determined by the data $\{B(X, k_n)\}_{n=1}^{\infty}$ where k_n is any sequence with a finite limit point, or a sequence tending to $\pm\infty$ with an appropriate asymptotic spacing. The coefficients in (7.5) will typically decay at a rate $O(1/n)$ and no faster, unless additional smoothness assumptions are made on q (or more precisely on the X periodic extension of q .)

A simple iterative scheme is the Newton-Kantorovich method

$$q_{n+1} = q_n - D\Lambda(0)^{-1}(\Lambda(q_n) - g) \quad (7.6)$$

where the Fréchet derivative $D\Lambda(0)$ is easily found to be

$$D\Lambda(0)\zeta(t) = \pm \frac{1}{2}\zeta\left(\frac{X+t}{2}\right) \quad (7.7)$$

The resulting scheme is thus explicitly

$$q_{n+1}(x) = q_n(x) \pm 2g(2x - X) \mp 2\Lambda(q_n)(2x - X) \quad (7.8)$$

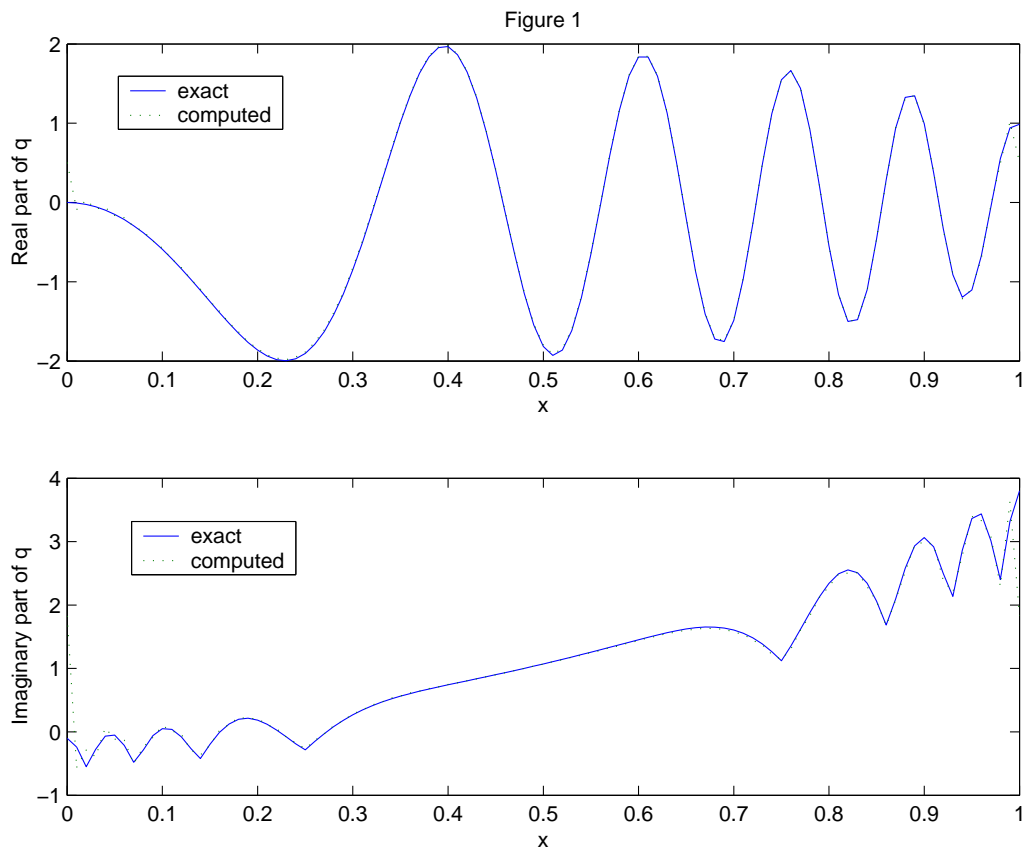
initialized, for example, by $q_0(x) = 0^4$.

If the coupling coefficient q is sufficiently small then convergence of q_n to q in $L^2(0, X)$ can be shown, essentially by the inverse function theorem and the fact that Λ can be shown to have suitable differentiability properties. The proof is very similar e.g. to that of Corollary 3.1 of [18].

In general the scheme is not globally convergent. In the case of $\mathbf{P1}_+$ the inverse problem has a unique solution which must also be the unique solution of $\Lambda(q) = g$, so if the sequence q_n converges at all then it must be to the solution q . In the case of $\mathbf{P1}_-$ the solution is not unique in general, so the sequence q_n may converge to some other solution of the inverse problem.

