1 Important Fourier Transforms

\[ \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx \]

General \( f \)                        Distributions
\[
\begin{array}{lll}
  f(x) & \hat{f}(k) & 1 \\
  f(x-a) & e^{-ak}\hat{f}(k) & \delta(x) \\
  e^{2\pi iax} f(x) & \hat{f}(k-2\pi a) & e^{iax} \\
  f(ax) & \frac{1}{|a|}\hat{f}\left(\frac{k}{a}\right) & \cos(ax) \\
  \hat{f}(x) & f(-k) & \sin(ax) \\
  d^n f(x) / dx^n & (ik)^n \hat{f}(k) & x^n \\
  x^n f(x) & i^n d^n \hat{f}(k) / dk^n & \frac{1}{i^n} \\
  (f \ast g)(x) & \sqrt{2\pi}\hat{f}(k)\hat{g}(k) & \text{sign}(x) \\
  f(x)g(x) & (\hat{f}\ast\hat{g})(k) / \sqrt{2\pi} & S\left(\frac{1}{i\pi k}\right) \delta(k) \\
  f(x) = \overline{f(x)} \text{ (real)} & \hat{f}(-k) = \overline{\hat{f}(k)} \\
  f(x) \text{ (real, even)} & \hat{f}(k) \text{ (real, even)} \\
  f(x) \text{ (real, odd)} & \hat{f}(k) \text{ (imaginary, odd)} \\
  \overline{f(x)} & \overline{\hat{f}(-k)}
\end{array}
\]

2 Important Maclaurin Series

Trigonometric Functions
\[
\begin{align*}
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\end{align*}
\]

Hyperbolic Functions
\[
\begin{align*}
\sinh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\
\cosh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\end{align*}
\]

Exponential Function
\[
\begin{align*}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\end{align*}
\]
Natural Logarithm (for $|x| < 1$)

\[ \log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - ... \]

\[ \log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - ... \]

Geometric Series (for $|x| < 1$)

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \]

Binomial Series (for $|x| < 1$, $\alpha \in \mathbb{C}$)

\[ (1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \]

This includes the square root series for $\alpha = \frac{1}{2}$ and the infinite geometric series for $\alpha = -1$.

\[ (1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + ... \]

3 Calculus Techniques

3.1 Coordinate Transformations

Making of linear transformation of coordinates

\[ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \]

Involves the Jacobian

\[ \frac{\partial(\alpha, \beta)}{\partial(x, t)} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

so

\[ dxdt = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|^{-1} d\alpha d\beta \]

3.2 Integration and Derivatives

Mean Value Theorem for Integrals

\[ \int_{a}^{b} f(x)dx = (b - a)f(\theta), \quad \theta \in (a, b) \]

Directional Derivative

\[ \nabla_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v}, \quad ||v||_2 = 1 \]

This operator is linear, obeys the product rule $\nabla_{\vec{v}}(fg) = f\nabla_{\vec{v}} f + g\nabla_{\vec{v}} g$ and obeys the chain rule $\nabla_{\vec{v}}(h \circ g)(\vec{x}) = h'(g(\vec{x}))\nabla_{\vec{x}} g(\vec{x})$. Example: Unit normal vectors $\frac{df}{dn} = \nabla_{\vec{n}} f$
3.3 Divergence Theorem

\[ \int_{\Omega} \vec{\nabla} \cdot \vec{g} \, dx = \int_{\partial \Omega} \vec{g} \cdot \vec{n} \, ds \]

The first integral is an integral over \( \Omega \), the second integral is a line integral around the boundary of \( \Omega \).

3.4 Green’s Identity

\[ \int \int_{\Omega} (f \Delta g - g \Delta f) \, dx = \int_{\partial \Omega} \left( f \frac{\partial h}{\partial n} - h \frac{\partial f}{\partial n} \right) \, ds \]

The first integral is an integral over \( \Omega \), the second integral is a line integral around the boundary of \( \Omega \).

