The following notation will facilitate the calculation of $\beta$. Let $\mu_0$ denote the null value of $\mu$ and let $\mu_a$ denote the actual value of the mean in $H_a$. Let $\beta(\mu_a)$ be the probability of a Type II error if the actual value of the mean is $\mu_a$ and let $\text{PWR}(\mu_a)$ be the power at $\mu_a$. Note that $\text{PWR}(\mu_a) = 1 - \beta(\mu_a)$. Although we never really know the actual mean, we select feasible values of $\mu$ and determine $\beta$ for each of these values. This will allow us to determine the probability of a Type II error occurring if one of these feasible values happens to be the actual value of the mean.

The decision whether or not to accept $H_0$ depends on the magnitude of $\beta$ for one or more reasonable values for $\mu_a$. Alternatively, researchers calculate the power curve for a test of hypotheses. Recall that the power of the test at $\mu_a$ $\text{PWR}(\mu_a)$ is the probability the test will detect that $H_0$ is false when the actual value of $\mu$ is $\mu_a$. Hence, we want tests of hypotheses in which $\text{PWR}(\mu_a)$ is large when $\mu_a$ is far from $\mu_0$.

For a one-tailed test, $H_0: \mu \leq \mu_0$ or $H_0: \mu \geq \mu_0$, the value of $\beta$ at $\mu_a$ is the probability that $z$ is less than

$$z_\alpha - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}}$$

This probability is written as

$$\beta(\mu_a) = P \left[ z < z_\alpha - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}} \right]$$

The value of $\beta(\mu_a)$ is found by looking up the probability corresponding to the number $z_\alpha - |\mu_0 - \mu_a|/\sigma/\sqrt{n}$ in Table 1 in the Appendix.

Formulas for $\beta$ are given here for one- and two-tailed tests. Examples using these formulas follow.

1. One-tailed test:

$$\beta(\mu_a) = P \left( z \leq z_\alpha - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}} \right) \quad \text{PWR}(\mu_a) = 1 - \beta(\mu_a)$$

2. Two-tailed test:

$$\beta(\mu_a) \approx P \left( z \leq z_{\alpha/2} - \frac{|\mu_0 - \mu_a|}{\sigma/\sqrt{n}} \right) \quad \text{PWR}(\mu_a) = 1 - \beta(\mu_a)$$
**EXAMPLE 5.8**

Compute \( \beta \) and power for the test in Example 5.7 if the actual mean number of improperly issued tickets is 395.

**Solution** The research hypothesis for Example 5.7 was \( H_0: \mu > 380 \). Using \( \alpha = .01 \) and the computing formula for \( \beta \) with \( \mu_0 = 380 \) and \( \mu_a = 395 \), we have

\[
\beta(395) = P \left[ z < z_{.01} \left| \frac{\mu_0 - \mu_a}{\sigma/\sqrt{n}} \right. \right] = P \left[ z < 2.33 \left| \frac{380 - 395}{35.2/\sqrt{50}} \right. \right]
\]

\[
= P[z < 2.33 - 3.01] = P[z < -.68]
\]

Referring to Table 1 in the Appendix, the area corresponding to \( z = -.68 \) is .2483. Hence, \( \beta(395) = .2483 \) and \( \text{PWR}(395) = 1 - .2483 = .7517 \).

Previously, when \( \bar{y} \) did not fall in the rejection region, we concluded that there was insufficient evidence to reject \( H_0 \) because \( \beta \) was unknown. Now when \( \bar{y} \) falls in the acceptance region, we can compute \( \beta \) corresponding to one (or more) alternative values for \( \mu \) that appear reasonable in light of the experimental setting. Then provided we are willing to tolerate a probability of falsely accepting the null hypothesis equal to the computed value of \( \beta \) for the alternative value(s) of \( \mu \) considered, our decision is to accept the null hypothesis. Thus, in Example 5.8, if the actual mean number of improperly issued tickets is 395, then there is about a .25 probability (1 in 4 chance) of accepting the hypothesis that \( \mu \) is less than or equal to 380 when in fact \( \mu \) equals 395. The city manager would have to analyze the consequence of making such a decision. If the risk was acceptable then she could state that the audit has determined that the mean number of improperly issued tickets has not increased. If the risk is too great, then the city manager would have to expand the audit by sampling more than 50 officers. In the next section, we will describe how to select the proper value for \( n \).
EXAMPLE 5.9

As the public concern about bacterial infections increases, a soap manufacture quickly promoted a new product to meet the demand for an antibacterial soap. This new product has a substantially higher price than the “ordinary soaps” on the market. A consumer testing agency notes that ordinary soap also kills bacteria and questions whether the new antibacterial soap is a substantial improvement over ordinary soap. A procedure for examining the ability of soap to kill bacteria is to place a solution containing the soap onto a petri dish and then add E. coli bacteria. After a 24-hour incubation period, a count of the number of bacteria colonies on the dish is taken. From previous studies using many different brands of ordinary soaps, the mean bacteria count is 33 for ordinary soap products. The consumer group runs the test on the antibacterial soap using 35 petri dishes. The results from the 35 petri dishes is a mean bacterial count of 31.2 with a standard deviation of 8.4. Do the data provide sufficient evidence that the antibacterial soap is more effective than ordinary soap in reducing bacteria counts? Use $\alpha = .05$.

