Explain your answers carefully!

1. (25 points) Define the translation operator $Tu(x) = u(x-1)$ on $L^2(\mathbb{R})$.
   (a) Find $T^*$.
   (b) Show that $T$ is unitary.
   (c) Show that $\sigma_p(T) = \emptyset$.

**Solution:** By simple change of variable

$$< Tu, v >= \int_{-\infty}^{\infty} u(x)\overline{v(x+1)}\, dx$$

so $T^*u(x) = u(x+1)$ and $TT^*u = T^*Tu = u$. We know that $\lambda \in \sigma(T)$ implies $|\lambda| = 1$ for a unitary operator (and the proof of this is especially simple for $\lambda \in \sigma_p(T)$). Here are two ways to show that $T$ has no eigenvalues:

i) If $Tu = \lambda u$ then $|u(x-1)| = |u(x)|$, i.e. $|u(x)|$ is periodic with period 1. But no nonzero periodic function can belong to $L^2(\mathbb{R})$ since the $L^2$ norm could be written as $\sum_{n=-\infty}^{\infty} \int_n^{n+1} |u(x)|^2\, dx$ and all of the terms are equal to the same nonzero constant.

ii) If $u(x-1) = \lambda u(x)$ take the Fourier transform of both sides to get $(e^{-iy} - \lambda)\hat{u}(y) = 0$. Since $\{y : e^{-iy} = \lambda\}$ has measure zero we conclude that $\hat{u}(y) = 0$ a.e., so $u = 0$.

2. (20 points) Let

$$Tu(x) = \frac{1}{x}\int_0^x u(y)\, dy \qquad u \in L^2(0,1)$$

Show that $(0,2) \subset \sigma_p(T)$ and that $T$ is not compact. (Suggestion: look for eigenfunctions in the form $u(x) = x^\alpha$.)

**Solution:** If we substitute $u = x^\alpha$ into $Tu = \lambda u$ we get $\lambda = \frac{1}{\alpha+1}$ provided that $\alpha > -1$. In order that $u$ be an eigenfunction we need

$u \in L^2(0,1)$ which happens if $\alpha > -\frac{1}{2}$, which corresponds to $0 < \lambda < 2$, as needed. Since $T$ has uncountably many nonzero eigenvalues it cannot be compact.

3. (15 points) Let $T$ be compact, and $Q(x) =< Tx, x >$. If $x_n \stackrel{w}{\to} x$ show that $Q(x_n) \to Q(x)$. (This property of $Q$ is called *weak continuity*.)

**Solution:** We can write

$$Q(x_n) - Q(x) =< Tx_n - Tx, x_n > + < Tx, x_n - x >$$

Weak convergence of $x_n$ to $x$ immediately implies that the second term on the right tends to zero. Since $T$ is compact $Tx_n \to Tx$, and the weakly convergent sequence $x_n$ must be bounded, $||x_n|| \le M$, so

$$| < Tx_n - T_x, x_n > | \le M||Tx_n - Tx|| \to 0$$

as needed.

4. (10 points) If $T$ is self-adjoint, $S$ is symmetric and $T \subset S$, show that $T = S$. (Thus a self-adjoint operator has no proper symmetric extension).

**Solution:** $S$ symmetric implies $S \subset S^*$ and $T \subset S$ implies $S^* \subset T^*$. Thus

$$T \subset S \subset S^* \subset T^* = T$$

so all 4 operators coincide.

5. (30 points) Consider the integral equation

$$\int_0^1 x^2 y^3 u(y)\, dy - \lambda u(x) = f(x)$$

(a) For what values of $\lambda \in \mathbb{C}$ does there exist a unique solution for every $f \in L^2(0,1)$?

(b) For the remaining values of $\lambda$ find the solvability conditions which $f$ must satisfy in order for a solution to exist.

**Solution:** If $\lambda \neq 0$ a unique solution exists for every $f \in L^2$ provided $\lambda$ is not an eigenvalue of $T$. To find such eigenvalues we substitute $u = Cx^2$ and find that $\lambda = \frac{1}{6}$ is the only one. Since the operator is compact either existence or uniqueness or both will fail for $\lambda = 0$ also. (Or you can observe that 0 is always an eigenvalue when the kernel is degenerate.)

The solvability condition for $\lambda = \frac{1}{6}$ is that $f$ be orthogonal to the null space of $T^*$ which is the integral operator with kernel $y^2x^3$, and so we obtain the condition $f \perp x^3$. When $\lambda = 0$ we see by direct inspection that a solution will exist if and only if $f = Cx^2$ for some $C$. Alternatively in this case, we know that $R(T)$ is closed, since it is finite dimensional, $N(T^*) = \{x^2\}^\perp$ by direct inspection, hence $R(T) = N(T^*)^\perp = \text{span}\,\{x^2\}$, leading to the same conclusion.