3.5 Laplace Operator

3.5.1 Polar Coordinates

\[ \nabla^2 = \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \]

If the system is radially symmetric, this becomes

\[ \nabla^2 = \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \]

3.5.2 Spherical Coordinates

\[ \nabla^2 = \Delta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \]

If the system is radially symmetric, this becomes

\[ \nabla^2 = \Delta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) \]

3.6 Jacobian Factors

3.6.1 Polar Coordinates

\[ \int_{\Omega} f(x, y) \, dx \, dy = \int_{\Omega} f(x, y, z) \, r \, dr \, d\theta \]

3.6.2 Spherical Coordinates

\[ \int_{\Omega} f(x, y, z) \, dx \, dy \, dz = \int_{\Omega} f(x, y, z) \rho^2 \, d\rho \, d\theta \, d\phi \]

3.7 Divergence and Curl

....
3.8 Sequences and Series

3.8.1 Sequences

3.8.2 Series

The partial sum of a geometric series is given by \((r \neq 1)\)

\[
a + ar + ar^2 + ar^4 + ... + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = \frac{a(1 - r^n)}{1 - r}
\]

If and only if \(|r| < 1\), then as \(n \to \infty\),

\[
a + ar + ar^2 + ar^4 + ... = \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}
\]

Convergence  A series \(S\) converges to a limit \(L\) if and only if the sequence of partial sums \(S_K\) converges to \(L\).

- The \(p\)-series \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges for \(r > 1\) and diverges for \(r \leq 1\).

- The harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) diverges.

- If the sequence \(\{b_n\}\) converges to the limit \(L\) as \(n \to \infty\), then the telescoping series \(\sum_{n=1}^{\infty} (b_n - b_{n+1})\) converges to \(b_1 - L\)

For function series,

- A function series converges pointwise on \(\Omega\) if it converges for each \(x \in \Omega\). That is, pointwise convergence is defined as

\[
S_N(x) = \sum_{n=1}^{N} f_n(x) \to S(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall \ x \in \Omega
\]

- A function series converges uniformly on \(\Omega\) if it converges pointwise and remainder from the partial series sum converges to 0 as \(n \to \infty\) independent of \(x\). That is, it converges if

\[
\forall \epsilon > 0, \exists N \text{ s.t. } n > N \implies |S_n(x) - f(x)| < \epsilon
\]

4 Trigonometric Functions

\[
\int_{\Omega} \cos(\alpha x)e^{\beta x} dx = (\beta^2 + \alpha^2) (\beta \cos(\alpha x) + \alpha \sin(\alpha x)) e^{\beta x}
\]

\[
\int_{\Omega} \sin(\alpha x)e^{\beta x} dx = (\beta^2 + \alpha^2) (\beta \sin(\alpha x) - \alpha \cos(\alpha x)) e^{\beta x}
\]

\[
e^{ix} = \cos(x) + i \sin(x) \quad \cos(x) = \frac{1}{2} (e^{-ix} + e^{ix}) \quad \sin(x) = \frac{i}{2} (e^{-ix} - e^{ix})
\]

4.1 Pythagorean Identities

\[
\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x
\]
4.2 Sum-Difference Formulas

\[
\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \\
\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \\
\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}
\]

4.3 Double Angle Formula

\[
\sin(2u) = 2 \sin u \cos u \\
\cos(2u) = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u \\
\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}
\]

4.4 Sum to Product Formulas

\[
\sin u \pm \sin v = 2 \sin \left(\frac{u \pm v}{2}\right) \cos \left(\frac{u - v}{2}\right) \\
\cos u + \cos v = 2 \cos \left(\frac{u + v}{2}\right) \cos \left(\frac{u - v}{2}\right) \\
\cos u - \cos v = -2 \sin \left(\frac{u + v}{2}\right) \sin \left(\frac{u - v}{2}\right)
\]

4.5 Differentiation

\[
\frac{d}{dx}(\tan x) = \sec^2 x \\
\frac{d}{dx}(\cot x) = -\csc^2 x \\
\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \\
\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}
\]

