Problems from Math 519/520 (M. Smiley)

Textbook references are to Green’s Functions and Boundary Value Problems, 2nd ed, by I. Stakgold

1. Let \( p, q, f \in C(\alpha, \beta) \) and \( x_0 \in (\alpha, \beta) \). Suppose that \( y_1(x) \) and \( y_2(x) \) are the solutions of the 2nd order linear homogeneous equation \( y'' + p(x)y' + q(x)y = 0 \), on the interval \((\alpha, \beta)\), that satisfy
\[
y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.
\]
This choice of initial conditions insures that \( y_1(x) \) and \( y_2(x) \) are linearly independent on \((\alpha, \beta)\). Using variation of parameters one finds that
\[
Y(x) = \int_{x_0}^{x} k(x, s)f(s) \, ds, \quad \text{where} \quad k(x, s) = \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)},
\]
is a solution of the nonhomogeneous problem
\[
y'' + p(x)y' + q(x)y = f(x) \tag{1}
\]
on the interval \((\alpha, \beta)\). Find the solution of the nonhomogeneous problem (1) that satisfies \( y(x_0) = a, \ y'(x_0) = b \).

2. The solution of
\[
y'' + y = f(x), \quad 0 < x < \frac{\pi}{2}, \quad y(0) = 0, \ y\left(\frac{\pi}{2}\right) = 0,
\]
can be written in the form
\[
y(x) = \int_{0}^{\frac{\pi}{2}} k(x, s)f(s) \, ds,
\]
where \( k(x, s) \) is defined on the square \( 0 \leq x, s \leq \frac{\pi}{2} \). Determine \( k(x, s) \).

3. Boundary value problems may have zero, one, or an infinite number of solutions. To which category does each of the following boundary value problems belong?
\[
y'' + y = 0, \quad 0 < x < \pi, \quad y(0) = 0, \ y(\pi) = 0,
\]
\[
y'' + y = 1, \quad 0 < x < \frac{\pi}{2}, \quad y(0) = 0, \ y\left(\frac{\pi}{2}\right) = 0,
\]
\[
y'' + y = 0, \quad 0 < x < \pi, \quad y(0) = 0, \ y(\pi) = 1,
\]

4. Show that \( \{\sin\left(\frac{nx\pi}{a}\right)\}_{n=1}^{\infty} \) is an orthogonal set of functions on the interval \((0, a)\). That is, show
\[
\int_{0}^{a} \sin\left(\frac{nx\pi}{a}\right) \sin\left(\frac{mx\pi}{a}\right) \, dx = \begin{cases} 0, & m \neq n \\ \frac{a}{2}, & m = n \end{cases}.
\]
5. Let \( w(t, x) = \sum_{n=1}^{\infty} b_n \exp(-\left(\frac{n\pi}{a}\right)^2 kt) \sin\left(\frac{n\pi x}{a}\right) \), where \( \{b_n\}_{n=1}^{\infty} \) is a bounded sequence of numbers.
   a) Show that \( w(t, x) \) is a continuous function on \((t_0, \infty) \times (0, a)\) for any \( t_0 > 0 \).
   b) Show that the series corresponding to \( w_t \) and \( w_{xx} \) are uniformly convergent on \((t_0, \infty) \times (0, a)\) for any \( t_0 > 0 \).
   c) Show that \( w(t, x) \to 0 \), uniformly on \((0, a)\) as \( t \to \infty \). Hint: Use comparison to a geometric series with ratio \( r = \exp(-\pi^2 kt/a^2) \).

6. Let \( y \in C[0, \pi] \cap C^1(0, \pi) \) be a function satisfying i) \( y(0) = y(\pi) = 0 \) and ii) \( |y'(x)| \) is bounded on \((0, \pi)\).
   a) Use L'Hôpital's rule to show that
      \[
      \lim_{x \to 0^+} \frac{[y(x)]^2}{x} = 0, \quad \lim_{x \to \pi^-} \frac{[y(x)]^2}{x - \pi} = 0.
      \]
   b) Let \( z(x) = [y(x)]^2 \cot x, 0 < x < \pi \). Show that
      \[
      z' = (y')^2 - y^2 - (y' - y \cot x)^2
      \]
      and that \( z(x) \to 0 \) as either \( x \to 0^+ \) or \( x \to \pi^- \). Conclude that
      \[
      \int_0^\pi [y(x)]^2 \, dx \leq \int_0^\pi [y'(x)]^2 \, dx.
      \]
      When can equality hold?
   c) Let \( w \in C[0, a] \cap C^1(0, a) \) be a function satisfying i) \( w(0) = w(a) = 0 \) and ii) \( |w'(x)| \) is bounded on \((0, a)\). Use the result above to show
      \[
      \left(\frac{\pi}{a}\right)^2 \int_0^a [w(x)]^2 \, dx \leq \int_0^a [w'(x)]^2 \, dx.
      \]
      The technique used in this exercise is given in the book *Inequalities* by Hardy, Littlewood and Pólya.

