1. Consider a particle of mass $m$ and charge $q$ in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. At time $t = 0$ the particle is in its ground state ($n = 0$). An electric field is applied for a time $\tau$ so that the perturbation is

$$W(t) = \begin{cases} -qEx & \text{for } 0 \leq t \leq \tau, \\ 0 & \text{otherwise}, \end{cases}$$

where $E$ is the constant field strength.

(a) Use first-order time-dependent perturbation theory to calculate the probability of finding the particle in the $n = 1$ state at time $\tau$. Express any dependence on $\tau$ as a real trigonometric function rather than an exponential with a complex argument.

(b) Use first-order time-dependent perturbation theory to calculate the probability of finding the particle in the $n = 2$ state at time $\tau$.

(c) Use second order perturbation theory to calculate the probability of finding the particle in the $n = 2$ state at time $\tau$. How does this result compare to the answer in part (a)?

2. The results of first-order time-dependent perturbation theory are not valid for a sinusoidal perturbation when $t$ is large. In this problem, you will use the resonance approximation to get an exact solution that is valid for large time. The resonance condition $\omega \approx \omega_{fi}$ implies that only the two discrete states $|\phi_i\rangle$ and $|\phi_f\rangle$ are effectively coupled by $W(t)$ in Eq. B-11 on p. 1288 in Cohen-Tannoudji et al. (or the boxed equation on p. 14 in lecture note 21), resulting in two equations that involve only $b_i$ and $b_f$:

$$i\hbar\frac{db_f(t)}{dt} = e^{i\omega_{fi}t}\tilde{W}_{fi}(t)b_i(t)$$  \hspace{1cm} (1)

and

$$i\hbar\frac{db_i(t)}{dt} = e^{-i\omega_{fi}t}\tilde{W}_{if}(t)b_f(t).$$  \hspace{1cm} (2)

Also note that we can write the perturbing Hamiltonian as

$$W(t) = W \cos \omega t = W \left(\frac{e^{i\omega t}}{2} + \frac{e^{-i\omega t}}{2}\right).$$

Neglecting the $A_+$ term relative to the $A_-$ term in the resonance approximation is equivalent to dropping the $\frac{1}{2}e^{i\omega t}$ part of $\cos \omega t$. If you make this so-called rotating wave approximation at the beginning of the calculation, Eqs. 1 and 2 can be solved exactly for $b_i(t)$ and $b_f(t)$ with no need for perturbation theory and no assumption about the strength of the field. (Exactly the same arguments apply to $W(t) = W \sin \omega t$.)
(a) Solve Eqs. 1 and 2 in the rotating wave approximation for the initial conditions \( b_i(0) = 1 \) and \( b_f(0) = 0 \). Express your results \( b_i(t) \) and \( b_f(t) \) in terms of the Rabi flopping frequency,

\[
\omega_r \equiv \frac{1}{2} \sqrt{(\omega - \omega_{fi})^2 + \frac{|W_{if}|^2}{\hbar^2}}.
\]

(b) Determine the transition probability \( P_{fi}(t) \) and show that it never exceeds 1. Confirm that \( |b_i(t)|^2 + |b_f(t)|^2 = 1 \).

(c) Check that \( P_{fi}(t) \) reduces to the perturbation theory result (Eqs. C-11 and C-12 on p. 1294 in Cohen-Tannoudji et al. or top equation on p. 7 in lecture note 22) when the perturbation is "small", and state precisely what small means in this context, as a constraint on \( W \).

(d) At what time \( t \) does the transition probability \( P_{if} \) first become zero?

5. A system of hydrogen atoms in the ground state is contained between the plates of a parallel-plate capacitor. A voltage pulse is applied to the capacitor to produce a homogeneous, exponentially-decaying electric field:

\[
E = \begin{cases} 
0 & \text{for } t < 0 \\
E_0 \exp(-t/\tau) & \text{for } t > 0 
\end{cases}
\]

(a) Show that, after a long time, the fraction of atoms in the \( 2P \) state is, to first order in time-dependent perturbation theory,

\[
\frac{2^{15}}{3^{10}} \frac{a_0^2 e^2 E_0^2}{\hbar^2 (\omega^2 + 1/\tau^2)},
\]

where \( a_0 \) is the Bohr radius, and \( \hbar \omega \) is the energy difference between the \( 2P \) state and the ground state. (Hint: \( \langle 210 | z | 100 \rangle = a_0^{2^{15}/3} \))

(b) What is the fraction of atoms in the \( 2S \) state?

Due Friday, March 23, in class. Scores for late problem sets will be divided by 2.