The log-transformation: A practice problem—Solution

Problem 31, page 81, *"The Statistical Sleuth"*

Iron Supplementation. A randomized experiment was performed on mice to determine whether two forms of iron, Fe^{3+} and Fe^{4+}, are retained differently. If one type is retained especially well, then it may be more useful as a dietary supplement for humans. (Data from J. Rice, *Mathematical Statistics and Data Analysis* (Pacific Grove, CA: Wadsworth, (1987), p. 356.) The mice were given the iron orally. The iron was radioactively labeled so that the initial amount and the amount retained after a fixed interval could be measured. The measurements of interest are the percentages of iron retained in each mouse after the time period had elapsed. The researchers wished to know if there is higher retention for one type than for the other. If so, which type? and by how much?

Use graphical and numerical methods to analyze the data. Answer the questions of interest. State your answers in one or two concise sentences that contain appropriate measures of uncertainty. Try to hold statistical jargon to a minimum.

Suppl.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Fe^{3+}	0.71	2.56	4.39	1.66	2.60	4.50	2.01	3.31	5.07	2.16	3.64	5.26	2.42	3.74	8.15	2.42	3.74	8.24
Fe^{4+}	2.20	4.27	6.97	2.69	4.53	6.97	3.54	5.32	7.52	3.75	6.18	8.36	3.83	6.22	11.65	4.08	6.33	12.45

Solution: We first need to formulate the testing hypothesis, in this case:

$$H_0 : Both\ types\ of\ iron\ are\ retained\ in\ the\ same\ way$$

and

$$H_a : There\ is\ a\ higher\ retention\ for\ one\ type\ of\ iron\ than\ the\ other.$$

Additionaly, if the H_0 is rejected, we would like to know which one of the supplements has higher retention and by how much.

After a visual inspection of the data (side-by-side boxplots or stem-and-leaf display, etc.), we decide that the logarithmic transformation is appropriate (for us to be able to employ the two sample t-test to test our hypothesis of interest).

If we denote by Z the log-transformed data, we obtain that the sample mean for the first log-transformed sample is $\bar{Z}_1 = 1.161$ and $\bar{Z}_2 = 1.680$ for the second sample. Therefore, $Z_1 - Z_2 = -0.519$. The standard deviations are $s_1^Z = 0.585$ and $s_2^Z = 0.465$. It follows that the pooled standard deviation is $s_p^Z = 0.528$ and standard error of the difference between means of the log-transformed data is $s.e._{diff}^Z = 0.1762$. One can immediately compute a 95% confidence interval for the difference in means of the log-transformed data, which equals $(-0.877, -0.161)$. If we calculate the t-ratio for the transformed data, we obtain that $t = -2.946$. For 34 degrees of freedom, we obtain a one-sided $p - value$ of 0.0029. Based on this value, we conclude that there is convincing evidence to reject the null hypothesis.

What does this mean in the original context that the scientific problem was formulated in?

Firstly, based on the observed $p - value$ and due to the symmetry in the log-transformed populations, we conclude that there is evidence of a significant difference in the retention of the two types of iron. But which one is retained better and by how much? The answer to that problem is based on the point estimate of the difference in means and confidence interval for the log-transformed data. As before, our answer is based on the assumption that the transformed populations are symmetric. The estimate of the ratio between the two population medians (second over the first) is $e^{-0.519} = 0.595$. This indicates that retention of the second iron supplement is higher than of the first one. Therefore the second one is retained better.

And by how much? In order to report a correct answer to this question, we need to calculate the confidence interval for the ratio of the two population medians, and that is simply $(e^{-0.877}, e^{-0.161})$ which equals: $(0.416, 0.851)$. We are ready now to give the complete answer to this problem.

There is convincing evidence that there is a higher retention for the second iron supplement than for the first. The median percentage of iron retained in each mouse after administering the second supplement is about 1.68 times larger than the median retention percentage of the first supplement, (with a 95% confidence interval of 1.17 and 2.40).

Note here that we used the confidence interval and point estimate based on the inverse ratio of the medians, second over the first, since we wanted our analysis to match the question under investigation: which supplement had a better retention percentage, as opposed to which one was worse. These are equivalent, of course, but one answers the question directly, the other requires the reader to figure out the answer on their own.