31. If $T \in \mathcal{B}(\mathcal{H})$ is compact and $\mathcal{H}$ is of infinite dimension, show that $0 \in \sigma(T)$.

32. Let $\{\phi_j\}_{j=1}^n, \{\psi_j\}_{j=1}^n$ be linearly independent sets in $L^2(\Omega)$,

$$K(x, y) = \sum_{j=1}^n \phi_j(x) \psi_j(y)$$

be the corresponding degenerate kernel and $T$ be the corresponding integral operator. Show that the problem of finding the nonzero eigenvalues of $T$ always amounts to a matrix eigenvalue problem. In particular, show that $T$ has at most $n$ nonzero eigenvalues. Find $\sigma_p(T)$ in the case that $K(x, y) = 6 + 12xy + 60x^2y^3$ and $\Omega = (0, 1)$. (Feel free to use Matlab or some such thing to solve the resulting matrix eigenvalue problem.)

33. Let

$$Tu(x) = \frac{1}{x} \int_0^x u(y) \, dy \quad u \in L^2(0, 1)$$

Show that $(0, 2) \subset \sigma_p(T)$ and that $T$ is not compact. (Suggestion: look for eigenfunctions in the form $u(x) = x^\alpha$.)

34. Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of nonzero real numbers satisfying

$$\sum_{j=1}^\infty \lambda_j^2 < \infty$$

Construct a symmetric Hilbert-Schmidt kernel $K$ such that the corresponding integral operator has eigenvalues $\lambda_j, j = 1, 2 \ldots$ and for which $0$ is an eigenvalue of infinite multiplicity. (Suggestion: look for such a $K$ in the form $K(x, y) = \sum_{j=1}^\infty \lambda_j u_j(x) u_j(y)$ where $\{u_j\}$ are orthonormal, but not complete, in $L^2(\Omega)$.)

35. Let $T$ be the integral operator with kernel $K(x, y) = e^{-|x-y|}$ on $L^2(-1, 1)$. Find all of the eigenvalues and eigenfunctions of $T$. (Suggestion: $Tu = \lambda u$ is equivalent to an ODE problem. Don’t forget about boundary conditions. The eigenvalues may need to be characterized in terms of the roots of a certain nonlinear function.)