Real Analysis
by
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Contents

1 Set Theory

1.1 Introduction

1. If $\{x : x \neq x\} \neq \emptyset$, then there exists x such that $x \neq x$. Contradiction.

2. $\emptyset \subset \{\text{green-eyed lions}\}$.

3. $X \times (Y \times Z) = \{\langle x, \langle y, z\rangle\rangle\}$, $(X \times Y) \times Z = \{\langle\langle x, y\rangle, z\rangle\}$; $\langle x, \langle y, z\rangle\rangle \leftrightarrow \langle\langle x, y\rangle, z\rangle \leftrightarrow \langle x, y, z\rangle$.

4. Suppose $P(1)$ is true and $P(n) \Rightarrow P(n+1)$ for all n. Suppose that $\{n \in \mathbb{N} : P(n) \text{ is false}\} \neq \emptyset$. Then it has a smallest element m. In particular, $m > 1$ and $P(m)$ is false. But $P(1) \Rightarrow P(2) \Rightarrow \cdots \Rightarrow P(m)$. Contradiction.

5. Given a nonempty subset S of natural numbers, let $P(n)$ be the proposition that if there exists $m \in S$ with $m \leq n$, then S has a smallest element. $P(1)$ is true since 1 will then be the smallest element of S. Suppose that $P(n)$ is true and that there exists $m \in S$ with $m \leq n+1$. If $m \leq n$, then S has a smallest element by the induction hypothesis. If $m = n+1$, then either m is the smallest element of S or there exists $m' \in S$ with $m' < m = n+1$, in which case the induction hypothesis again gives a smallest element.

1.2 Functions

6. ($\Rightarrow$) Suppose f is one-to-one. For each $y \in f[X]$, there exists a unique $x_y \in X$ such that $f(x_y) = y$. Fix $x_0 \in X$. Define $g : Y \to X$ such that $g(y) = x_y$ if $y \in f[X]$ and $g(y) = x_0$ if $y \in Y \setminus f[X]$. Then g is a well-defined function and $g \circ f = id_X$.

($\Leftarrow$) Suppose there exists $g : Y \to X$ such that $g \circ f = id_X$. If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$. i.e. $x_1 = x_2$. Thus f is one-to-one.

7. ($\Rightarrow$) Suppose f is onto. For each $y \in Y$, there exists $x_y \in X$ such that $f(x_y) = y$. Define $g : Y \to X$ such that $g(y) = x_y$ for all $y \in Y$. Then g is a well-defined function and $f \circ g = id_Y$.

($\Leftarrow$) Suppose there exists $g : Y \to X$ such that $f \circ g = id_Y$. Given $y \in Y$, $g(y) \in X$ and $f(g(y)) = y$. Thus f is onto.

8. Let $P(n)$ be the proposition that for each n there is a unique finite sequence $\langle x_1^{(n)}, \ldots, s_n^{(n)}\rangle$ with $x_1^{(n)} = a$ and $x_{i+1}^{(n)} = f_i(x_1^{(n)}, \ldots, x_i^{(n)})$. Clearly $P(1)$ is true. Given $P(n)$, we see that $P(n+1)$ is true by letting $x_i^{(n+1)} = x_i^{(n)}$ for $1 \leq i \leq n$ and letting $x_{n+1}^{(n+1)} = f_n(x_1^{(n+1)}, \ldots, x_n^{(n+1)})$. By letting $x_n = x_n^{(n)}$ for each n, we get a unique sequence $\langle x_i\rangle$ from X such that $x_1 = a$ and $x_{i+1} = f_i(x_1, \ldots, x_i)$.

1.3 Unions, intersections and complements

9. $A \subset B \Rightarrow A \subset A \cap B \Rightarrow A \cap B = A \Rightarrow A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A) = (A \cap B) \cup (B \setminus A) = B \Rightarrow A \subset A \cup B = B$.

10. $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in B$ or $C \Leftrightarrow x \in A$ and B or $x \in A$ and $C \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B$ and $C \Leftrightarrow x \in A$ or B and $x \in A$ or $C \Leftrightarrow x \in (A \cup B) \cap (A \cup C)$. Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

11. Suppose $A \subset B$. If $x \notin B$, then $x \notin A$. Thus $B^c \subset A^c$. Conversely, if $B^c \subset A^c$, then $A = (A^c)^c \subset (B^c)^c = B$.

12a. $A \Delta B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \Delta A$.

$A\Delta(B\Delta C) = [A \setminus ((B \setminus C) \cup (C \setminus B))] \cup [((B \setminus C) \cup (C \setminus B)) \setminus A] = [A \cap ((B \setminus C) \cup (C \setminus B))^c] \cup [((B \setminus C) \cup (C \setminus B)) \cap A^c] = [A \cap ((B^c \cup C) \cap (C^c \cup B))] \cup [((B \cap C^c) \cup (C \cap B^c)) \cap A^c] = [A \cap (B^c \cup C) \cap (B \cup C^c)] \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) = (A \cap B^c \cap C^c) \cup (A \cap B \cap C) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$.

