Z-Tests for Population mean $\mu$ (when $\sigma$ is known)

1. State Hypotheses: $H_0: \mu = \mu_0$ versus $H_a: \mu > \mu_0$
   $H_a: \mu < \mu_0$
   $H_a: \mu \neq \mu_0$

2. Test Statistic:

   \[
   z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}
   \]

   where

   $\bar{x}$ is the observed sample mean, $\mu_0$ is the mean under $H_0$, $n$ is the sample size, $\sigma$ is known standard deviation.

3. p-value: probability of getting more extreme test statistic than $z$ given $H_0$ is true

   $H_a: \mu > \mu_0$  \hspace{1cm} p-value = $P(Z > z)$

   $H_a: \mu < \mu_0$  \hspace{1cm} p-value = $P(Z < z)$

   $H_a: \mu \neq \mu_0$  \hspace{1cm} p-value = $2P(Z > |z|) = 2P(Z < -|z|)$

4. Decision: Reject $H_0$ if $p-value \leq \alpha$

5. Conclusion: In terms of the problem and $H_a$

   If we reject $H_0 \Rightarrow$ statistically significant evidence to support the claim made in $H_a$

   If we fail to reject $H_0 \Rightarrow$ no statistically significant evidence to support the claim made in $H_a$
EXAMPLES FOR TESTING A POPULATION MEAN $\mu$, WHEN $\sigma$ IS KNOWN
($z$ - procedure)

EXAMPLE 1:

Developing a new diet (to lose weight) and measure the # of lbs lost. Took a random sample of 36 people on the diet and found the average # of pounds lost to be 2.5 lbs. From past experience we know the population standard deviation to be 6 lbs. Is the diet effective? Conduct a test at the $\alpha = 0.01$ level of significance.

1. State the null and alternative hypotheses:

\[ H_0: \mu = 0 \quad \text{vs} \quad H_a: \mu > 0 \]

2. Compute the value of the test statistic:

\[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{2.5 - 0}{6 / \sqrt{36}} = \frac{2.5}{1} = 2.5 \]

3. Find the p-value associated with the test statistic:

\[ p\text{-value} = P(z > 2.5) = P(z > 2.5) \]

\[ = 1 - P(z \leq 2.5) \]

\[ = 1 - .9938 = 0.0062 \]

4. Decision:

\[ \text{reject } H_0 \text{ if } p\text{-value} \leq \alpha = 0.01 \]

\[ 0.0062 \leq 0.01 \Rightarrow \text{reject } H_0 \]

5. There is statistically significant evidence (at the $\alpha = 0.01$ level) to conclude that the diet is effective or mean weight loss is greater than zero.
EXAMPLE 2:

A researcher believes that the mean score $\mu$ of all third graders in a district is lower than the national mean, which is 32. We know the scores are approximately normal with population standard deviation of 11 for this school district. Test the claim of the researcher at the $\alpha = 0.05$ level, using a random sample of 44 students in which the sample mean is 28.91.

1. State the null and alternative hypotheses:

$$H_0 : \mu = 32 \quad \text{vs.} \quad H_a : \mu < 32$$

2. Compute the value of the test statistic:

$$z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{28.91 - 32}{\frac{11}{\sqrt{44}}} = -1.86$$

3. Find the p-value associated with the test statistic:

$$p-value = P(Z < z)$$

$$= P(Z < -1.86)$$

$$= 0.0314$$

4. Decision:

Reject $H_0$ if $p-value \leq \alpha = 0.05$

$0.0314 < 0.05 \Rightarrow$ reject $H_0$

5. There is statistically significant evidence (at the $\alpha = 0.05$ level) to conclude that the true score of all 3rd graders in this district is less than 32 (National Average).

Note: We could not have rejected $H_0$ had $\alpha$ been 0.01.

$p-value > 0.01 \Rightarrow$ fail to reject
EXAMPLE 3:

A manufacturer of a sprinkler system used for fire protection in office buildings claims that the true average system-activation temperature is 130°F. A random sample of \( n = 9 \) systems when tested yielded a sample mean activation temperature of 131.08°F. If the distribution of activation temperatures is normal with \( \sigma = 1.5°F \), does the data contradict the manufacturer's claim at the 0.01 significance level?

1. State the null and alternative hypotheses:

\[ H_0: \mu = 130 \quad \text{vs.} \quad H_a: \mu \neq 130 \]

2. Compute the value of the test statistic:

\[
Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{131.08 - 130}{1.5/\sqrt{9}} = 2.16
\]

3. Find the p-value associated with the test statistic:

\[
\text{p-value} = 2 \cdot P(Z < -2.16) + 2 \cdot P(Z > 2.16) = 2 \cdot 0.0154 = 0.0308
\]

4. Decision:

\[ \text{Reject } H_0 \text{ if } \text{p-value} \leq \alpha \]

\[ \Rightarrow \text{p-value of 0.0308 } > \alpha = 0.01 \Rightarrow \text{fail to reject } H_0 \]

5. There is no statistically significant evidence to conclude that the true average activation temperature differs from the design value of 130°F.