Explain your answers carefully! 10 points each.

1. If A, B are measurable sets of finite measure, $m(A) = m(B)$, and

$$A \subset E \subset B$$

show that E is measurable.

Solution: Since A, B are measurable, so is $A \backslash B$ and

$$m(B) = m(A) + m(B \backslash A)$$

so that $m(B \backslash A) = 0$. We have $E \backslash A \subset B \backslash A$ so by monotonicity $m^*(E \backslash A) \leq m^*(B \backslash A) = 0$ so that $E \backslash A$ is measurable. Finally it follows that $E = A \cup (E \backslash A)$ is measurable.

2. Let $f \in L^1(0, \infty)$ and

$$g(y) = \int_0^\infty f(x) \sin(xy)\, dx$$

Show that g is a bounded and continuous function of y on all of $\mathbb{R}$.

Solution: We have

$$|g(y)| \leq \int_0^\infty |f(x) \sin(xy)|\, dx \leq \int_0^\infty |f(x)|\, dx$$

so that g is bounded on $\mathbb{R}$. If $y_n \to y$, then

$$h_n(x) := f(x) \sin(xy_n) \to h(x) := f(x) \sin(xy) \quad \text{a.e.}$$

The Dominated Convergence Theorem may then be applied, using $|h_n(x)| \leq |f(x)|$, to show

$$g(y_n) = \int_0^\infty h_n(x)\, dx \to \int_0^\infty h(x)\, dx = g(y)$$

so that g is continuous at y.

3. If $f \in BV[a,b]$, show that $f^2 \in BV[a,b]$. Is the converse true?

Solution: *Solution 1:* If $f = g - h$ where g, h are monotone increasing on $[a,b]$ then $f^2 = G - H$ where $G = g^2 + h^2, H = 2gh$. If $g \in BV$ then it is bounded below by some constant C, and we can express $g = (g-C) + C$ and therefore $g^2 = (g-C)^2 + C^2 + 2C(g-C)$. Since $g - C \geq 0$ and monotone, it follows that $(g-C)^2$ is monotone, and so g^2 is either monotone or the difference of monotone functions. Similarly for h^2 and $2gh$, so that f^2 can also be expressed as a difference of monotone functions.

Solution 2: Since $f \in BV[a,b]$ there exists M such that $|f(x)| \leq M$ on $[a,b]$. If $\mathcal{P}$ is any partition of $[a,b]$ we then have

$$\sum_{k=1}^{n} |f^2(x_{k+1}) - f^2(x_k)| = \sum_{k=1}^{n} |f(x_{k+1}) - f(x_k)||f(x_{k+1}) + f(x_k)|$$

Therefore

$$V(f^2, \mathcal{P}) \leq 2MV(f, \mathcal{P}) \leq 2MV(f)$$

and so

$$V(f^2) \leq 2MV(f) < \infty$$

The converse is false, for example if $f(x) = -1$ for $x \in \mathbb{Q}$, $f(x) = 1$ otherwise, then $f \notin BV$ but f^2 is.

(Thanks to Andy for pointing out the error in the original solution.)

4. If $\int_A f\,dx = 0$ for every measurable set $A \subset E$, show that $f = 0$ a.e. on E.

Solution: Choose $A = \{x \in E : f(x) \geq 0\}$. Then A is measurable, $f \geq 0$ on A and $\int_A f(x)\,dx = 0$ from which it follows that $f = 0$ a.e. on A. (This was a homework problem). Similarly $f = 0$ a.e. on the set $\{x \in E : f(x) \leq 0\}$, and so $f = 0$ a.e. on E.

5. Prove the *Tchebychev inequality*: If $f \in L^1(E)$ and $\alpha > 0$ then

$$m\{x \in E : |f(x)| \geq \alpha\} \leq \frac{1}{\alpha} \int_E |f(x)|\, dx$$

Solution: Let $E_\alpha = \{x \in E : |f(x)| \geq \alpha\}$. Then

$$\int_E |f(x)|\, dx \geq \int_{E_\alpha} |f(x)|\, dx \geq \int_{E_\alpha} \alpha\, dx = \alpha m(E_\alpha)$$

which is equivalent to the given statement.

6. Suppose that $E_1, E_2, \ldots E_n$ are measurable subsets of $(0,1)$, and that every point of $(0,1)$ belongs to at least 3 of the sets E_j. Prove that at least one of the sets has Lebesgue measure larger than or equal to $\frac{3}{n}$. (Suggestion: look at the function $\sum_{j=1}^n \chi_{E_j}(x)$.)

Solution: The assumptions imply that

$$\sum_{j=1}^n \chi_{E_j}(x) \geq 3$$

for all x. Integrating over $(0,1)$ it follows that

$$\sum_{j=1}^n m(E_j) \geq 3$$

But if $m(E_j) < \frac{3}{n}$ for all n then we would have $\sum_{j=1}^n m(E_j) < 3$, a contradiction.