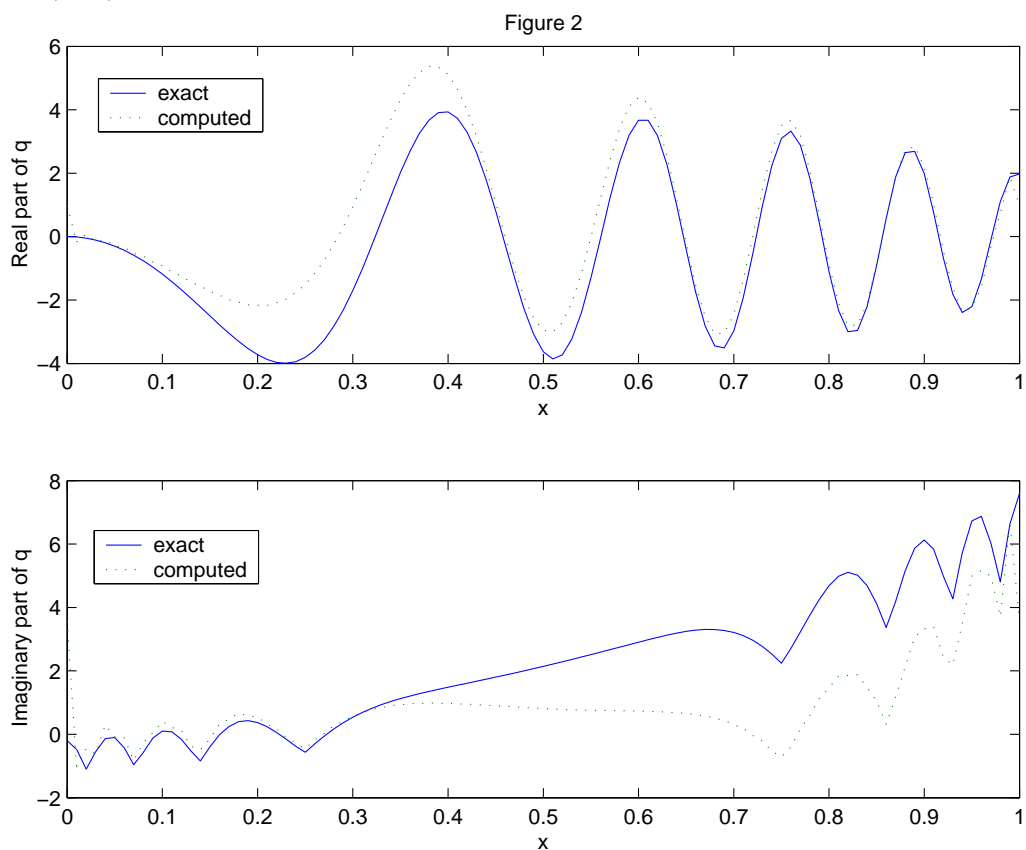
⁴In which case $q_1(x) = D\Lambda(0)^{-1}g(x) = \pm 2g(2x - X)$ is the same as the Born approximation mentioned above, which could also be used as the initial guess.

In Figure 1 a reconstruction is displayed together with the exact coupling coefficient for $\mathbf{P1}_-$. In this example $X = 1$, and the data $B(1, n\pi)$ is used for $|n| \leq 50$ to approximate $b(1, t)$ using (7.5). The direct problem (1.1)-(1.2) was solved using `ode45` in `Matlab`, and the time domain map Λ was approximated by means of a straightforward finite difference scheme in characteristic coordinates, with grid size extrapolation to improve the accuracy. The relative error in q is about 15%, but essentially all of this may be attributed to the error in $b(1, t)$ due to the truncation error in (7.5). That is to say if we used more (or fewer) terms we will always find that the relative error in the computed q is comparable to the relative error in $b(1, t)$. The results are very similar if we repeat the calculation for $\mathbf{P1}_+$.



For a second example we replace q in the previous case by $2q$. We now find that the sequence q_n does not converge at all in the case of $\mathbf{P1}_+$, while in the case of $\mathbf{P1}_-$ the sequence converges, but to a different solution displayed in Figure 2, consistent with the earlier discussion of nonuniqueness.

It seems clear that some alternative optimization approach could be used to obtain the original q , provided a sufficiently accurate initial guess were available. We note that in the first example the identity (6.19) is satisfied, while in the second example it is satisfied by the computed $q(x)$, not by the original q , leading one to surmise that the result of the computational method just described will be the unique choice of q with the prescribed data (1.3) but with no bound states.



