One-Way Analysis of Variance & F-test

Goal: compare the means of *several* populations/treatments

- There are I populations/treatments to compare

- Each population/treatment has a mean μ_i & standard deviation σ_i (or variance σ_i^2) (where $i = 1, \ldots, I$)

- Take an independent random sample from each population/treatment (in experiments, treatments are randomly assigned to experimental units to define each treatment group)

- n_i =size of sample i, $\bar{Y}_i$=average of sample i, s_i = standard deviation of sample i, (computed for each $i = 1, \ldots, I$)

- Use $\bar{Y}_i$, s_i estimate population parameters μ_i, σ_i for each group $i = 1, \ldots, I$

- **Assumptions:** The distribution of observations in each population or treatment group are *approximately normal* (mound-shaped) around the mean μ_i. In addition, the spreads or *standard deviations of all populations/treatment groups are equal* (or approximately equal) to a common value σ. That is, assume

$$\sigma_1 = \sigma_2 = \cdots = \sigma_I \;=\; \sigma$$

- Estimate common standard deviation σ with *pooled* estimate of standard deviation:

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2 + \cdots + (n_I-1)s_I^2}{n_1+n_2+\cdots+n_I - I}} = \sqrt{\frac{\sum_{i=1}^{I}(n_i-1)s_i^2}{n-I}}$$

(each s_i should estimate the same standard deviation $\sigma_i = \sigma$ so we pool these I sample standard deviations together); n =total number of observations collected= $n_1 + n_2 + \cdots n_I$

- **State Hypotheses:**

  H_0: $\mu_1 = \mu_2 = \cdots = \mu_I$ *no difference among population/treatment means*

  H_a: not all μ_i are equal

- **F-Test (based on F-ratio):**

  1. Compute *Mean Square Between I* groups (denoted MSB) as

     $$MSB = \frac{n_1(\bar{Y}_1 - \bar{Y})^2 + n_2(\bar{Y}_2 - \bar{Y})^2 + \cdots + n_I(\bar{Y}_I - \bar{Y})^2}{I-1} = \frac{\sum_{i=1}^{I} n_i(\bar{Y}_i - \bar{Y})^2}{I-1}$$

     where $\bar{Y}$ is the sample mean of all n combined observations.
     (i.e., measure how far each sample mean $\bar{Y}_i$ is from a common value $\bar{Y}$)

  2. Compute *Mean Square Within I* groups or *Mean Square Error* (denoted MSW or MSE) as

     $$MSW = MSE = s_p^2$$

  3. Compute F-test statistic as a ratio

     $$F = \frac{MSB}{MSW}$$

     If H_0 is true, then the F-test statistic behaves like an observation from an $F-$distribution with numerator degrees of freedom $df_1 = I - 1$ and denominator degrees of freedom $df_2 = n - I$. We can then judge if our data supports H_0 or H_a by where our statistic F falls among F-ratios from an $F-$distribution with appropriate degrees of freedom.

  4. Compute p-value as

     p-value = % of $F-$ratios larger than our F from $F-$distribution with numerator degrees of freedom $df_1 = I - 1$ and denominator degrees of freedom $df_2 = n - I$
     (area under $F-$curve with $df_1 = I - 1$, $df_2 = n - I$ to right of F)

     See Table A.4 for $F-$distributions when computing p-values by hand.

     Note large F test statistics are evidence against H_0 in favor of H_a. That is, large F statistics imply small p-values (that only a small proportion of F-ratios are more extreme/larger than our F)