Explain your answers carefully!

1. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$, and let $f : X \to (0, \infty)$ be measurable. Show that

$$\left(\int_X f\,d\mu\right)^2 \left(\int_X \frac{1}{f^2}\,d\mu\right) \ge 1$$

(Suggestion: apply Hölder's inequality to $f^\alpha f^{-\alpha}$ for a good choice of α.)

Solution: We have $1 = f^\alpha f^{-\alpha}$ so by Hölder's inequality

$$1 = \int_X 1\,d\mu \le \left(\int_X f^{\alpha p}\,d\mu\right)^{1/p} \left(\int_X f^{-\alpha q}\,d\mu\right)^{1/q}$$

for any conjugate exponents p, q. Choose $\alpha = \frac{2}{3}$, $p = \frac{3}{2}$ and $q = 3$ and then cube both sides to get the stated inequality.

2. (a) Let X be an uncountable set, $\mu^*(E) = 0$ if E is countable, and $\mu^*(E) = 1$ if E is uncountable. Show that μ^* is an outer measure on X.

   (b) What are the μ^* measurable sets?

Solution: a) Obviously $\mu^*(\emptyset) = 0$, and if $A \subset B$ and $\mu^*(B) < \mu^*(A)$ then $\mu^*(A) = 1, \mu^*(B) = 0$, so A is an uncountable subset of a countable set B, which is impossible. Thus $\mu^*(A) \le \mu^*(B)$ must hold. Finally, suppose $A = \cup_{j=1}^\infty A_j$. If A is countable then $\mu^*(A) = 0$ while if A is uncountable then at least one of the sets A_j is also uncountable, and either way we must have

$$\mu^*(A) \le \sum_{j=1}^\infty \mu^*(A_j)$$

b) If E is μ^* measurable then

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for any $A \subset X$. In particular, choosing $A = X$ we must have $\mu^*(E) + \mu^*(E^c) = 1$ so that one of E, E^c must be countable. Thus a measurable set E must either be countable or have a countable complement. The converse statement that any such E is μ^* measurable is simple to check.

3. (a) Show that for any $\epsilon > 0$ there exists a closed set $F_\epsilon \subset [0, 1]$ which is nowhere dense and $m(F_\epsilon) \ge 1 - \epsilon$. (Suggestion: consider sets of the form $\cup_{n=1}^\infty B(r_n, \frac{\epsilon}{2^n})$, where r_n is some ordering of the rational numbers in $(0, 1)$.)

(b) Show that there exists a subset of $[0, 1]$ which is of first category but has Lebesgue measure 1.

Solution: a) Let $\{r_n\}_{n=1}^{\infty}$ be an ordering of the rationals in $(0, 1)$. Then

$$G_\epsilon = \cup_{n=1}^{\infty} B(r_n, \frac{\epsilon}{2^{n+1}})$$

is an open set with

$$m(G_\epsilon \cap (0, 1)) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

Thus $F_\epsilon = [0, 1] \cap G_\epsilon^c$ is a closed subset of $[0, 1]$ with measure $m(F_\epsilon) \geq 1 - \epsilon$, and is nowhere dense because any open set contains a rational number, which is in F_ϵ^c.

b) If we choose $\epsilon_n \to 0$ and $F = \cup_{n=1}^{\infty} F_{\epsilon_n}$ then F is a countable union of nowhere dense sets, so of first category. It is clearly measurable (a Borel set even) and $1 \geq m(F) \geq m(F_{\epsilon_n}) \geq 1 - \epsilon_n$, so $m(F) = 1$ must hold.

4. Let $T : X \to Y$ be a bounded linear operator and suppose that T^{-1} exists. Show that T^{-1} is closed.

Solution: To show that T^{-1} is closed, it is enough to show that if $y_n \to y$ in Y and $T^{-1}(y_n) \to x$ in X then $T^{-1}(y) = x$. Set $x_n = T^{-1}(y_n)$. We then have that $x_n \to x$ in X and $T(x_n) = y_n \to y$ in Y. Since we assume that T is bounded, we must have $T(x) = y$, that is, $x = T^{-1}(y)$ as needed. (The same conclusion would follow if we only assumed that T was closed.)