7. a) Give an example of a function \( \phi(x) \) that satisfies i) \( \phi \in C^1(-\infty, +\infty) \), ii) \( 0 < \phi(x) \leq 1 \) on \(|x| < 1\), and iii) \( \phi(x) = 0 \) on \(|x| > 1\).
   b) Let \( x_0, \epsilon \in \mathcal{R} \) with \( \epsilon > 0 \). Give an example of a function \( \psi(x) \) that satisfies i) \( \psi \in C^1(-\infty, +\infty) \), ii) \( 0 < \psi(x) \leq 1 \) on \(|x - x_0| < \epsilon\), and iii) \( \psi(x) = 0 \) on \(|x - x_0| > \epsilon\).

8. Suppose that \( h \in C[0, 1] \) and \( < h, \phi' >= 0 \) for every \( \phi \in C^1[0, 1] \) satisfying \( \phi(0) = \phi(1) = 0 \). Show that \( h \) is a constant function without assuming a priori that \( h \) is differentiable. (See problem 5.2 on page 39 for a hint.)

9. Let \( f_n(x) = n^\alpha x e^{-n|x|} \) and \( 1 \leq p < \infty \). Show that the sequence \( \{f_n\}_{n=1}^{\infty} \) converges to the zero function in \( L^p(0, 1) \) for all \( \alpha \in \left[0, 1 + \frac{1}{p}\right] \). Hint: The Gamma function \( \Gamma(x) \) is defined for all \( x > 0 \) by
      \[
      \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.
      \]

10. Let \( f(x) \) be piecewise continuous on \((0, 1)\), and let \( u(x) \) be the solution of:
       \(-u'' = f \) on \((0, 1)\), \( u(0) = u(1) = 0 \). Use the Schwartz inequality to show that
\[ \|u\|_2 \leq I(g)\|f\|_2, \] where \( I(g) \) denotes a double integral involving the Green's function for the problem. State and evaluate the double integral.

11. Let \( 0 < \xi < s \) be given real numbers. Find the (distribution) solution of the boundary value problem
\[
-\frac{d^2g}{dx^2} = \delta(x - \xi), \quad 0 < x < s, \quad g(0) = g(s) = 0
\]
by first finding a general (distribution) solution of the differential equation.

12. Provide a solution to problem 4.4 on page 95. Hint: Use the Green’s function determined in the previous problem.

13. Let \( \phi(x) \) be defined by \( \phi(x) = 0 \) for \( |x| \geq 1 \) and \( \phi(x) = \exp(\psi(x)) \) for \( |x| < 1 \), where \( \psi(x) = 1/(x^2 - 1) \). Show that \( \phi \in C^\\infty(\mathbb{R}) \). Hint: Use induction to show that, for all integers \( m \geq 0 \), \( \phi^{(m)}(x) = p_m(x)[\psi(x)]^m\phi(x) \) for \( |x| < 1 \), where \( p_m \) is a polynomial.

14. Show that \( g(x) = e^{-|x|/4\pi|x|} \) satisfies \( -\Delta g + q^2 g = \delta(x) \) in the sense of distributions on \( \mathbb{R}^3 \).

15. Let \( a > 0 \) be a fixed number and \( f(x) = (4\pi)^{-\frac{3}{2}} \exp(-|x|^2/4), \quad x \in \mathbb{R}^3 \). Show that the family of functions \( \{f_\epsilon(x)\}_{\epsilon > 0} \) defined by \( f_\epsilon(x) = (ae)^{-\frac{3}{2}} f(x/\sqrt{a\epsilon}) \) is a delta family as \( \epsilon \to 0^+ \). Hint: Use Theorem 3, page 123.