Solution  Let $\mu$ be the population mean bacterial count for the antibacterial soap and $\sigma$ be the population standard deviation. The 5 parts to our statistical test are as follows.

\[ H_0: \quad \mu \geq 33 \]
\[ H_a: \quad \mu < 33 \]

T.S.: \[ z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} = \frac{31.2 - 33}{8.4/\sqrt{35}} = -1.27 \]

R.R.: For $\alpha = .05$, we will reject the null hypothesis if $z \leq -z_{.05} = -1.645$.

Check assumptions and draw conclusions: With $n = 35$, the sample size is probably large enough that the Central Limit Theorem would justify our assuming that the sampling distribution of $\bar{y}$ is approximately normal. Because the observed value of $z$, $-1.27$, is not less than $-1.645$, the test statistic does not fall in the rejection region. We reserve judgment on accepting $H_0$ until we calculate the chance of a Type II error $\beta$ for several values of $\mu$. falling in the alternative hypothesis, values of $\mu$ less than 33. In other words, we conclude that there is insufficient evidence to reject the null hypothesis and hence there is not sufficient evidence that the antibacterial soap is more effective than ordinary soap. However, we next need to calculate the chance that the test may have resulted in a Type II error.

EXAMPLE 5.10

Refer to Example 5.9. Suppose that the consumer testing agency thinks that the manufacturer of the antibacterial soap will take legal action if the antibacterial soap has a population mean bacterial count that is considerably less than 33, say 28. Thus, the consumer group wants to know the probability of a Type II error in its test if the population mean $\mu$ is 28 or smaller—that is, determine $\beta(28)$ because $\beta(\mu) \leq \beta(28)$ for $\mu \leq 28$. 

Solution Using the computational formula for $\beta$ with $\mu_0 = 33$, $\mu_a = 28$, and $\alpha = .05$, we have

$$
\beta(38) = P \left[ z \leq z_{0.05} - \frac{\mu_0 - \mu_a}{\sigma / \sqrt{n}} \right] = P \left[ z \leq 1.645 - \frac{33 - 28}{8.4 / \sqrt{35}} \right] \\
= P[z \leq -1.88]
$$

The area corresponding to $z = -1.88$ in Table 1 of the Appendix is .0301. Hence,

$$
\beta(28) = .0301 \quad \text{and} \quad \text{PWR}(28) = 1 - .0301 = .9699
$$

Because $\beta$ is relatively small, we accept the null hypothesis and conclude that the antibacterial soap is not more effective than ordinary soap in reducing bacteria counts.

The manufacturer of the antibacterial soap wants to determine the chance that the consumer group may have made an error in reaching its conclusions. The manufacturer wants to compute the probability of a Type II error for a selection of potential values of $\mu$ in $H_a$. This would provide them with an indication of how likely a Type II error may have occurred when in fact the new soap is considerably more effective in reducing bacterial counts in comparison to the mean count for ordinary soap, $\mu = 33$. Repeating the calculations for obtaining $\beta(28)$, we obtain the values in Table 5.4.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>33</th>
<th>32</th>
<th>31</th>
<th>30</th>
<th>29</th>
<th>28</th>
<th>27</th>
<th>26</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta(\mu)$</td>
<td>.9500</td>
<td>.8266</td>
<td>.5935</td>
<td>.3200</td>
<td>.1206</td>
<td>.0301</td>
<td>.0049</td>
<td>.0005</td>
<td>.0000</td>
</tr>
<tr>
<td>PWR(\mu)</td>
<td>.0500</td>
<td>.1734</td>
<td>.4065</td>
<td>.6800</td>
<td>.8794</td>
<td>.9699</td>
<td>.9951</td>
<td>.9995</td>
<td>.9999</td>
</tr>
</tbody>
</table>

Figure 5.11 is a plot of the $\beta(\mu)$ values in Table 5.4 with a smooth curve through the points. Note that as the value of $\mu$ decreases, the probability of Type II error decreases to 0 and the corresponding power value increases to 1.0. The company could examine this curve to determine whether the chances of Type II error are reasonable for values of $\mu$ in $H_a$ that are important to the company. From Table 5.4 or Figure 5.11, we observe that $\beta(28) = .0301$, a relatively small number. Based on the results from Example 5.9, we find that the test statistic does not fall in the rejection region. The manufacturer has decided that if the true population