Midterm 2 Example Questions

Solve LPs using Simplex

1. Consider the following LP:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \le 6 \\ & x_1 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(a) Convert the LP to standard form.

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 = 6 \\ & x_1 + x_4 = 4 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

(b) Starting with x_1 and x_2 as nonbasic variables, solve the problem using the Simplex algorithm. Explain why you terminated the algorithm.

max	x_1	x_2	x_3	x_4	
c	2	1	0	0	**b**
	3	2	1	0	6
A	1	0	0	1	4
	N	N	B	B	
$\mathbf{x}^{(0)}$	0	0	6	4	
$\Delta\mathbf{x}_1$	1	0	-3	-1	$\mathbf{c}\Delta\mathbf{x}_1$=2>0
$\Delta\mathbf{x}_2$	0	1	-2	0	$\mathbf{c}\Delta\mathbf{x}_2$=1>0
			λ = 2	λ = 4	
	B	N	N	B	
$\mathbf{x}^{(1)}$	2	0	0	2	
$\Delta\mathbf{x}_1$	-0.6667	1	0	0.66667	$\mathbf{c}\Delta\mathbf{x}_1$=-0.333<0
$\Delta\mathbf{x}_2$	-0.3333	0	1	0.3333	$\mathbf{c}\Delta\mathbf{x}_2$=-0.6667<0

Optimal solution found!

$$x^* = \begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix}$$

(c) Now, assume that the first constraint is dropped. Using a Simplex algorithm solution, show what happens to the optimal solution. Explain.

max	x_1	x_2	x_4	
c	2	1	0	**b**
A	1	0	1	4
	N	N	B	
$\mathbf{x}^{(0)}$	4	0	0	
$\Delta\mathbf{x}_1$	1	0	-1	$\mathbf{c}\Delta\mathbf{x}_1=2>0$
$\Delta\mathbf{x}_2$	0	1	0	$\mathbf{c}\Delta\mathbf{x}_2=1>0$

Can move in improving direction Δx_2 forever

Problem unbounded!

2. Consider the following LP

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \le 2 \\ & x_1 + x_2 \le 6 \\ & x_1 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

a. (5%) Convert the LP to standard form:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1 + x_5 = 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

b. (25%) Starting with x_1, x_2 as non-basic, solve the problem using the Simplex algorithm. Explain why you terminated the algorithm.

	x_1	x_2	x_3	x_4	x_5		
c	2	1	0	0	0		
A	-2	1	1	0	0	2	
	1	1	0	1	0	6	
	1	0	0	0	1	4	
	N	N	B	B	B		
$\mathbf{x}^{(0)}$	0	0	2	6	4		
$\Delta\mathbf{x}_1$	1	0	2	-1	-1		reduced cost = 2
$\Delta\mathbf{x}_2$	0	1	-1	-1	0		reduced cost = 1
				$\lambda = 6$	$\lambda = 4$		
	B	N	B	B	N		
$\mathbf{x}^{(1)}$	4	0	10	2	0		
$\Delta\mathbf{x}_1$	0	1	-1	-1	0		reduced cost = 1
$\Delta\mathbf{x}_2$	-1	0	-2	1	1		reduced cost = -2
			$\lambda = 10$	$\lambda = 2$			
	B	B	B	N	N		
$\mathbf{x}^{(2)}$	4	2	8	0	0		
$\Delta\mathbf{x}_1$	0	-1	1	1	0		reduced cost = -1
$\Delta\mathbf{x}_2$	-1	1	-3	0	1		reduced cost = -1

Both of the reduce costs are now negative, which means that we have found the optimal solution (4,2,8,0,0), with performance of 2 * 4 + 1 * 2 = 10.

Note on grading part (b) above:

> Find the correct Simplex directions (5pt)
> Calculate the reduced cost and select a direction (5pt)
> Calculate and select the step size (5pt)
> Move to a new solution (5pt)
> Terminate the algorithm correctly (5pt)

c. (10%) From the same starting point as in b) above, find the most improving direction. Compare this direction with the Simplex direction that you chose on Step 1 of b) above. Which direction is better? Explain.

