Solve LPs using Simplex

1. Consider the following LP:

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad 3x_1 + 2x_2 \leq 6 \\
& \quad x_1 \leq 4 \\
& \quad x_1, x_2 \geq 0 \\
\end{align*}
\]

(a) Convert the LP to standard form.

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad 3x_1 + 2x_2 + x_3 = 6 \\
& \quad x_1 + x_4 = 4 \\
& \quad x_1, x_2, x_3, x_4 \geq 0 \\
\end{align*}
\]
(b) Starting with $x_1$ and $x_2$ as nonbasic variables, solve the problem using the Simplex algorithm. Explain why you terminated the algorithm.

\[
\begin{array}{cccccc}
\text{max} & x_1 & x_2 & x_3 & x_4 \\
\hline
\text{c} & 2 & 1 & 0 & 0 & b \\
\hline
\text{A} & 3 & 2 & 1 & 0 & 6 \\
\hline
\end{array}
\]

\[
\begin{array}{cccccc}
N & N & B & B \\
\hline
x^{(0)} & 0 & 0 & 6 & 4 \\
\hline
\Delta x_1 & 1 & 0 & -3 & -1 & c\Delta x_1=2>0 \\
\hline
\Delta x_2 & 0 & 1 & -2 & 0 & c\Delta x_2=1>0 \\
\hline
\lambda = 2 & \lambda = 4 \\
\hline
B & N & N & B \\
\hline
x^{(1)} & 2 & 0 & 0 & 2 \\
\hline
\Delta x_1 & -0.6667 & 1 & 0 & 0.66667 & c\Delta x_1=-0.333<0 \\
\hline
\Delta x_2 & -0.3333 & 0 & 1 & 0.3333 & c\Delta x_2=-0.6667<0 \\
\end{array}
\]

Optimal solution found!

\[
x^* = (2 \ 0 \ 0 \ 2)
\]
(c) Now, assume that the first constraint is dropped. Using a Simplex algorithm solution, show what happens to the optimal solution. Explain.

<table>
<thead>
<tr>
<th>max</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>N</td>
<td>N</td>
<td>B</td>
<td></td>
</tr>
</tbody>
</table>

$x^{(0)} = 4 \ 0 \ 0$

$\Delta x_1 = 1 \ 0 \ -1 \ c\Delta x_1 = 2 > 0$

$\Delta x_2 = 0 \ 1 \ 0 \ c\Delta x_2 = 1 > 0$

Can move in improving direction $\Delta x_2$ forever

Problem unbounded!
2. Consider the following LP

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad -2x_1 + x_2 \leq 2 \\
& \quad x_1 + x_2 \leq 6 \\
& \quad x_1 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

a. (5%) Convert the LP to standard form:

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad -2x_1 + x_2 + x_3 = 2 \\
& \quad x_1 + x_2 + x_4 = 6 \\
& \quad x_1 + x_5 = 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

b. (25%) Starting with \(x_1, x_2\) as non-basic, solve the problem using the Simplex algorithm. Explain why you terminated the algorithm.

\[
\begin{array}{ccccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
c & 2 & 1 & 0 & 0 & 0 \\
\hline
A & -2 & 1 & 1 & 0 & 0 & 2 \\
 & 1 & 1 & 0 & 1 & 0 & 6 \\
 & 1 & 0 & 0 & 0 & 1 & 4 \\
\hline
x^{(0)} & N & N & B & B & B \\
\hline
\Delta x_1 & 1 & 0 & 2 & -1 & -1 & \text{reduced cost} = 2 \\
\Delta x_2 & 0 & 1 & -1 & -1 & 0 & \text{reduced cost} = 1 \\
\hline
x^{(1)} & B & N & B & B & N \\
\hline
\Delta x_1 & 0 & 1 & -1 & -1 & 0 & \text{reduced cost} = 1 \\
\Delta x_2 & -1 & 0 & -2 & 1 & 1 & \text{reduced cost} = -2 \\
\hline
\lambda = 6 & \lambda = 4 \\
\hline
x^{(2)} & B & B & B & N & N \\
\hline
\Delta x_1 & 0 & -1 & 1 & 1 & 0 & \text{reduced cost} = -1 \\
\Delta x_2 & -1 & 1 & -3 & 0 & 1 & \text{reduced cost} = -1 \\
\hline
\end{array}
\]

Both of the reduce costs are now negative, which means that we have found the optimal solution \((4, 2, 8, 0, 0)\), with performance of \(2 \times 4 + 1 \times 2 = 10\).
Note on grading part (b) above:

- Find the correct Simplex directions (5pt)
- Calculate the reduced cost and select a direction (5pt)
- Calculate and select the step size (5pt)
- Move to a new solution (5pt)
- Terminate the algorithm correctly (5pt)

c. (10%) From the same starting point as in b) above, find the most improving direction. Compare this direction with the Simplex direction that you chose on Step 1 of b) above. Which direction is better? Explain.