16. i) Find the solution of the wave equation for the semi-infinite string
\[
 Du - c^2 u_{xx} = 0, \quad 0 < t, x < \infty,
\]
\[
u(0, x) = f(x), \quad u_t(0, x) = g(x), \quad 0 < x < \infty,
\]
\[
u(t, 0) = h(t), \quad 0 < t < \infty.
\]
ii) Plot the solution \( u(t, x) \) as a function of \( x \) at times \( t = 2 \) and \( t = 5 \) when the data for the problem is \( f(x) = \max\{0, 1 - |x - 3|\} \), \( g(x) = 0 \) for \( x > 0 \), \( h(t) = 0 \) for \( t > 0 \), and \( c = 1 \).

iii) Plot the solution \( u(t, x) \) as a function of \( x \) at times \( t = 1 \) and \( t = 5 \) when the data for the problem is \( f(x) = 0 \), \( g(x) = 0 \) for \( x > 0 \), \( h(t) = \sin(\pi t/2) \) for \( t > 0 \), and \( c = 1 \).

17. i) If \( f, g \in C^m(\mathbb{R}) \) then
\[
 \int f^{(m)}(x)g(x) \, dx = \sum_{i=1}^{m} (-1)^{i-1} f^{(m-i)}(x)g^{(i-1)}(x) + (-1)^m \int f(x)g^{(m)}(x) \, dx.
\]
Use induction to verify this statement.

ii) Let \( L = \sum_{m=0}^{p} a_m(x)D^m \) be a linear variable coefficient ordinary differential operator on the real line. Verify the Lagrange identity
\[
v Lu - u L^* v = \frac{d}{dx} \sum_{m=1}^{p} \sum_{i=1}^{m} (-1)^{i-1} D^{i-1}(a_m v)D^{m-i} u,
\]
for functions \( u, v \in C^p(\mathbb{R}) \).

18. Let \( a > 0 \) be a fixed number and \( K(t, x) = (4\pi at)^{-\frac{3}{2}} \exp(-|x|^2/4at), (t, x) \in \mathbb{R} \times \mathbb{R}^3 \). Show that \( E(t, x) = H(t)K(t, x) \) is a causal fundamental solution for the diffusion operator \( L = \partial_t - a\Delta \), with pole at \( (t, x) = (0, 0) \); thus \( LE(t, x) = \delta(t, x) \) and \( E(t, x) = 0 \) for \( (t, x) \in (-\infty, 0) \times \mathbb{R}^3 \). The causal fundamental solution \( E(t, x) \), with pole at \( (t, x) = (0, 0) \), for the time-dependent diffusion equation in an absorbing medium satisfies

\[
\frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial x^2} + q^2 E = \delta(t, x),
\]

with \( E(t, x) = 0 \) for \( (t, x) \in (-\infty, 0) \times \mathbb{R} \). Find \( E(t, x) \) by changing dependent variables \( E = \exp(-q^2t)F \), keeping in mind that both \( E \) and \( F \) are distributions.

20. Show that \( U(t, x) = \frac{1}{4}H(t - |x|) \) is a causal fundamental solution for the 1-dimensional wave operator \( \partial^2_{tt} - \partial^2_{xx} \) with pole at \( (t, x) = (0, 0) \). Hint: See Example 7 on page 190.

21. Find the causal fundamental solution for the ordinary differential operator \( L = \frac{d^2}{dx^2} + 1 \) with pole at \( x = \xi \).

22. Let \( L = a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \), where \( a_0, a_1, a_2 \in C^2[a, b] \), and define the boundary value operators \( B_iu = \alpha_i u(a) + \alpha_i' u'(a) + \beta_i u(b) + \beta_i' u'(b) \), for \( i = 1, 2 \).

i) Show that \( M = \{ u \in C^1[a, b] : B_1u = B_2u = 0 \} \) is a closed linear subspace of the Banach space \( C^1[a, b] \). The norm is the standard one:

\[
||u||_{C^1[a, b]} = \max_{x \in [a, b]} |u(x)| + \max_{x \in [a, b]} |u'(x)|.
\]

ii) Show that \( M^* = \{ v \in C^1[a, b] : J(u, v)|_a^b = 0, \forall u \in M \} \) is also a closed linear subspace of \( C^1[a, b] \), where \( J(u, v) = a_2(vu' - uv') + (a_1 - a_2')uv \).

iii) Suppose that \( \alpha_{i1} = \beta_{i1} = 1 \) and all other coefficients \( \alpha_{ij}, \beta_{ij} \) are zero, so that \( B_1u = u(a), B_2u = u(b) \). Show that \( \{ \sin(k \pi(x - a)/(b - a)) \}_{k=1}^N \) is a linearly independent set of functions in \( M \) for any \( N \) (and hence conclude \( M \) is an infinite-dimensional subspace of \( C^1[a, b] \)).

