1. (30%) A food processing plant manufactures hot dogs and hot dog buns. They grind their own flour for the hot dog buns at a maximum rate of 200 pounds per week. Each hot dog bun requires 0.1 pounds of flour. They currently have a contract with a supplier that specifies that a delivery of 800 pounds of pork product is delivered every Monday. Each hot dog requires 0.25 pound of pork product. All other ingredients in the hot dogs and hot dog buns are in plentiful supply. Finally, the labor force consists of 5 employees working full time (40 hours/week). Each hot dog requires 3 minutes of labor, and each hot dog bun requires 2 minutes of labor. Each hot dog yields a profit of $0.20, and each bun yields a profit of $0.10. How many hot dogs and how many hot dog buns should be produced per week to maximize the profit?

(a) Formulate a linear programming problem for this problem.

\[
\begin{align*}
    x_1 &= \text{Number of hot dogs produced} \\
    x_2 &= \text{Number of buns produced} \\
    \text{max} &\quad 0.20x_1 + 0.10x_2 \\
    \text{s.t.} &\quad 0.1x_2 \leq 200 \quad [\text{Flour constraint}] \\
    &\quad 0.25x_1 \leq 800 \quad [\text{Pork constraint}] \\
    &\quad \frac{3}{60}x_1 + \frac{2}{60}x_2 \leq 200 \quad [\text{Labor constraint}] \\
    &\quad x_1, x_2 \geq 0
\end{align*}
\]
(b) Graph the feasible region and identify the optimal solution.
2. (35%) Consider the following LP:

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{s.t.} & \quad 3x_1 + 2x_2 \leq 6 \\
& \quad x_1 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Starting at \( \mathbf{x}^{(0)} = (0,0) \), do one iteration of improving search using the steepest ascent direction (that is, the direction that improves the fastest). Is the new solution that you found a global optimum? Is it a local optimum? Explain.

See figure below:

The most improving direction is \( \Delta \mathbf{x} = (2,1) \) [Found by taking the partial derivatives of the objective function.]

Moving in this direction will always be improving, so we move until we are no longer feasible. This will happen when we hit constraint \( 3x_1 + 2x_2 \leq 6 \) [See fig.]

Calculate

\[
3(2\lambda) + 2(\lambda) = 6
\]

\[
\Rightarrow \lambda = \frac{6}{8} = 0.75
\]

Hence, the next solution is \( \mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \lambda \Delta \mathbf{x} = (0,0) + 0.75 \cdot (2,1) = (1.5,0.75) \).

This is not a local optimum (and hence not a global optimum), because there still exists a feasible improving direction, for example, \( \Delta \mathbf{x} = (0.5,-0.75) \) that is improving because \( (0.5,-0.75) \cdot (2,1) = 1-0.75 = 0.25 > 0 \).
3. (35%) Classic Candles makes three models of elegant Christmas candles by hand. Santa models require 0.10 day of molding, 0.35 days of decorating, and 0.08 day of packaging and produce $16 of profit per unit sold. Corresponding values for the Christmas tree model are 0.10, 0.15, 0.03, and $9, while those of the gingerbread house model are 0.25, 0.40, 0.05, and $27. Classic Candles wants to maximize profit on what it makes over the next 20 working days with its 1 molder, 3 decorators, and 1 packager, assuming that everything that is made can be sold. Formulate a LP to find the optimal production plan for Classic Candles.

\[
\begin{align*}
    x_1 &= \text{Number of Santa models produced} \\
    x_2 &= \text{Number of Christmas tree model produced} \\
    x_3 &= \text{Number of gingerbread house model produced} \\
    \text{max} & \quad 16x_1 + 9x_2 + 27x_3 \\
    \text{s.t.} & \quad 0.10x_1 + 0.10x_2 + 0.25x_3 \leq 20 \quad \text{[Molding constraint]} \\
    & \quad 0.35x_1 + 0.15x_2 + 0.40x_3 \leq 60 \quad \text{[Decorating constraint]} \\
    & \quad 0.08x_1 + 0.03x_2 + 0.05x_3 \leq 20 \quad \text{[Packaging constraint]} \\
    & \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Alternatively, you can specify the constraints in number of hours (instead of days), assuming 8-hour workdays:

\[
\begin{align*}
    x_1 &= \text{Number of Santa models produced} \\
    x_2 &= \text{Number of Christmas tree model produced} \\
    x_3 &= \text{Number of gingerbread house model produced} \\
    \text{max} & \quad 16x_1 + 9x_2 + 27x_3 \\
    \text{s.t.} & \quad 0.80x_1 + 0.80x_2 + 2.00x_3 \leq 160 \quad \text{[Molding constraint]} \\
    & \quad 2.80x_1 + 1.20x_2 + 3.20x_3 \leq 480 \quad \text{[Decorating constraint]} \\
    & \quad 0.64x_1 + 0.24x_2 + 0.40x_3 \leq 160 \quad \text{[Packaging constraint]} \\
    & \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