4.6 Integration

\[
\int \sec^2 x = \tan x + C \\
\int \csc^2 x = -\cot x + C
\]

5 Hyperbolic Functions

\[
\sinh x = \frac{e^x - e^{-x}}{2} \\
\cosh x = \frac{e^x + e^{-x}}{2} \\
\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]
6 Areas and Volumes

6.1 Two Dimensions

<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoid</td>
<td>$\frac{b_1 + b_2}{2} \cdot h$</td>
<td>sum of sides</td>
</tr>
</tbody>
</table>

6.2 Three Dimensions

For shapes with height $h$, base $b$, radius $r$,

<table>
<thead>
<tr>
<th>Shape</th>
<th>Volume</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cone</td>
<td>$\frac{1}{3} \pi r^2$</td>
<td>$\pi r^2 + \pi r s = \pi r^2 \sqrt{r^2 + h^2}$</td>
</tr>
<tr>
<td>Pyramid</td>
<td>$\frac{1}{3} b h$</td>
<td></td>
</tr>
<tr>
<td>Sphere</td>
<td>$\frac{4}{3} \pi r^3$</td>
<td>$4\pi r^2$</td>
</tr>
</tbody>
</table>

6.3 N Dimensions

<table>
<thead>
<tr>
<th>Shape</th>
<th>Volume</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>$\frac{\pi^{n/2}}{\Gamma(n/2+1)} r^N$</td>
<td>$\frac{N\pi^{n/2}}{\Gamma(n/2+1)} r^{N-1}$</td>
</tr>
</tbody>
</table>
7 Named Functions

7.1 Gamma Function
For a positive integer \( n \),
\[
\Gamma(n) = (n - 1)!
\]
This function is also defined for all complex numbers except negative integers and zero. For complex numbers with a positive real part, the Gamma function is defined as the improper integral
\[
\Gamma(t) = \int_{0}^{\infty} x^{t-1} e^{-x} \, dx
\]

8 Fourier

8.1 Parseval’s Identity
Given the Fourier coefficients of \( f \), \( c_n \),
\[
\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx
\]

8.2 Plancherel Theorem
If \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), then \( \hat{f} \in L^2(\mathbb{R}) \), and the FT is an isometry wrt \( \| \cdot \|_{L^2(\mathbb{R})} \)
\[
\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |\hat{f}(k)|^2 \, dk
\]

8.3 Poisson Summation Formula
For \( \phi \in S \)
\[
\sqrt{2\pi} \sum_{n=-\infty}^{n} \phi(2\pi n) = \sum_{n=-\infty}^{n} \hat{\phi}(n)
\]

9 Famous Inequalities

9.1 Jensen’s Inequality
If \( \phi \) is convex on \( \mathbb{R} \) then
\[
\phi \left( \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right) \leq \frac{1}{b-a} \int_{a}^{b} \phi(f(t)) \, dt
\]
If \( \phi \) is concave on \( \mathbb{R} \) then
\[
\phi \left( \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right) \geq \frac{1}{b-a} \int_{a}^{b} \phi(f(t)) \, dt
\]

9.2 Normed Linear Space Inequalities
\( X \) is a vector space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| x \| = \sqrt{\langle x, x \rangle} \). Take any \( x, y \in X \).
9.3 Young’s Inequality

9.2.1 Cauchy Schwartz Inequality

\[ |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \] or equivalently, \[ |\langle x, y \rangle| \leq \|x\| \|y\| \]

9.2.2 Parallelogram Law

\( X \) is an normed inner product space if and only if

\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]

9.2.3 Pythagorean Theorem

\[ \|x^2 + y^2\| = \|x^2\| + \|y^2\| \]

9.2.4 Bessel’s Inequality

For an infinite dimensional basis, \( S_N = \sum \langle x, e_n \rangle e_n = P_{M_N}x \) where \( M_N = L \{e_1, e_2, ..., e_n\} \), which implies

\[ \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \]

9.3 Young’s Inequality

For \( \epsilon, a, b > 0, 1 \leq p, q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[ ab \leq \epsilon \frac{a^p}{p} + \epsilon \frac{b^q}{q} \]

