49. Let \( M = \{0, 1, 2, \ldots, 9\} \), and consider the discrete dynamical system defined by the map 
\[
\phi(k) = k^2 + 1 \mod 10
\]
Find all fixed points, periodic points and periodic orbits. Answer the same question if \( \phi(k) = k^2 + 4 \mod 10 \).

50. Let \( M = [0, 1] \) and consider the discrete dynamical system defined by the map 
\[
\phi(x) = 10x \mod 1
\]
   a) This map is called the decimal shift map. Why is this terminology appropriate? (Suggestion: think about how the decimal representation of \( \phi(x) \) relates to that of \( x \).)
   b) Find all fixed points and show they are unstable. (Don’t try to linearize here, it is easier than that.)
   c) Show that there exist periodic orbits of every period.

51. Let \( \Phi = \Phi(t, x) \) be a flow on \([0, \infty) \times M \) where \( M \subset \mathbb{R}^n \). A point \( x \in M \) is called nonwandering if for any open set \( \mathcal{O} \) containing \( x \), there is a sequence of times \( t_n \to +\infty \) such that \( \Phi(t_n, \mathcal{O}) \cap \mathcal{O} \neq \emptyset \).
   a) Show that that the set of nonwandering points is a closed invariant set.
   b) Find the set of nonwandering points for the system \( x' = y, y' = -x \).

52. The Hermite equation of order \( \alpha \) is 
\[
u'' - 2tu' + 2\alpha u = 0
\]
   a) Derive the two power series solutions
\[
\begin{align*}
u_1(t) &= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^k \alpha(\alpha - 2) \cdots (\alpha - 2k + 2)}{(2k)!} t^{2k} \\
u_2(t) &= t + \sum_{k=1}^{\infty} (-1)^k \frac{2^k(\alpha - 1)(\alpha - 3) \cdots (\alpha - 2k + 1)}{(2k + 1)!} t^{2k+1}
\end{align*}
\]
   b) If \( \alpha = n \) is a nonnegative integer, show that there exists a solution which is a polynomial of degree \( n \), unique up to a constant multiple. (When normalized so that the coefficient of \( t^n \) is \( 2^n \), this is the \( n \)’th Hermite polynomial, denoted \( H_n(t) \).)

53. Prove that \( J_0 \), the Bessel function of order zero, satisfies 
\[
J_0(t) = \frac{1}{\pi} \int_0^\pi \cos (t \sin \theta) \, d\theta
\]
Do this by showing that the right hand side satisfies Bessel’s equation of order zero and that it has the right behavior as \( t \to 0 \).