The most improving direction is the gradient \( \Delta x = [2,1,0,0,0] \) (2pt). This direction gives more immediate improvement than the simplex direction \( \Delta x^{(1)} = [1,0,2,-1,-1] \) as seen by calculating (3pt):

\[
\Delta x \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5 > 2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \Delta x^{(1)} \cdot \nabla f
\]

However, while \( \Delta x^{(1)} \) is feasible by construction, \( \Delta x \) is not feasible as can be seen from the constraints (3pt):

\[
A\Delta x = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 + 1 \\ 2 + 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Hence, the most improving direction would have to be transformed before it can be used, and neither can be said to be better (2pt).
Sensitivity Analysis

1. The NCAA is making plans for distributing tickets to the upcoming basketball championships. The up to 10,000 seats available will be divided between the media, the competing universities, and the general public. Media people are admitted free, but the NCAA receives $45 per ticket from universities and $100 per ticket from the general public. At least 500 tickets must be reserved for the media, and at least half as many tickets should go to the competing universities as to the general public. Within these restrictions, the NCAA wishes to find the allocation that raises the most money. We have formulated the following LP to solve the problem, and the LINDO output is below.

\[
\begin{align*}
\text{max} & \quad 45x_2 + 100x_3 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 10000 \\
& \quad \frac{1}{2}x_2 \geq 0 \\
& \quad x_1 \geq 500 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

OBJECTIVE FUNCTION VALUE
1) 775833.3

VARIABLE VALUE REDUCED COST
X2 3166.666748 0.000000
X3 6333.333496 0.000000
X1 500.000000 0.000000

ROW SLACK OR SURPLUS DUAL PRICES
2) 0.000000 81.666664
3) 0.000000 -36.666668
4) 0.000000 -81.666664
5) 500.000000 0.000000
6) 3166.666748 0.000000
7) 6333.333496 0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES
VARIABLE CURRENT ALLOWABLE ALLOWABLE
X2 45.000000 55.000000 244.999985
X3 100.000000 INFINITY 55.000000
X1 0.000000 81.666664 INFINITY

RIGHHAND SIDE RANGES
ROW CURRENT ALLOWABLE ALLOWABLE
RHS INCREASE DECREASE
2 10000.000000 INFINITY 9500.000000
3 0.000000 9500.000000 4750.000000
4 500.000000 9500.000000 500.000000
5 0.000000 500.000000 INFINITY
6 0.000000 3166.666748 INFINITY
7 0.000000 6333.333496 INFINITY
Please answer the following questions (all worth equal points):

a. What is the marginal cost to the NCAA of each seat guaranteed to the media?

This is simply the dual price of the third constraint = $81.67 (10pt)

b. Suppose that there is an alternative arrangement for the dome where the games will be played that can provide 15,000 seats. How much additional revenue would be gained from the expanded seating? How much would it be for 20,000 seats?

Look at the dual price of the capacity constraint = $81.67. This dual price is valid for any increase, hence the additional revenue will be (15000-10000) * $81.67 = $408,300 and $816,600, respectively (10pt).

c. Since television revenue provides most of the income for NCAA events, another proposal would reduce the price of general public tickets to $50. How much revenue would be lost from this change? What if the price were $30?

If we reduce it to $50, we still sell the same number of tickets to each party (within allowable range for basis to remain the same), so the revenue reduction is ($100 - $50) * 6333 = $316,700 (Spt).

If we reduce it to $30, the basis changes so the dual price is no longer valid (5pt). By taking the maximum allowable decrease, we can say that the profit changes by at least $55 * 6333 = $348,300 (lower bound on the decrease).

d. To accommodate high demand from student supporters of the participating universities, the NCAA is considering marketing a new “scrunch seat” that consumes only 80% of the regular bleacher seat but counts fully against the “university ≥ half public” rule. Could an optimal solution allocate any such seats at a ticket price of $35? At a price of $25?