In particular, if \( p = q = 2 \),

\[ ab \leq \epsilon \frac{a^2}{2} + \frac{1}{\epsilon} \frac{b^2}{2} \]

Further, if \( \epsilon = 1 \),

\[ ab \leq \frac{a^2}{2} + \frac{b^2}{2} \]

9.4 Young’s Convolution Inequality

If \( \phi, \psi \in C_0^\infty \), then for \( 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( 1 \leq p, q, r \leq \infty \)

\[ \|\phi * \psi\|_{L^r(\mathbb{R}^N)} \leq \|\phi\|_{L^p(\mathbb{R}^N)} \|\psi\|_{L^q(\mathbb{R}^N)} \]

Example: \( \|\phi * \psi\|_{L^p(\mathbb{R}^N)} \leq \|\phi\|_{L^p(\mathbb{R}^N)} \|\psi\|_{L^1(\mathbb{R}^N)} \)

9.5 Holder Inequality

Integral version

For \( u, v \) measurable, and \( 1 \leq p, q \leq \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \)

\[ \int_{\Omega} |u(x)v(x)| \, dx \leq \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |v(x)|^q \, dx \right)^{\frac{1}{q}} \]

\[ \|u(x)v(x)\|_{L^1(\Omega)} = \|u(x)\|_{L^p(\Omega)} \|v(x)\|_{L^q(\Omega)} \]
9.6 Minkowski Inequality

For $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum |a_k b_k| \leq \left( \sum |a_k|^p \right)^{\frac{1}{p}} \left( \sum |b_k|^q \right)^{\frac{1}{q}}$$

9.6 Minkowski Inequality

For $u, v$ measurable, and $1 \leq p \leq \infty$

$$\left( \int_{\Omega} |u(x) + v(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |v(x)|^p dx \right)^{\frac{1}{p}}$$

$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$

Sum Version

For $1 \leq p \leq \infty$

$$\left( \sum |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left( \sum |a_k|^p \right)^{\frac{1}{p}} + \left( \sum |b_k|^p \right)^{\frac{1}{p}}$$

10 Common Theorems

10.1 Stone-Weierstrass Theorem

(also known as Weierstrass Approximation Theorem)

Every continuous function on a closed interval can be uniformly approximated by a polynomial function.

10.2 Heine-Borel Theorem

Every closed and totally bounded subset of a complete matrix space is compact.

For a subset $S \subset \mathbb{R}^N$, the following are equivalent

- $S$ is closed and bounded
- $S$ is compact

10.3 Arzela Ascoli Theorem

Consider a sequence of real-valued continuous functions $\{f_n\}$ defined on a closed and bounded interval $[a, b]$ of the real line. There exists a subsequence $\{f_{n_k}\}$ that converges uniformly if and only if this sequence is uniformly bounded and equicontinuous.

10.4 Fubini’s Theorem

Given measurable spaces $A, B$, and if $f$ is $A \times B$ measurable, and if the integral with respect to a product measure satisfies

$$\int_{A \times B} |f(x, y)| d(x, y) < \infty$$

then the integral with respect to a product measure is equal to the iterated integrals

$$\int_{A \times B} f(x, y) d(x, y) = \int_A \left( \int_B f(x, y) dy \right) dx = \int_B \left( \int_A f(x, y) dx \right) dy$$
Corollary
If \( f \) satisfies the above conditions and additionally \( f(x, y) = h(x)g(y) \), then

\[
\int_{A \times B} f(x, y) d(x, y) = \int_A h(x) dx \int_B g(y) dy
\]

10.5 Lax Milgram Theorem

If \( a(\cdot, \cdot) \) be a bilinear form on \( H \) which is

- **bounded:** \( |a(u, v)| \leq C \| u \|_H \| v \|_H \)
- **coercive:** \( |a(u, u)| \geq c \| u \|^2_H \)

then for any \( f \in H^* \) there is a unique solution \( u \in H \) to the equation \( a(u, v) = \langle f, v \rangle \) and also \( \| u \| \leq \frac{1}{c} \| f \|^2 \).