23. Let \( X, \| \cdot \| \) be a normed linear space. Show that:

1. \( \| \cdot \| : X \to [0, \infty) \) is a Lipschitz function on \( X \) with constant \( \rho = 1 \).

2. addition is a continuous operation; that is \( u_n + v_n \to u + v \), as \( n \to \infty \), whenever \( u_n \to u \) and \( v_n \to v \), as \( n \to \infty \).

3. scalar multiplication is a continuous operation; that is \( s_n u_n \to su \), as \( n \to \infty \), whenever \( s_n \to s \) and \( u_n \to u \), as \( n \to \infty \).

24. Let \( x \in \mathbb{R}^n \) and define \( ||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \), for \( p \geq 1 \). Hölder’s inequality states that if \( p, q \in (1, \infty) \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \) then

\[
\sum_{i=1}^{n} |x_i y_i| \leq ||x||_p ||y||_q, \quad \forall x, y \in \mathbb{R}^n.
\]
Verify Hölder’s inequality. Hint: See problem 3.7 on page 264.

25. The function \( \| \cdot \|_p \) is a norm on \( \mathbb{R}^n \) for all \( p \in [1, \infty] \). This is easy to show if \( p = 1 \) or \( \infty \). Show that \( \| \cdot \|_p \) is a norm on \( \mathbb{R}^n \) if \( 1 < p < \infty \).

Hint: The triangle inequality in this case is Minkowski’s inequality
\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p, \quad x, y \in \mathbb{R}^n.
\]
Verify Minkowski’s inequality by applying Hölder’s inequality to the terms on the right of the inequality
\[
\sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i + y_i|^{p-1}|x_i| + \sum_{i=1}^{n} |x_i + y_i|^{p-1}|y_i|.
\]

26. Use the contraction mapping theorem to prove the following.

The Implicit Function Theorem Let \( f(x, y) \) and \( f_y(x, y) \) be continuous functions on the rectangle \( \mathcal{R} = (a, b) \times (c, d) \subset \mathbb{R}^2 \) containing the point \((x_0, y_0)\). If \( f(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) \neq 0 \) then there are positive numbers \( \alpha, \beta > 0 \) determining subintervals \([x_0 - \alpha, x_0 + \alpha] \subset (a, b), [y_0 - \beta, y_0 + \beta] \subset (c, d)\) such that there is a unique continuous function \( y = \psi(x) \), defined on \([x_0 - \alpha, x_0 + \alpha]\), satisfying \( \psi(x_0) = y_0 \) and \( f(x, \psi(x)) = 0 \), \( x \in [x_0 - \alpha, x_0 + \alpha] \).

Hint: Problem 4.12 on page 283 treats a simpler version of the implicit function theorem. For the version considered here you need to consider a subset \( B = \{y \in C[x_0 - \alpha, x_0 + \alpha] : |y(x) - y_0| \leq \beta, x \in [x_0 - \alpha, x_0 + \alpha]\} \) of the Banach space \( C[x_0 - \alpha, x_0 + \alpha] \). See also problem 4.9 on page 282.

27. Let \([a, b] \subset \mathbb{R}\) be a bounded interval and \( f : [a, b] \to [a, b] \) be a continuous function satisfying \( |f(x) - f(y)| < |x - y| \) for all \( x, y \in [a, b] \) with \( x \neq y \). Show that \( f \) has a unique fixed point in \([a, b]\). Hint: See problem 4.2 on page 280.

28. Show that the function \( f(x) = \ln(1 + e^x) \) satisfies \( |f(x) - f(y)| < |x - y| \) for all \( x \neq y \). Does the previous result apply?

