

Physics 481: Quantum Mechanics II

Spring Semester 2006
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Lecture XI

General case: Now consider an arbitrary potential. Let the single-particle energies be E_1, E_2, E_3, \dots , with degeneracies d_1, d_2, d_3, \dots (i.e., there are d_n distinct one-particle states with the same single-particle energy E_n). Note that in three dimensions, the degeneracies can be quite large.

Suppose we put N particles (all with the same mass) into this potential. As before, we are interested in the possible configurations (N_1, N_2, N_3, \dots) , for which there are N_1 particles with energy E_1 , N_2 particles with energy E_2 , etc. We will want to ask two questions:

1. How many different ways can a particular configuration be achieved? Or, more precisely, how many distinct N -particle states correspond to this particular configuration? The answer $Q(N_1, N_2, N_3, \dots)$ depends on whether the particles are distinguishable, identical fermions or identical bosons. We will spend the remainder of this lecture answering this question.
2. Which is the most probable configuration? That is, which one can be achieved in the largest number of different ways? It is that configuration for which $Q(N_1, N_2, N_3, \dots)$ is a maximum. We will find the most probable configuration in the next lecture, using Lagrange multipliers.

Counting states

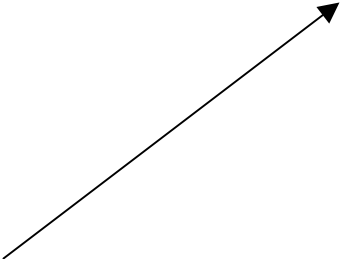
Treat the configuration (N_1, N_2, N_3, \dots) as having N_1 particles in the *first bin*, N_2 in the *second bin*, etc. The n th bin has d_n slots, due to degeneracy. The energy only depends on the bin, not the slot.

1. Distinguishable particles:

Given N particles total, how many ways are there to select the N_1 particles to be placed in the *first bin*?

Answer: the binomial coefficient, " N choose N_1 ":

$$\binom{N}{N_1} = \frac{N!}{(N - N_1)! N_1!} = \frac{N!}{N_1!(N - N_1)!}$$

 # of different permutations of the N_1 particles; divide by $N_1!$ since order does not matter.

N ways to pick first particle, $N - 1$ ways to pick second particle, \dots , $(N - (N_1 - 1))$ ways to pick last of the N_1 particles.

$$= N(N - 1)(N - 2) \dots (N - (N_1 - 1)) = N! / (N - N_1)!$$

Now how many different ways can these N_1 particles be arranged in the d_1 one-particle states all with energy E_1 ?

Each of the N_1 particles has d_1 choices; thus, there are $(d_1)^{N_1}$ possibilities in all.

Therefore, the # number of ways to put N_1 particles, selected from a total of N particles, into a bin containing d_1 distinct options, is

$$\frac{N!d_1^{N_1}}{N_1!(N - N_1)!}$$

The same argument holds for bin 2 (E_2) except that there are now only $(N - N_1)$ particles left to work with:

$$\frac{(N - N_1)! d_2^{N_2}}{N_2!(N - N_1 - N_2)!}, \text{ and so on.}$$

Thus, the total # of different ways that a particular configuration can be achieved is

$$\begin{aligned} Q(N_1, N_2, N_3, \dots) &= \frac{N! d_1^{N_1}}{N_1!(N - N_1)!} \frac{(N - N_1)! d_2^{N_2}}{N_2!(N - N_1 - N_2)!} \frac{(N - N_1 - N_2)! d_3^{N_3}}{N_3!(N - N_1 - N_2 - N_3)!} \dots \\ &= N! \frac{d_1^{N_1} d_2^{N_2} d_3^{N_3} \dots}{N_1! N_2! N_3! \dots} \\ &= N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!} \end{aligned}$$

2) Identical Fermions

If there was no degeneracy in the problem (i.e., there is only one single-particle state corresponding to each one-particle energy E_n), then the only allowed configurations would be those for which $N_n = 0$ or 1 since only one fermion can occupy any given state. Moreover, because the particles are indistinguishable, it doesn't matter which particles are in which occupied states — the antisymmetrization requirement means that there is just one N -particle state in which a specific set of one-particle states is occupied. Every possible configuration would then be equally likely.

However, we must consider the more general case when the single-particle energy E_n has degeneracy d_n ; i.e., there are d_n distinct one-particle states with the same energy E_n .

Therefore, up to d_n particles can have the same energy E_n . For $N_n \leq d_n$, the # of different ways to choose the N_n occupied states in the n th bin is " d_n choose N_n ":

$$\binom{d_n}{N_n} = \frac{d_n!}{N_n!(d_n - N_n)!}.$$

Thus, the total # of different ways that a particular configuration can be achieved is

$$Q(N_1, N_2, N_3, \dots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n - N_n)!}.$$

Define the factorial of a negative # to be infinite to restrict $N_n \leq d_n$.

3) Identical bosons

As for fermions, the symmetrization requirement means that there is just one N -particle state in which a specific set of one particle states is occupied, but this time there is no restriction on the number of particles that can share the same one-particle state. Again, the complication comes in when we consider the degeneracy of each bin or energy. We must answer the question: for the n th bin, how many different ways can N_n identical particles be assigned to d_n different slots? (For bosons, N_n can be greater than d_n .)

Let dots represent particles and crosses represent partitions, so that, for example, if $d_n = 5$ and $N_n = 7$,

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would represent one possible assignment of 7 particles to 5 slots. There are N_n dots and $(d_n - 1)$ crosses (partitioning the dots into d_n groups).

If the individual dots and crosses were labeled, there would be $(N_n + d_n - 1)!$ different ways to arrange them. But the dots are all equivalent since the particles are identical. And the crosses are all equivalent. Therefore we must divide by $N_n!$ and $(d_n - 1)!$. So there are

$$\frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} = \binom{N_n + d_n - 1}{N_n}$$

distinct ways of assigning the N_n particles to the d_n one-particle states in the n th bin.

Thus, the total # of different ways that a particular configuration can be achieved is

$$Q(N_1, N_2, N_3, \dots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!}.$$

Summary

1. Distinguishable particles

$$Q(N_1, N_2, N_3, \dots) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$$

2. Identical fermions

$$Q(N_1, N_2, N_3, \dots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n - N_n)!}$$

3. Identical bosons

$$Q(N_1, N_2, N_3, \dots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!}$$