8 Acknowledgment

The author would like to thank Anand Gopinath, Aurelia Minut and Fadil Santosa for helpful conversations.

References

- [1] M. J. Ablowitz and H. Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics*. SIAM, 1981.
- [2] T. Aktosun. Bound states and inverse scattering for the Schrödinger equation in one dimension. *J. Math. Phys.*, 35(12):6231–6236, 1994.
- [3] T. Aktosun, M. Klaus, and C. van der Mee. Direct and inverse scattering for selfadjoint Hamiltonian systems on the line. *Integral Equations Operator Theory*, 38(2):129–171, 2000.
- [4] T. Aktosun and V. G. Papanicolaou. Recovery of a potential from the ratio of reflection and transmission coefficients. *Preprint*, 2003.
- [5] K. Chadan and P. C. Sabatier. *Inverse problems in quantum scattering theory*. Texts and Monographs in Physics. Springer-Verlag, 2nd edition, 1989.
- [6] M. Klaus D. B. Hinton, A. K. Jordan and J. K. Shaw. Inverse scattering on the line for a Dirac system. *J. Math. Phys.*, 32:3015–3030, 1991.
- [7] P. Deift and E. Trubowitz. Inverse scattering on the line. *Comm. Pure Appl. Math.*, 32(2):121–251, 1979.
- [8] L. D. Faddeev. Properties of the S -matrix of the one-dimensional Schrödinger equation. *Trudy Mat. Inst. Steklov.*, 73:314–336, 1964.
- [9] R. Feced, M. N. Zervas, and M. A. Muriel. An efficient inverse scattering algorithm for the design of nonuniform fiber Bragg gratings. *IEEE J. Quant. Elect.*, 35:1105–1115, 1999.
- [10] S. A. Furman and A. V. Tikhonravov. *Optics of Multilayer Systems*. Editions Frontières, Gif-sur-Yvette, 1992.
- [11] C. Laliew, S. Lovseth, X. Zhang, and A. Gopinath. A linearized optical directional-coupler modulator at 1.3 μm . *IEEE J. Lightwave Tech.*, 18:1244–1249, 2000.

- [12] B. M. Levitan and I. S. Sargsjan. *Sturm-Liouville and Dirac operators*, volume 59 of *Mathematics and its Applications (Soviet Series)*. Kluwer, 1991.
- [13] T. Li, C. Laliew, and A. Gopinath. An iterative transfer matrix inverse scattering technique for the synthesis of co-directional optical couplers and filters. *IEEE J. Quant. Electronics*, 38:375–379, 2002.
- [14] N. N. Novikova and V. M. Markushevich. Uniqueness of the solution of the one-dimensional problem of inverse scattering for potentials located on the positive semi-axis. *Computational Seismology*, 18:164–172, 1987.
- [15] E. Peral, J. Capmany, and J. Marti. Iterative solution to the Gelfand-Levitan-Marchenko coupled equations and application to synthesis of fiber gratings. *IEEE J. Quant. Elect.*, 32:2078–2084, 1996.
- [16] J. C. Portinari. Finite range solutions to the one dimensional inverse scattering problem. *Ann. Phys.*, 45:445–451, 1967.
- [17] Rakesh. A one-dimensional inverse problem for a hyperbolic system with complex coefficients. *Inverse Problems*, 17(5):1401–1417, 2001.
- [18] W. Rundell and P. E. Sacks. On the determination of potentials without bound state data. *J. Comput. Appl. Math.*, 55(3):325–347, 1994.
- [19] J. Skaar, L. Wang, and T. Erdogan. On the synthesis of fiber Bragg gratings by layer peeling. *IEEE J. Quant. Elect.*, 37:165–173, 2001.
- [20] G.-H. Song and S.-Y. Shin. Design of corrugated waveguide filters by the Gelfand-Levitan-Marchenko inverse scattering method. *J. Opt. Soc. Am.*, 2:1905–1915, 1985.
- [21] J. Sylvester, D. Winebrenner, and F. Gylys-Colwell. Layer stripping for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(3):736–754, 1996.

- [22] K. A. Winick and J. E. Roman. Design of corrugated waveguide filters by Fourier transform techniques. *IEEE J. Quant. Elect.*, 26:1918–1929, 1990.
- [23] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.
- [24] X. Zhou. Zakharov-Shabat inverse scattering. In R. Pike and P. Sabatier, editors, *Scattering: Scattering and inverse scattering in pure and applied science*, pages 1707–1716. Academic Press, 2002.