29. The set of points in the plane \( \mathbb{R}^2 \) is a Hilbert space when the Euclidean norm \( \|x\| = \sqrt{x_1^2 + x_2^2} \) is used, which we assume in the exercise. For \( m \in \mathbb{R} \) define \( M = \{x \in \mathbb{R}^2 : x_2 = mx_1\} \). It is easy to see that \( M \) is a closed subspace of \( \mathbb{R}^2 \). Define a mapping \( Q : \mathbb{R}^2 \to M \) by \( Qx = (x_1, mx_1) \).

1. Show that \( Q \) is a linear operator.

2. Show that \( Q \) is a projection operator by showing that it is idempotent (i.e. \( Q^2 = Q \)).

3. Show that \( Q \) is bounded and determine the norm \( \|Q\| \) of \( Q \).

4. Let \( P \) denote the orthogonal projection operator onto \( M \). Determine an algebraic rule for \( P \) (analogous to the one for \( Q \)).

5. Show that \( P = Q \) if and only if \( m = 0 \).
30. Let \((a, b) \subset \mathbb{R}\) be a bounded interval. A function \(u(x)\) defined on \((a, b)\) is said to be Hölder continuous on \((a, b)\) with exponent \(\alpha \in (0, 1]\) if there is a constant \(C\) such that
\[
|u(x) - u(y)| \leq C|x - y|^{\alpha}, \quad \forall x, y \in (a, b).
\] (2)
If \(\alpha = 1\) this is the same as Lipschitz continuity. Let \(C^{0,\alpha}(a, b) = \{u \in C(a, b) : u\) satisfies (2) \} denote the set of functions that are Hölder continuous on \((a, b)\) with exponent \(\alpha\). If \(u \in C^{0,\alpha}(a, b)\) then the number
\[
[u]_{0,\alpha} = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} : x, y \in (a, b), \ x \neq y \right\}
\]
is well-defined.

1. Let \(u(x) = \sqrt{x}\). Show that \(u \in C^{0,\frac{1}{2}}(0, 1)\) but \(u \notin C^{0,1}(0, 1)\).

2. Show that if \(0 \leq \alpha < \beta \leq 1\) then \(C^{0,\beta}(a, b) \subset C^{0,\alpha}(a, b)\).

3. Show that \(C^{0,\alpha}(a, b)\) is a linear subspace of \(C(a, b)\).

4. Show that \(C^{0,\alpha}(a, b)\) is a Banach space with the norm
\[
\|u\|_{0,\alpha} = \|u\|_{\infty} + [u]_{0,\alpha}.
\]

5. Let \(0 \leq \alpha < \beta \leq 1\). Then the inclusion mapping \(i : C^{0,\beta}(a, b) \rightarrow C^{0,\alpha}(a, b)\) is a linear operator. Show that it is an imbedding by showing that it is bounded .

31. Let \(a \in (0, 1)\) and define \(\delta_a\) to be the distribution \(\delta_a(\phi) = \phi(a), \phi \in C_c^\infty(0, 1)\). It can be shown that \(H^1_0(0, 1) \subset C[0, 1]\) is an imbedding. That is, the inclusion map is a well-defined bounded linear functional. Hence
\[
|\delta_a(u)| = |u(a)| \leq \|u\|_{\infty} \leq C|u|_{1,2}, \quad \forall u \in H^1_0(0, 1).
\] (3)
Recall that a norm and inner product on \(H^1_0(0, 1)\) is defined by
\[
<u, v>_{1,2} = \int_0^1 u'(x)v'(x) \, dx, \quad |u|_{1,2} = \sqrt{<u, u>_{1,2}}.
\]
The inequality (3) shows that \(\delta_a \in [H^1_0(0, 1)]^*\) is a continuous linear functional on \(H^1_0(0, 1)\). Therefore, according to the Riesz representation theorem there is a unique function \(v \in H^1_0(0, 1)\) such that \(\delta_a(u) = <u, v>_{1,2}\) for all \(u \in H^1_0(0, 1)\). Find \(v\).

32. Let \(X\) be a Banach space with norm \(\| \cdot \|\) and let \(B(X)\) denote the set of all bounded linear operators \(T : X \rightarrow X\). With vector addition and scalar multiplication defined by \((T_1 + T_2)(x) = T_1(x) + T_2(x)\) and \((sT)(x) = s(T(x))\) respectively, it is easy to verify that \(B(X)\) is a linear space.

