46. If \((a, b)\) is any bounded interval in \(\mathbb{R}\), prove that \(W^{1,p}(a, b)\) is compactly embedded in \(L^p(a, b)\) for any \(1 < p \leq \infty\), that is to say, any bounded set in \(W^{1,p}(a, b)\) is precompact in \(L^p(a, b)\). (Suggestion: Use the Arzela-Ascoli theorem.)

47. When we rewrite

\[ a_2(x)u'' + a_1(x)u' + a_0(x)u = \lambda u \]

the latter is often referred to as the Liouville normal form. Consider the eigenvalue problem

\[ x^2u'' + xu' + u = \lambda u \quad 1 < x < 2 \]

\[ u(1) = u(2) = 0 \]

a) Find the Liouville normal form.

b) What is the orthogonality relationship satisfied by the eigenfunctions?

c) Find the eigenvalues and eigenfunctions. (You may find the original form of the equation easier to work with than the Liouville normal form when computing the eigenvalues and eigenfunctions.)

48. Consider the Sturm-Liouville equation in the Liouville normal form,

\[ -(p(x)u')' + q(x)u = \lambda \rho(x)u \quad a < x < b \]

where \(p, \rho \in C^2([a, b]), q \in C([a, b]), p, \rho > 0 \) on \([a, b]\). Let

\[ \sigma(x) = \frac{\sqrt{\rho(x)}}{p(x)} \]

\[ \eta(x) = (p(x)\rho(x))^{1/4} \]

\[ L = \int_a^b \sigma(s) ds \]

\[ \phi(x) = \frac{1}{L} \int_a^x \sigma(s) ds \]

If \(\psi = \phi^{-1}\) (the inverse function of \(\phi\)) and \(v(z) = \eta(\psi(z))u(\psi(z))\) show that \(v\) satisfies

\[ -v'' + Q(z)v = \mu v \quad 0 < z < 1 \quad (*) \]

for some \(Q\) depending on \(p, \rho, q, \) and \(\mu = L^2\lambda\). (This is mainly a fairly tedious exercise with the chain rule. Focus on making the derivation as clean as possible and be sure to say exactly what \(Q(z)\) is. The point of this is that every eigenvalue problem for a second order ODE is equivalent to one with an equation of the form (\(*)\), provided that the coefficients have sufficient smoothness. The map \(u(x) \to v(z)\) is sometimes called the Liouville transformation, and the ODE (\(*)\) is the canonical form for a 2nd order ODE eigenvalue problem.)

49. Using separation of variables, compute explicitly the Dirichlet eigenvalues and eigenfunctions of \(-\Delta\) when the domain is a rectangle \((0, A) \times (0, B)\) in \(\mathbb{R}^2\). Verify directly that the first eigenvalue is simple, and that the first eigenfunction is of constant sign. Can there be other eigenvalues of multiplicity greater than one? (Hint: Your answer should depend on whether the ratio \(A/B\) is rational or irrational).

50. If \(\{\psi_n\}_{n=1}^\infty\) are eigenfunctions of the Laplacian making up an orthonormal basis of \(L^2(\Omega)\), let \(\zeta_n = \psi_n/\sqrt{\lambda_n}\) (\(\lambda_n\) the corresponding eigenvalue). Show that \(\{\zeta_n\}_{n=1}^\infty\) is an orthonormal basis of \(H^1_0(\Omega)\).