The most improving direction is the gradient $\Delta\mathbf{x}$ = [2,1,0,0,0] (**2pt**). This direction gives more immediate improvement than the simplex direction $\Delta\mathbf{x}^{(1)}$=[1,0,2,-1,-1] as be seen by calculating (**3pt**):

$$\Delta\mathbf{x} \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5 > 2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \Delta\mathbf{x}^{(1)} \cdot \nabla f$$

However, while $\Delta\mathbf{x}^{(1)}$ is feasible by construction, $\Delta\mathbf{x}$ is not feasible as can be seen from the constraints (**3pt**):

$$\mathbf{A}\Delta\mathbf{x} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+1 \\ 2+1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the most improving direction would have to be transformed before it can be used, and neither can be said to be better (**2pt**).

Sensitivity Analysis

1. The NCAA is making plans for distributing tickets to the upcoming basketball championships. The up to 10,000 seats available will be divided between the media, the competing universities, and the general public. Media people are admitted free, but the NCAA receives $45 per ticket from universities and $100 per ticket from the general public. At least 500 tickets must be reserved for the media, and at least half as many tickets should go to the competing universities as to the general public. Within these restrictions, the NCAA wishes to find the allocation that raises the most money. We have formulated the following LP to solve the problem, and the LINDO output is below.

$$
\begin{aligned}
\max \quad & 45x_2 + 100x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \le 10000 \\
& x_2 - \frac{1}{2}x_3 \ge 0 \\
& x_1 \ge 500 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

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        OBJECTIVE FUNCTION VALUE
        1)      775833.3

  VARIABLE        VALUE          REDUCED COST
        X2       3166.666748          0.000000
        X3       6333.333496          0.000000
        X1        500.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         81.666664
        3)         0.000000        -36.666668
        4)         0.000000        -81.666664
        5)       500.000000          0.000000
        6)      3166.666748          0.000000
        7)      6333.333496          0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       X2       45.000000        55.000000       244.999985
       X3      100.000000         INFINITY        55.000000
       X1        0.000000        81.666664         INFINITY