1. Show that \(\|T\| = \sup \{\|Tx\|/\|x\| : x \in X, \ x \neq 0\}\) is a norm on \(B(X)\).

2. Show that \(B(X)\) is complete and therefore a Banach space.
3. Show that the subset $\mathcal{H}(X) = \{T \in B(X) : T \text{ has a bounded inverse on } X\}$ is an open subset of $B(X)$. (Hint: Use a Neumann series argument.)

33. Consider the operator $A$ in $L^2(-1,1)$ (square-integrable real-valued functions) defined by

$$(Au)(x) = -\frac{2}{3}u(x) + \int_{-1}^{1} y^2 u(y) \, dy.$$ 

Determine $\mathcal{N}(A)$, $\mathcal{R}(A)$, $A^*$, $\mathcal{N}(A^*)$, $\mathcal{R}(A^*)$. Does the identity $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$ hold in this case? When $f \in \mathcal{R}(A)$ find the minimum norm solution of $Au = f$. (This is essentially part (b) of problem 5.4 on page 346)

34. Let $Au = -u''$ be the closed densely-defined operator in $L^2(0,1)$ (square-integrable complex-valued functions) with domain $\mathcal{D}(A) = \{u \in H^2(0,1) : u(0) = 0, \: u(1) + u'(1) = 0\}$.

1. Show that all the eigenvalues of $A$ are real, and characterize them well enough to resolve the next two problems.

2. Show that the eigenvalues of $A$ are bounded from below but not from above.

3. Determine a fairly accurate estimate of the smallest eigenvalue.

35. Let $A_1$, $A_2$, $A_3$ denote the closed densely-defined unbounded linear operators in $L^2(0,1)$ (square-integrable complex-valued functions) defined by

$$A_1 u = i \frac{du}{dt}, \quad \mathcal{D}(A_1) = H^1(0,1)$$
$$A_2 u = i \frac{du}{dt}, \quad \mathcal{D}(A_2) = \{u \in H^1(0,1) : u(0) = u(1)\}$$
$$A_3 u = i \frac{du}{dt}, \quad \mathcal{D}(A_3) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}$$

As usual $i = \sqrt{-1}$ is the imaginary unit. Determine the adjoint operators $A_1^*$, $A_2^*$, $A_3^*$. Are any of these operators symmetric or self-adjoint?

36. Consider the integral operator $K : L^2(0,1) \to L^2(0,1)$ defined by

$$(Ku)(x) = \int_{0}^{1} (1 + xy) u(y) \, dy.$$ 

1. Find all the non-zero eigenvalues of $K$ and the corresponding eigenfunctions.

2. Determine $\mathcal{N}(K)$.

3. Discuss the solvability of the Fredholm equation $Ku - \lambda u = f$, for all $\lambda \in \mathbb{C}$. Include a description of all solutions and any compatibility conditions required of $f$. 

Hint: The kernel $k(x, y) = 1 + xy$ is separable.

37. Consider the integral operator $K : L^2(0, 1) \rightarrow L^2(0, 1)$ defined by

$$(Ku)(x) = \int_0^1 k(x, y) u(y) \, dy = \int_0^x u(y) \, dy$$

defined by the kernel $k(x, y) = 1$ if $x > y$ and $k(x, y) = 0$ otherwise.

1. Determine the adjoint operator $K^*$.

2. Determine the operators $R = KK^*$ and $L = K^*K$ and show that they are symmetric non-negative integral operators.

3. Show that the eigenvalue problem for $L$ is equivalent to a boundary value problem.

4. Find the “singular system” describe in exercise 3.2 on page 400 for the operator $K$

Hint: See exercises 3.2 and 3.4 on pages 400-401.

38. Assuming $\beta \in (0, 1)$, find the smallest eigenvalue of the regular Sturm-Liouville problem

$$\phi'' + \lambda \phi = 0, \quad 0 < x < 1,$$

$$\phi(0) = 0, \quad \phi(1) - \beta \phi'(1) = 0.$$ 

Using this example show that there are regular Sturm-Liouville problems having negative eigenvalues which are arbitrarily large in magnitude.