5. Define

$$h(x, y) = \begin{cases} 1 & x^2 y \in \mathbb{Q} \\ 0 & x^2 y \notin \mathbb{Q} \end{cases}$$

Evaluate the integral of h with respect to two dimensional Lebesgue measure over the unit square $[0, 1] \times [0, 1]$, and clearly explain your reasoning. You may assume that h is measurable. (Suggestion: show that for $x \neq 0$, $y \to h(x, y)$ is zero a.e.)

Solution: If $x = 0$ then clearly $h(x, y) = 1$ for all y. But for any $x \neq 0$ there is a one to one correspondence between the rational numbers in $[0, 1]$ and the points $y \in [0, 1]$ for which $x^2 y$ is rational, namely by the map $y \to x^2 y$. Thus

$h(x,y) = 0$ a.e. in y for all $x \neq 0$. Since $h \geq 0$ we may apply Tonelli's theorem to evaluate the integral

$$\iint_{[0,1]\times[0,1]} h = \int_0^1 \left(\int_0^1 h(x,y)\, dy \right) dx = 0$$

since the inner integral

$$\int_0^1 h(x,y)\, dy = 0 \quad \text{a.e. } x$$

Alternatively, you could show that $h = \chi_E$ where E is the union of the sets $E_r = \{(x,y) \in [0,1]\times[0,1] : y = r/x^2\}$ for $r \in \mathbb{Q} \cap [0,1]$, which is a countable union of sets of measure zero.

6. If $x_n \stackrel{w}{\to} x$ in a Hilbert space H, show that $||x|| \leq \liminf_{n\to\infty} ||x_n||$. (Suggestion: start with $0 \leq ||x_n - x||^2$.)

Solution: We have

$$0 \leq ||x_n - x||^2 = ||x||^2 + ||x_n||^2 - 2\text{Re} < x_n, x >$$

or

$$2\text{Re} < x_n, x > -||x||^2 \leq ||x_n||^2$$

Since $< x_n, x > \to < x, x > = ||x||^2$ the left side is convergent to $||x||^2$, so if we take the lim inf of both sides it follows that

$$||x||^2 \leq \liminf_{n\to\infty} ||x_n||^2$$

7. Let $\mathbf{X} = C[0,1]$, E be the subspace of polynomials of degree less than or equal to 1, and $I(f) = f'(1)$ for $f \in E$.
   (a) Show that I is a continuous linear functional on E with $||I|| = 2$. (If you can't derive the exact value for the norm, at least show that it is finite.)
   (b) Show that there exists a finite measure μ on $[0,1]$, with $||\mu|| = 2$ such that

$$\int_0^1 f(x)\, d\mu = f'(1)$$

   whenever $f \in E$.
   (c) Is μ a positive measure?

Solution: a) For any $f \in E$ we have $f(x) = mx + b$ for some constants m, b and clearly $|I(f)| = |m|$. On the other hand $||f|| = \max\{|f(0)|, |f(1)|\} = \max\{|b|, |m + b|\}$. If $|b| \geq \frac{|m|}{2}$ then $||f|| \geq |b| \geq \frac{|m|}{2}$ so $|I(f)| \leq 2||f||$, while if $|b| < \frac{|m|}{2}$ then $||f|| \geq |m + b| > \frac{|m|}{2}$, so again we get $|I(f)| \leq 2||f||$. Choosing $f = 1/2 - x$ we see that $||I|| = 2$.

b) By the Hahn-Banach Theorem there exists an extension of I to all of $C[0, 1]$ preserving the norm. By the Riesz Representation Theorem for $C([0, 1]$ the extended bounded linear functional on $C([0, 1]$ is represented by integration against a finite measure μ on $[0, 1]$, with $||\mu|| = ||I|| = 2$. In particular the integral of f must coincide with $I(f)$ on E.

c) The measure is not positive, since $I(f) = f'(1)$ may be negative when $f \geq 0$.