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2     10000.000000         INFINITY      9500.000000
        3         0.000000      9500.000000      4750.000000
        4       500.000000      9500.000000       500.000000
        5         0.000000       500.000000         INFINITY
        6         0.000000      3166.666748         INFINITY
        7         0.000000      6333.333496         INFINITY
```

Please answer the following questions (all worth equal points):

a. What is the marginal cost to the NCAA of each seat guaranteed to the media?

This is simply the dual price of the third constraint = $81.67 (10pt)

b. Suppose that there is an alternative arrangement for the dome where the games will be played that can provide 15,000 seats. How much additional revenue would be gained from the expanded seating? How much would it be for 20,000 seats?

Look at the dual price of the capacity constraint = $81.67. This dual price is valid for any increase, hence the additional revenue will be (15000-10000)* $81.67 = $408,300 and $816,600, respectively (10pt).

c. Since television revenue provides most of the income for NCAA events, another proposal would reduce the price of general public tickets to $50. How much revenue would be lost from this change? What if the price were $30?

If we reduce it to $50, we still sell the same number of tickets to each party (within allowable range for basis to remain the same), so the revenue reduction is ($100 - $50) * 6333 = $316,700 (5pt).

If we reduce it to $30, the basis changes so the dual price is no longer valid (5pt). By taking the maximum allowable decrease, we can say that the profit changes by *at least* $55 * 6333 = $348,300 (lower bound on the decrease).

d. To accommodate high demand from student supporters of the participating universities, the NCAA is considering marketing a new "scrunch seat" that consumes only 80% of the regular bleacher seat but counts fully against the "university ≥ half public" rule. Could an optimal solution allocate any such seats at a ticket price of $35? At a price of $25?

This corresponds to adding a new variable (x_4), and you should think about the effect of setting this variable equal to one (2 pt for setting up the constraints):

$$\left.\begin{array}{l} x_1 + x_2 + x_3 + 0.8x_4 \le 10000 \\ x_2 - \frac{1}{2}x_3 + 1 \ge 0 \\ x_1 \ge 500 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1 + x_2 + x_3 + 0.8 \le 10000 \\ x_2 - \frac{1}{2}x_3 + 1 \ge 0 \\ x_1 \ge 500 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 + x_2 + x_3 \le 9999.2 \\ x_2 - \frac{1}{2}x_3 \ge -1 \\ x_1 \ge 500 \end{array}$$

The *cost of tightening* the first constrain by 0.8 is $81.67*0.8 = $65.34, while the *benefit of relaxing* the second constraint by 1 is $36.67 (dual prices). Hence the new ticket (allowing x_4 to be at least one) becomes attractive at $65.34 - $36.67 = $28.67 (8 pt). Hence we would not sell those tickets at $25 but we would sell them at $35.

2. As a result of a recent decision to stop production of toy guns that look too real, the SuperSlayer Toy Company is planning to focus its production on two futuristic models: beta zappers and freeze phasers. Beta zappers produce $2.50 in profit for the company and freeze phasers $1.60. The company is contracted to sell 10 thousand beta zappers and 15 thousand freeze phasers in the next month, but all that are produced can be sold. Production of either model involves three crucial steps: extrusion, trimming, and assembly. Beta zappers use 5 hours of extrusion time per thousand units, 1 hour of trimming time, and 12 hours of assembly. Corresponding values for freeze phasers are 9, 2, and 15. There are 320 hours of extrusion time, 300 hours of trimming time, and 480 hours of assembly time available over the next month.

$$\begin{aligned} \max \quad & 2500x_1 + 1600x_2 \\ \text{s.t.} \quad & x_1 \geq 10 \\ & x_2 \geq 15 \\ & 5x_1 + 9x_2 \leq 320 \\ & x_1 + 2x_2 \leq 300 \\ & 12x_1 + 15x_2 \leq 480 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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        OBJECTIVE FUNCTION VALUE
        1)       77125.00
  VARIABLE         VALUE          REDUCED COST
        X1        21.250000           0.000000
        X2        15.000000           0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        21.250000           0.000000
        3)         0.000000       -1525.000000
        4)        78.750000           0.000000
        5)       248.750000           0.000000
        6)         0.000000         208.333328
        7)        21.250000           0.000000
        8)        15.000000           0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:
                               OBJ COEFFICIENT RANGES
 VARIABLE          CURRENT        ALLOWABLE         ALLOWABLE
                    COEF          INCREASE          DECREASE
       X1     2500.000000         INFINITY       1220.000000
       X2     1600.000000      1525.000000          INFINITY

                               RIGHTHAND SIDE RANGES
      ROW          CURRENT        ALLOWABLE         ALLOWABLE
                     RHS          INCREASE          DECREASE
        2         0.000000        21.250000          INFINITY
        3        15.000000        17.000000         15.000000
        4       320.000000         INFINITY         78.750000
        5       300.000000         INFINITY        248.750000
        6       480.000000       189.000000        255.000000
        7         0.000000        21.250000          INFINITY
        8         0.000000        15.000000          INFINITY
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a) Is the optimum solution sensitive to the exact value of trimming hours available? If not, at what number of hours capacity would it become relevant?

No there is a slack of 248.75. Hence, it is relevant at 300 – 248.75 = 51.25 hours.

b) How much should SuperSlayer be willing to pay for an additional hour of extrusion time? For an additional hour of assembly time?

Look at the dual prices. It is 0 and 208.33 for extrusion time and assembly time, respectively. This is what they should be willing to pay.

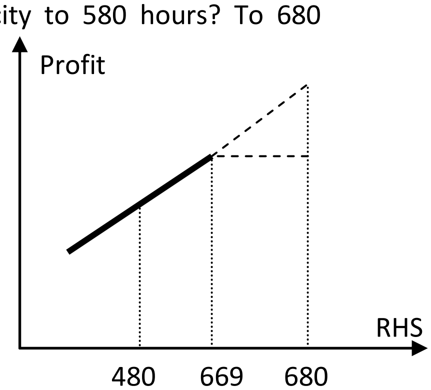
c) What would be the profit effect of increasing assembly capacity to 580 hours? To 680 hours?

Increase to 580 hours (increase of 100) is within the allowable increase (189), so it is simply 100 x 208.33 = $20,833

Increase to 680 is outside the allowable increase, but we can bound it with:

At least 189 x 208.33 = $39,374

At most 200 x 208.33 = $41,666



d) What would be the profit effect of increasing the profit margin on beta zappers by $1500 per thousand? What would be the effect of a decrease in that amount?

Increase of $1500 is inside the allowable increase (infinity). Hence profit would increase by $1500 x 21.25 = $31,875

Decrease of $1500 is outside the allowable decrease of 1220. However, we can again bound it. The decrease in profit would be

At least $1220 x 21.25 = $25,925