Rotation Of Axes
And
The General Form Of A Conic Section

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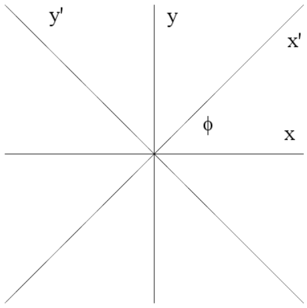
Supplement for Math 142

Rotation Of Axes

Consider a set of axes (x', y') formed by rotating the standard (x, y) axes through an angle ϕ. If a point has coordinates (a, b) relative to the original (x, y) axes and coordinates (a', b') relative to the new (x', y') axes, we can find the relation between these coordinates.



If the point has polar coordinates (r, θ) in the rotated system, then it has polar coordinates $(r, \theta + \phi)$ in the original system. Then

$$
\begin{aligned}
a' &= r\cos\theta & b' &= r\sin\theta \\
a &= r\cos(\theta+\phi) & b &= r\sin(\theta+\phi) \\
&= r\cos\theta\cos\phi - r\sin\theta\sin\phi & &= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= a'\cos\phi - b'\sin\phi & &= b'\cos\phi + a'\sin\phi
\end{aligned}
$$

Solving the linear system gives the formulas for going the other way. Thus we have

$$
\boxed{
\begin{aligned}
a &= a'\cos\phi - b'\sin\phi & a' &= a\cos\phi + b\sin\phi \\
b &= a'\sin\phi + b'\cos\phi & b' &= -a\sin\phi + b\cos\phi
\end{aligned}
}
$$

The General Form Of A Conic Section

The expression $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where the constants $A, B, C, D, E,$ and F are real numbers, is the general form of a conic section. In particular, we do not want $A, B,$ and C all equal to 0. If $A = B = C = 0$, we have a degenerate conic section.

The Discriminant Of A Conic Section

The quantity $\eta = B^2 - 4AC$ is called the discriminant of the conic section. This value is invariant (unchanged) for rotation. In other words, if we rotate the conic section through any angle ϕ, the discriminant is unchanged. We can use η to tell what type of conic section we are looking at.

I) $\eta < 0$, the conic section is an ellipse. In particular if $B = 0$ and $A = C$, the conic section is a circle.

II) $\eta = 0$, the conic section is a parabola.

III) $\eta > 0$, the conic section is a hyperbola.

Graphing The General Form On A Graphic Calculator

If we wish to use a graphic calculator for graphing an arbitrary conic section, we must solve the general form for y. We can do this by use of the quadratic formula. First we collect terms to get a quadratic in y (we treat x as a fixed value in solving this).

1)General Form	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$	$= 0$
2)Collect Terms and Factor	$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F)$	$= 0$
3)Quadratic Formula	$y = \frac{-(Bx+E) \pm \sqrt{(Bx+E)^2 - 4C(Ax^2+Dx+F)}}{2C}$	

Thus, to graph a general conic section we need to graph the two equations:

1) $y = \frac{-(Bx+E)}{2C} + \frac{\sqrt{(Bx+E)^2-4C(Ax^2+Dx+F)}}{2C}$

2) $y = \frac{-(Bx+E)}{2C} - \frac{\sqrt{(Bx+E)^2-4C(Ax^2+Dx+F)}}{2C}$

Rotating A Conic Section (Eliminating The xy Term)

Our concern with rotating a conic section is to eliminate the xy term in the general form. This term is going to cause difficulty in recognizing properties of the conic. However, by eliminating it through a rotation, we can get a more familiar conic. We can identify any properties that we wish to find for the conic with this simplified form, and then reverse the rotation to carry them back to our original conic.

To find the rotation that will accomplish this, we see what happens if we rotate through the angle ϕ. Substituting the rotation formulas into the general form, we obtain

$$\begin{array}{llllll} & A(x'\cos\phi - y'\sin\phi)^2 & + & B(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) & & \\ + & C(x'\sin\phi + y'\cos\phi)^2 & + & D(x'\cos\phi - y'\sin\phi) & & \\ + & E(x'\sin\phi + y'\cos\phi) & + & F & = & 0 \end{array}$$

Collecting terms, expanding, and factoring to match this with the general form, we get a new conic section $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ where

$$\begin{aligned} A' &= A\cos^2\phi + B\cos\phi\sin\phi + C\sin^2\phi \\ B' &= 2(C-A)\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi) \\ C' &= A\sin^2\phi - B\sin\phi\cos\phi + C\cos^2\phi \\ D' &= D\cos\phi + E\sin\phi \\ E' &= E\cos\phi - D\sin\phi \\ F' &= F \end{aligned}$$

We wish to find an angle ϕ such that $B' = 0$. We now need to solve this to find such a ϕ.

$$\begin{aligned} 2(C-A)\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi) &= 0 \\ (C-A)\sin(2\phi) + B\cos(2\phi) &= 0 \\ B\cos(2\phi) &= (A-C)\sin(2\phi) \\ B\cot(2\phi) &= A-C \\ \cot(2\phi) &= \tfrac{A-C}{B} \end{aligned}$$

Thus, assuming that $B \neq 0$, we need to pick ϕ such that $\cot(2\phi) = \frac{A-C}{B}$. If $B = 0$, we do not need to rotate at all. Picking ϕ in this way and then rotating through the angle ϕ will eliminate the xy term in the general form. We can identify any desired properties of the conic in this rotated system, and then rotate back in order to find the desired properties of our original conic.