This corresponds to adding a new variable ($x_4$), and you should think about the effect of setting this variable equal to one (2 pt for setting up the constraints):

$$x_1 + x_2 + x_3 + 0.8x_4 \leq 10000$$
$$x_2 - \frac{1}{2}x_3 + 1 \geq 0$$
$$x_1 \geq 500$$

The cost of tightening the first constrain by 0.8 is $81.67 \times 0.8 = $65.34, while the benefit of relaxing the second constraint by 1 is $36.67 (dual prices). Hence the new ticket (allowing $x_4$ to be at least one) becomes attractive at $65.34 - 36.67 = $28.67 (8 pt). Hence we would not sell those tickets at $25 but we would sell them at $35.
2. As a result of a recent decision to stop production of toy guns that look too real, the SuperSlayer Toy Company is planning to focus its production on two futuristic models: beta zappers and freeze phasers. Beta zappers produce $2.50 in profit for the company and freeze phasers $1.60. The company is contracted to sell 10 thousand beta zappers and 15 thousand freeze phasers in the next month, but all that are produced can be sold. Production of either model involves three crucial steps: extrusion, trimming, and assembly. Beta zappers use 5 hours of extrusion time per thousand units, 1 hour of trimming time, and 12 hours of assembly. Corresponding values for freeze phasers are 9, 2, and 15. There are 320 hours of extrusion time, 300 hours of trimming time, and 480 hours of assembly time available over the next month.

\[
\begin{align*}
\text{max} & \quad 2500x_1 + 1600x_2 \\
\text{s.t.} & \quad x_1 \geq 10 \\
& \quad x_2 \geq 15 \\
& \quad 5x_1 + 9x_2 \leq 320 \\
& \quad x_1 + 2x_2 \leq 300 \\
& \quad 12x_1 + 15x_2 \leq 480 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

OBJECTIVE FUNCTION VALUE
1) 77125.00

VARIABLE VALUE REDUCED COST
X1 21.250000 0.000000
X2 15.000000 0.000000

ROW SLACK OR SURPLUS DUAL PRICES
2) 21.250000 0.000000
3) 0.000000 -1525.000000
4) 78.750000 0.000000
5) 248.750000 0.000000
6) 0.000000 208.333328
7) 21.250000 0.000000
8) 15.000000 0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:
OBJ COEFFICIENT RANGES
VARIABLE CURRENT ALLOWABLE ALLOWABLE
COEF INCREASE DECREASE
X1 2500.000000 INFINITY 1220.000000
X2 1600.000000 1525.000000 INFINITY

RIGHTHAND SIDE RANGES
ROW CURRENT ALLOWABLE ALLOWABLE
RHS INCREASE DECREASE
2 0.000000 21.250000 INFINITY
3 15.000000 17.000000 15.000000
4 320.000000 INFINITY 78.750000
5 300.000000 INFINITY 248.750000
6 480.000000 189.000000 255.000000
7 0.000000 21.250000 INFINITY
8 0.000000 15.000000 INFINITY
a) Is the optimum solution sensitive to the exact value of trimming hours available? If not, at what number of hours capacity would it become relevant?

No there is a slack of 248.75. Hence, it is relevant at 300 – 248.75 = 51.25 hours.

b) How much should SuperSlayer be willing to pay for an additional hour of extrusion time? For an additional hour of assembly time?

Look at the dual prices. It is 0 and 208.33 for extrusion time and assembly time, respectively. This is what they should be willing to pay.

c) What would be the profit effect of increasing assembly capacity to 580 hours? To 680 hours?
Increase to 580 hours (increase of 100) is within the allowable increase (189), so it is simply 100 x 208.33 = $20,833

Increase to 680 is outside the allowable increase, but we can bound it with:

At least 189 x 208.33 = $39,374
At most 200 x 208.33 = $41,666

d) What would be the profit effect of increasing the profit margin on beta zappers by $1500 per thousand? What would be the effect of a decrease in that amount?

Increase of $1500 is inside the allowable increase (infinity). Hence profit would increase by $1500 x 21.25 = $31,875

Decrease of $1500 is outside the allowable decrease of 1220. However, we can again bound it. The decrease in profit would be

At least $1220 x 21.25 = $25,925