39. Consider the regular Sturm-Liouville problem

$$\frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + \left( q(x) + \lambda r(x) \right) \phi = 0, \quad a < x < b,$$

$$\phi(a) \cos \alpha - \phi'(a) \sin \alpha = 0, \quad \phi(b) \cos \beta + \phi'(b) \sin \beta = 0.$$ 

Show that if $\phi(x)$ and $\psi(x)$ are eigenfunctions corresponding to the same eigenvalue $\lambda$ then $\psi(x) = C\phi(x)$ for some constant $C$. (This shows that the eigenvalues of a regular Sturm-Liouville problem with unmixed boundary conditions are simple.)

Hint: Consider $z(x) = \phi'(a)\psi(x) - \psi'(a)\phi(x)$

40. Find the eigenvalues and eigenfunctions of the problem

$$\phi'' + \lambda \phi = 0, \quad 0 < x < 2\pi,$$

$$\phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi).$$

(This problem provides an example showing that if the boundary conditions are of mixed type then the eigenvalues may have geometric multiplicity two.)
41. Let \( B_a = \{ x \in \mathbb{R}^3 : \| x \|_2 < a \} \) denote the ball of radius \( a \) centered at the origin, and consider the initial-boundary value problem

\[
\begin{align*}
&u_t = \Delta u, \quad (t, x) \in (0, \infty) \times B_a \\
&u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial B_a \\
&u(0, x) = u_0(x), \quad x \in B_a.
\end{align*}
\]

If \( u_0 \) is radially symmetric (i.e. \( u_0(x) = f(\rho) \), where \( \rho = \| x \|_2 \)) then the solution \( u \) will also be radially symmetric (i.e \( u = u(t, \rho) \)). In this case one should use spherical coordinates. Since for a radially symmetric function \( u \) one has

\[
\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial u}{\partial \rho}
\]

the PDE becomes \( u_t = u_{\rho\rho} + \frac{2}{\rho} u_\rho \). Use separation of variables to find the radially symmetric solution when \( u_0 = f(\rho) \).

Hint: You will need to change variables \( \eta(\rho) = \rho \phi(\rho) \) in the eigenvalue problem to arrive at a more familiar problem.

42. Find the solution of the Cauchy problem

\[
yu_x + xu_y = x^2 + y^2, \quad u(x, 0) = \ln x, \quad x > 0,
\]

that is valid in the half-plane \( \{(x, y) \in \mathbb{R}^2 : x > 0\} \). Can the solution be extended to a larger region?

Hint: An easy way to solve the system \( x' = -y, \ y' = x \) is to compute \( x'' \) in terms of \( x \), solve for \( x \) and then compute \( y \).

43. Find a product solution \( u = \phi(x)\psi(y) \) of the Cauchy problem

\[
u_{xx} + u_{yy} = 0, \quad (x, y) \in \{(x, y) \in \mathbb{R}^2 : x > 0\}
\]

\[
u(0, y) = 0, \quad u_x(0, y) = \delta \sin(\alpha y).
\]

Clearly the data for this problem satisfies \( \| u(0, y) \|_{C(\mathbb{R})} + \| u_x(0, y) \|_{C(\mathbb{R})} \leq \delta \), for any \( \alpha \in \mathbb{R} \), where the norms are the usual maximum norms. Show that for any point \( (x_0, y_0) \), with \( x_0 > 0 \) and \( y_0 \neq 0 \), there is sequence \( \{\alpha_n\} \) that determines a sequence of initial data such that the corresponding solutions \( \{u_n(x, y)\} \) satisfy

\[
\lim_{n \to \infty} |u_n(x_0, y_0)| = \infty.
\]

This shows that the Cauchy problem for Laplace’s equation is an ill-posed problem, since solutions do not depend continuously on the data.

44. Let \( \mathcal{R} = (0, a) \times (0, b) \), where \( a/b \) is a rational number. Show that there is a non-trivial solution of the boundary value problem for the wave equation

\[
u_{xx} - u_{yy} = 0, \quad (x, y) \in \mathcal{R} \quad u(x, y) = 0, \quad (x, y) \in \partial \mathcal{R}.
\]

Thus the boundary value problem for the wave equation is an ill-posed problem since solutions are not unique.

45. A classic example of a PDE that changes type is Tricomi’s equation \( yu_{xx} + u_{yy} = 0 \). Determine the points in the \( xy \)-plane at which Tricomi’s equation is hyperbolic, parabolic and elliptic. In the hyperbolic region change to characteristic coordinates to obtain the canonical form.