MATH 520 Homework Spring 2014

6. (Problem 8.15 in text) If $H$ is a Hilbert space and $S, T \in \mathcal{B}(H)$, show that
   (i) $(S + T)^* = S^* + T^*$
   (ii) $(ST)^* = T^*S^*$
   (These properties, together with (iii) $(\lambda T)^* = \bar{\lambda}T^*$ for scalars $\lambda$ and (iv) $T^{**} = T$, which we have already proved, are the axioms for an involution on $\mathcal{B}(H)$, that is to say the mapping $T \mapsto T^*$ is an involution. The term involution is also used more generally to refer to any mapping which is its own inverse.)

7. Let $S_+, S_-$ be the left and right shift operators on $\ell^2$. Show that $S_+ = S_-^*$ and $S_- = S_+^*$.  

8. Let $T$ be the Volterra integral operator $Tu(x) = \int_0^x u(y) \, dy$, considered as an operator on $L^2(0,1)$. Find $T^*$ and $N(T^*)$. Is $R(T)$ closed?

9. (Problem 8.12 in text) Suppose $T \in \mathcal{B}(H)$ is self-adjoint and there exists a constant $c > 0$ such that $\|Tu\| \geq c\|u\|$ for all $u \in H$. Show that there exists a solution of $Tu = f$ for all $f \in H$. Show by example that the conclusion may be false if the assumption of self-adjointness is removed.

10. Let $M$ be the multiplication operator $Mu(x) = xu(x)$ in $L^2(0,1)$. Show that $R(M)$ is dense but not closed.