Explain your answers carefully!

Unless otherwise stated, $(X, \mathcal{B}, \mu)$ denotes a measure space with a positive measure μ.

1. Show that μ is σ-finite if and only if there exists $f \in L^1(\mu)$ with $f(x) > 0$ for every $x \in X$.

> **Solution:** If μ is σ-finite then we can find a countable collection X_n of disjoint measurable sets of finite measure such that $\cup_{n=1}^{\infty} X_n = X$. If $f(x) = \frac{1}{2^n \mu(X_n)}$ for $x \in X_n$, then $f(x) > 0$ for every $x \in X$ and $\int_X f(x)\,dx = \sum_{n=1}^{\infty} 2^{-n} < \infty$.
> Conversely if $f \in L^1(X)$ and $f > 0$ on X, let
> $$X_n = \{x : f(x) > \frac{1}{n}\}$$
> Then $X = \cup_{n=1}^{\infty} X_n$ and each X_n has finite measure since
> $$\int_X f(x)\,dx \geq \int_{X_n} f(x)\,dx \geq \frac{\mu(X_n)}{n}$$

2. Let μ be a finite positive measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$, and suppose that there exists M such that
$$\int_{\mathbb{R}^n} |f|\,d\mu \leq M \int_{\mathbb{R}^n} |f|\,dx$$
for any measurable function f. Show that $\mu << m$ and $\left|\frac{d\mu}{dm}\right| \leq M$ a.e.

> **Solution:** We may choose $f = \chi_E$ for any measurable set E to get
> $$\mu(E) \leq Mm(E)$$

In particular $m(E) = 0$ implies that $\mu(E) = 0$, so $\mu << m$. By the Radon-Nikodym Theorem there exist $h = \frac{d\mu}{dm} \in L^1(\mathbb{R}^n)$ such that

$$\mu(E) = \int_E h\,dx$$

for any measurable set E. If there exists a set of positive measure A and $\epsilon > 0$ such that $h \geq M + \epsilon$ on A, then we'd have $\mu(A) \geq (M + \epsilon)m(A)$, contradicting the above inequality. Thus $h \leq M$ a.e. and likewise $h \geq -M$ a.e., as needed.

The second part of the problem could also be done using the differentiation theory we discussed: Choosing $E = B(x, r)$ we get

$$\frac{\mu(B(x,r))}{m(B(x,r))} \leq M$$

for any x, r, so that the symmetric derivative $D\mu$ must satisfy $|D\mu(x)| \leq M$ for all x where it is defined. Since $D\mu = \frac{d\mu}{dm}$ a.e. we get the second conclusion.

3. A topological space (X, τ) is said to be *perfectly normal* if for any closed set $F \subset X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $F = f^{-1}(\{0\})$. Show that any perfectly normal space is normal. (Suggestion: If E, F are disjoint closed sets in X, find a continuous function f such that $f = 0$ on E, $f = 1$ on F.)

Solution: Let E, F be disjoint closed sets in X. By assumption we can find continuous functions e, f such that $E = e^{-1}(\{0\})$ and $F = f^{-1}(\{0\})$. Note that $e^2 + f^2 > 0$ on X since

$$e^2(x) + f^2(x) = 0 \Longrightarrow e(x) = f(x) = 0 \Longrightarrow x \in E \cap F = \emptyset$$

Thus $h = e^2/(e^2 + f^2)$ is continuous on X, $h = 0$ on E, $h = 1$ on F. It therefore follows that the two sets $\{h < 1/2\}, \{h > 1/2\}$ are disjoint open sets containing E, F respectively, so X is normal.

The statement of the problem should have included the assumption that the space is T_1. Note that any metric space is perfectly normal, since $f(x) = \rho(x, F)$ has the property required in the definition. The last part of the argument above shows that any T_1 space for which the conclusion of Urysohn's lemma is valid, must be normal.

4. Suppose that f is a measurable function on X such that

$$\int_X f^2\,d\mu = \int_X f^3\,d\mu = \int_X f^4\,d\mu = C$$

for some constant C. Show that $f = \chi_E$ for some measurable set $E \subset X$. (Suggestion: consider $\int_X f^2(1-f)^2\,d\mu$.)

Solution: We get

$$\int_X f^2(1-f)^2\,d\mu = \int_X (f^2 - 2f^3 + f^4)\,d\mu = 0$$

Since the integrand is nonnegative we must have $f^2(1-f)^2 = 0$ a.e[μ]. If we let $E = \{x : f(x) = 1\}$ then E is measurable, $f = 0$ a.e. on E^c and so $f = \chi_E$ a.e.

5. Let X be an uncountable set, and let $\mathcal{B}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable. Define $\mu(E) = 0$ if E is at most countable and $\mu(E) = 1$ if E^c is at most countable. Show that $(X, \mathcal{B}, \mu)$ is a measure space.

Solution: Clearly $\emptyset \in E$ since it is finite. If $E \in \mathcal{B}$ then E or E^c is at most countable, so $E^c, (E^c)^c = E$ is at most countable, so $E^c \in \mathcal{B}$. Suppose $E = \cup_{n=1}^{\infty} E_n$ where each $E_n \in \mathcal{B}$. If every E_n is countable then so is E. If there is some n for which E_n^c is countable then $E^c \subset E_n^c$ is countable. So in all cases $E \in \mathcal{B}$. Thus $\mathcal{B}$ is a σ-algebra.

It remains to show that μ is a measure. Clearly $\mu(\emptyset) = 0$. If E is a disjoint union then at most one of the sets E_n has a countable complement, since if E_j, E_k both have countable complements then each must itself be uncountable, and in particular E_j^c is uncountable since it contains E_k. Thus E_j is not measurable. If E_n is countable for all n then so is E, in which case $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = 0$, while if there exists one E_n with a countable complement then $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = 1$. Thus countable additivity holds in all cases.