Comparing two means

Goal: compare the means of two populations/treatments

- μ_1 & μ_2 represent two population means to compare

- σ_1 & σ_2 represent two population standard deviations

- Take a random sample from each population/treatment

- Two samples are *independent* (don't influence each other)

(recall: in matched pairs, two sets of observations *not* independent)

- Sample sizes: n_1 = sample 1 size; n_2 = sample 2 size

- Use $\bar{Y}_1$, $\bar{Y}_2$, s_1, s_2 to estimate population parameters

- **Sampling Distribution of $\bar{Y}_2 - \bar{Y}_1$**
  - Center/typical value is $\mu_2 - \mu_1$
  - Standard deviation: $\mathrm{SD}(\bar{Y}_2 - \bar{Y}_1) = \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$
  - If two population distributions are normal or sample sizes large $n_1, n_2 \geq 30$, then distribution of $\bar{Y}_2 - \bar{Y}_1$ is *normal*
  - If $\sigma_1 = \sigma_2 = \sigma$, then $\mathrm{SD}(\bar{Y}_2 - \bar{Y}_1) = \sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$

To compare 2 population/treatment means, we either give a *confidence interval* for their difference $\mu_2 - \mu_1$ or we test the hypothesis of no difference $H_0 : \mu_1 = \mu_2$

Standard Errors (*assuming* $\sigma_1 = \sigma_2 = \sigma$)

• To measure of how well $\bar{Y}_2 - \bar{Y}_1$ estimates $\mu_2 - \mu_1$, we estimate the spread $\text{SD}(\bar{Y}_2 - \bar{Y}_1) = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ with the **standard error of** $\bar{Y}_2 - \bar{Y}_1$ given by

$$\text{SE}(\bar{Y}_2 - \bar{Y}_1) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

• Estimate σ with *pooled* standard deviation: $s_p = \sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$

(s_1, s_2 both estimate same standard deviation σ)

• The ratio

$$t = \frac{(\bar{Y}_2 - \bar{Y}_1) - (\mu_2 - \mu_1)}{\text{SE}(\bar{Y}_2 - \bar{Y}_1)}$$

has an approximate $t-$distribution with $n_1 + n_2 - 2$ degrees of freedom

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Confidence Intervals for $\mu_2 - \mu_1$ (assuming $\sigma_1 = \sigma_2 = \sigma$)

• A $100(1-\alpha)\%$ confidence interval for $\mu_2 - \mu_1$ is given by:

$$\bar{Y}_2 - \bar{Y}_1 \pm t_{n_1+n_2-2}^{(1-\alpha/2)}\text{SE}(\bar{Y}_2 - \bar{Y}_1)$$

where $t_{n_1+n_2-2}^{(1-\alpha/2)}$ is the t-ratio larger than $100(1-\alpha/2)\%$ of $t-$ratios from a $t-$distribution with $n_1 + n_2 - 2$ degrees of freedom

• Same use/interpretation as before except confidence interval is for the *difference* $\mu_2 - \mu_1$ in population/treatment means

Two sample t-tests

1. **State Hypotheses:**

H_0: $\mu_1 = \mu_2$	(or $\mu_2 - \mu_1 = 0$)	*null hypothesis*
H_a: $\mu_2 > \mu_1$	(or $\mu_2 - \mu_1 > 0$)	*alternatives*
H_a: $\mu_2 < \mu_1$	(or $\mu_2 - \mu_1 < 0$)	
H_a: $\mu_2 \neq \mu_1$	(or $\mu_2 - \mu_1 \neq 0$)	

2. **Test Statistic (t-ratio):** $t = \dfrac{(\bar{Y}_2 - \bar{Y}_1) - (\mu_2 - \mu_1)}{\mathrm{SE}(\bar{Y}_2 - \bar{Y}_1)}$,

substituting value of $\mu_2 - \mu_1$ (i.e., $\mu_2 - \mu_1 = 0$) under null hypothesis H_0

3. **p-value:** probability of getting more extreme test statistic ($t-$ratio) than our observed t if H_0 is true

Consider t-distribution having degrees of freedom $df = n_1 + n_2 - 2$

H_a: $\mu_2 > \mu_1$ p-value = % of t-ratios larger than our t
= area under t-curve to right of t
= 1 - % of t-ratios smaller than our t

H_a: $\mu_2 < \mu_1$ p-value = % of t-ratios smaller than our t
= area under t-curve to left of t

H_a: $\mu_2 \neq \mu_1$ p-value = % of t-ratios absolutely larger than our $|t|$
= twice area under t-curve to right of $|t|$
= $2\times$(1 - % of t-ratios smaller than our $|t|$)

4. **Interpretation of p-value**: Small p-values are evidence against H_0 (i.e., the t-tatio t we observe seems "unusual" under the null hypothesis or "unusually far" from the center of t-distribution at zero).

5. **Conclusion:** In terms of the problem and H_a, roughly again

small p-value $\Rightarrow$ statistically significant evidence to prove H_a

large p-value $\Rightarrow$ *no* statistically significant evidence to prove H_a

Note: It is possible to also test $H_0 : \mu_2 - \mu_1 = D$, that the difference in means $\mu_2 - \mu_1$ equals some specified value D, where D may not necessarily be zero. (See page 43 of text.) One would then substitute the hypothesized value D for $\mu_2 - \mu_1$ in the test statistic t or $t-$ratio in Step 3 above. Usually though, the value $D = 0$ (for "no difference in means") is most common.