10.6 Fredholm Alternative

10.6.1 Operator Version

Given a compact integral operator \( K \), a nonzero \( \lambda \) is either an eigenvalue of \( K \) of lies in the domain of the resolvent.

\[
R_\lambda(K) = (K - \lambda I)^{-1}
\]

10.6.2 Integral Equation Version

Let \( K(x, y) \) be an kernel of the integral operator \( Tu = \lambda u - (K, u) \). If \( K(x, y) \) yields a compact integral operator, then the following theorem holds: For any nonzero \( \lambda \in \mathbb{C} \), either the integral equation

\[
\lambda \phi(x) - \int_a^b K(x, y) \phi(y) dy = f(x)
\]

has a solution for all \( f(x) \) OR the associated homogenous case \( f(x) = 0 \) has only trivial solutions. \( K(x, y) \) being Hilbert Schmidt is a sufficient but not necessary condition.

10.6.3 Linear Algebra Version

For \( A \in \mathbb{C}^{n \times m} \) and \( b \in \mathbb{C}^{m \times 1} \),

- Either \( Ax = b \) has a solution \( x \)
- OR: \( A^T \tilde{y} = 0 \) has a solution \( \tilde{y} \) with \( \tilde{y}^T b \neq 0 \).

That is, \( Ax = b \) has a solution if and only if for any \( \tilde{y} \) s.t. \( A^T \tilde{y} = 0, \tilde{y}^T b = 0 \).

10.7 Riesz Representation Theorem

Given a Hilbert space \( H \) and its dual space \( H' \). For all \( y \in H' \), there exists a unique \( \phi_y \) such that

\[
\phi_y(x) = \langle x, y \rangle
\]
10.8 Riemann Lebesgue Lemma

The Fourier Transform of any $L^1$ function vanishes at infinity.

Let $f \in L^1(\mathbb{R})$ and since $f \in L^1$ there exists a smooth function (say $g$), compactly supported (say on $[a, b]$) that approximates $f$. Thus let $\|f - g\|_{L^1} < \epsilon$. Since $g$ is smooth,

$$\hat{g}(k) = \int_a^b g(x)e^{-ixk}dx = \frac{g(b)e^{-ibk}}{-ik} - \frac{g(a)e^{-iak}}{-ik} + \int_a^b g'(x)e^{-ixk}i dx$$

So $|\hat{g}(k)| \to 0$ at $k \to \pm \infty$. Then

$$|\hat{f}(k)| = \left|\int f(x)e^{-ixk}dx\right| \leq \left|\int (f(x) - g(x))e^{-ixk}dx\right| + |\hat{g}(k)| \leq \int |f(x) - g(x)| dx + |\hat{g}(k)| < \epsilon + |\hat{g}(k)|$$

So as $k \to \pm \infty$, $\limsup_{k \to \pm \infty} = 0$

10.9 Eigenfunction Expansion Theorem

Let $K$ be a self adjoint compact operator and let $(\lambda_k, e_k)$ be the set of eigenpairs for $K$ where $\lambda_k \neq 0$ and $e_k$ are the eigenfunctions orthonormalized to $\|e_k\| = 1$.

Any function in the range of $K$ can be expanded in a Fourier series in the eigenfunctions of $K$ corresponding to nonzero eigenvalues. There eigenfunctions form an orthonormal basis for $\mathbb{R}(K)$ (but necessarily for $\mathcal{H}$).

Thus, for all $f \in \mathcal{H}$,

$$Kf = \sum (Kf, e_k) e_k = \sum (f, Ke_k) e_k = \sum \lambda_k \langle f, e_k \rangle e_k$$

where equality is in the $L^2$ sense.

If we include the eigenfunctions for $\lambda = 0$, we have a basis for $\mathcal{H}$. If $h$ is the projection if $f$ onto the nullspace of $K$, then an arbitrary function can be decomposed uniquely as

$$f = h + \sum \langle f, e_k \rangle e_k$$