$(A\Delta B)\Delta C = [((A \setminus B) \cup (B \setminus A)) \setminus C] \cup [C \setminus ((A \setminus B) \cup (B \setminus A))] = [((A \setminus B) \cup (B \setminus A)) \cap C^c] \cup [C \cap ((A \setminus B) \cup (B \setminus A))^c] = [((A \cap B^c) \cup (B \cap A^c)) \cap C^c] \cup [C \cap ((A^c \cup B) \cap (B^c \cup A))] = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$.

Hence $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

12b. $A \Delta B = \emptyset \Leftrightarrow (A \setminus B) \cup (B \setminus A) = \emptyset \Leftrightarrow A \setminus B = \emptyset$ and $B \setminus A = \emptyset \Leftrightarrow A \subset B$ and$B \subset A \Leftrightarrow A = B$.

12c. $A\Delta B = X \Leftrightarrow (A \setminus B) \cup (B \setminus A) = X \Leftrightarrow A \cap B = \emptyset$ and $A \cup B = X \Leftrightarrow A = B^c$.

12d. $A\Delta\emptyset = (A \setminus \emptyset) \cup (\emptyset \setminus A) = A \cup \emptyset = A$; $A\Delta X = (A \setminus X) \cup (X \setminus A) = \emptyset \cup A^c = A^c$.

12e. $(A\Delta B) \cap E = ((A \setminus B) \cup (B \setminus A)) \cap E = ((A \setminus B) \cap E) \cup ((B \setminus A) \cap E) = [(A \cap E) \setminus (B \cap E)] \cup [(B \cap E) \setminus (A \cap E)] = (A \cap E)\Delta(B \cap E)$.

13. $x \in (\bigcup_{A\in\mathcal{C}} A)^c \Leftrightarrow x \notin A$ for any $A \in \mathcal{C} \Leftrightarrow x \in A^c$ for all $A \in \mathcal{C} \Leftrightarrow x \in \bigcap_{A\in\mathcal{C}} A^c$.

$x \in (\bigcap_{A\in\mathcal{C}} A)^c \Leftrightarrow x \notin \bigcap_{A\in\mathcal{C}} A \Leftrightarrow x \in A^c$ for some $A \in \mathcal{C} \Leftrightarrow x \in \bigcup_{A\in\mathcal{C}} A^c$.

14. $x \in B \cap (\bigcup_{A\in\mathcal{C}} A) \Leftrightarrow x \in B$ and $x \in A$ for some $A \in \mathcal{C} \Leftrightarrow x \in B \cap A$ for some $A \in \mathcal{C} \Leftrightarrow x \in \bigcup_{A\in\mathcal{C}}(B \cap A)$. Thus $B \cap (\bigcup_{A\in\mathcal{C}} A) = \bigcup_{A\in\mathcal{C}}(B \cap A)$.

15. $(\bigcup_{A\in\mathfrak{A}} A) \cap (\bigcup_{B\in\mathfrak{B}} B) = \bigcup_{B\in\mathfrak{B}}((\bigcup_{A\in\mathfrak{A}} A) \cap B) = \bigcup_{B\in\mathfrak{B}}(\bigcup_{A\in\mathfrak{A}}(A \cap B)) = \bigcup_{A\in\mathfrak{A}}\bigcup_{B\in\mathfrak{B}}(A \cap B)$.

16a. If $x \in \bigcup A_\lambda$, then $x \in A_{\lambda_0}$ for some λ_0 and $f(x) \in f[A_{\lambda_0}] \subset \bigcup f[A_\lambda]$. Thus $f[\bigcup A_\lambda] \subset \bigcup f[A_\lambda]$. Conversely, if $y \in \bigcup f[A_\lambda]$, then $y \in f[A_{\lambda_0}]$ for some λ_0 so $y \in f[\bigcup A_\lambda]$. Thus $\bigcup f[A_\lambda] \subset f[\bigcup A_\lambda]$.

16b. If $x \in \bigcap A_\lambda$, then $x \in A_\lambda$ for all λ and $f(x) \in f[A_\lambda]$ for all λ. Thus $f(x) \in \bigcap f[A_\lambda]$ and $f[\bigcap A_\lambda] \subset \bigcap f[A_\lambda]$.

16c. Consider $f : \{1, 2, 3\} \to \{1, 3\}$ with $f(1) = f(2) = 1$ and $f(3) = 3$. Let $A_1 = \{1, 3\}$ and $A_2 = \{2, 3\}$. Then $f[A_1 \cap A_2] = f[\{3\}] = \{3\}$ but $f[A_1] \cap f[A_2] = \{1, 3\}$.

17a. If $x \in f^{-1}[\bigcup B_\lambda]$, then $f(x) \in B_{\lambda_0}$ for some λ_0 so $x \in f^{-1}[B_{\lambda_0}] \subset \bigcup f^{-1}[B_\lambda]$. Thus $f^{-1}[\bigcup B_\lambda] \subset \bigcup f^{-1}[B_\lambda]$. Conversely, if $x \in \bigcup f^{-1}[B_\lambda]$, then $x \in f^{-1}[B_{\lambda_0}]$ for some λ_0 so $f(x) \in B_{\lambda_0} \subset \bigcup B_\lambda$ and $x \in f^{-1}[\bigcup B_\lambda]$.

17b. If $x \in f^{-1}[\bigcap B_\lambda]$, then $f(x) \in B_\lambda$ for all λ and $x \in f^{-1}[B_\lambda]$ for all λ so $x \in \bigcap f^{-1}[B_\lambda]$. Thus $f^{-1}[\bigcap B_\lambda] \subset \bigcap f^{-1}[B_\lambda]$. Conversely, if $x \in \bigcap f^{-1}[B_\lambda]$, then $x \in f^{-1}[B_\lambda]$ for all λ and $f(x) \in \bigcap B_\lambda$ so $x \in f^{-1}[\bigcap B_\lambda]$.

17c. If $x \in f^{-1}[B^c]$, then $f(x) \notin B$ so $x \notin f^{-1}[B]$. i.e. $x \in (f^{-1}[B])^c$. Thus $f^{-1}[B^c] \subset (f^{-1}[B])^c$. Conversely, if $x \in (f^{-1}[B])^c$, then $x \notin f^{-1}[B]$ so $f(x) \in B^c$. i.e. $x \in f^{-1}[B^c]$. Thus $(f^{-1}[B])^c \subset f^{-1}[B^c]$.

18a. If $y \in f[f^{-1}[B]]$, then $y = f(x)$ for some $x \in f^{-1}[B]$. Since $x \in f^{-1}[B]$, $f(x) \in B$. i.e. $y \in B$. Thus $f[f^{-1}[B]] \subset B$. If $x \in A$, then $f(x) \in f[A]$ so $x \in f^{-1}[f[A]]$. Thus $f^{-1}[f[A]] \supset A$.

18b. Consider $f : \{1, 2\} \to \{1, 2\}$ with $f(1) = f(2) = 1$. Let $B = \{1, 2\}$. Then $f[f^{-1}[B]] = f[B] = \{1\} \subsetneq B$. Let $A = \{1\}$. Then $f^{-1}[f[A]] = f^{-1}[\{1\}] = \{1, 2\} \supsetneq A$.

18c. If $y \in B$, then there exists $x \in X$ such that $f(x) = y$. In particular, $x \in f^{-1}[B]$ and $y \in f[f^{-1}[B]]$. Thus $B \subset f[f^{-1}[B]]$. This, together with the inequality in Q18a, gives equality.

1.4 Algebras of sets

19a. $\mathcal{P}(X)$ is a σ-algebra containing $\mathcal{C}$. Let $\mathcal{F}$ be the family of all σ-algebras containing $\mathcal{C}$ and let $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\}$. Then $\mathcal{A}$ is a σ-algebra containing $\mathcal{C}$. Furthermore, by definition, if $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$, then $\mathcal{B} \supset \mathcal{A}$.

19b. Let $\mathcal{B}_1$ be the σ-algebra generated by $\mathcal{C}$ and let $\mathcal{B}_2$ be the σ-algebra generated by $\mathcal{A}$. $\mathcal{B}_1$, being a σ-algebra, is also an algebra so $\mathcal{B}_1 \supset \mathcal{A}$. Thus $\mathcal{B}_1 \supset \mathcal{B}_2$. Conversely, since $\mathcal{C} \subset \mathcal{A}$, $\mathcal{B}_1 \subset \mathcal{B}_2$. Hence $\mathcal{B}_1 = \mathcal{B}_2$.

20. Let $\mathcal{A}'$ be the union of all σ-algebras generated by countable subsets of $\mathcal{C}$. If $E \in \mathcal{C}$, then E is in the σ-algebra generated by $\{E\}$. Thus $\mathcal{C} \subset \mathcal{A}'$. If $E_1, E_2 \in \mathcal{A}'$, then E_1 is in some σ-algebra generated by some countable subset $\mathcal{C}_1$ of $\mathcal{C}$ and E_2 is in some σ-algebra generated by some countable subset $\mathcal{C}_2$ of $\mathcal{C}$. Then $E_1 \cup E_2$ is in the σ-algebra generated by the countable subset $\mathcal{C}_1 \cup \mathcal{C}_2$ so $E_1 \cup E_2 \in \mathcal{A}'$. If $F \in \mathcal{A}'$, then F is in some σ-algebra generated by some countable set and so is F^c. Thus $F^c \in \mathcal{A}'$. Furthermore, if $\langle E_i \rangle$ is a sequence in $\mathcal{A}'$, then each E_i is in some σ-algebra generated by some countable subset $\mathcal{C}_i$ of $\mathcal{C}$. Then $\bigcup E_i$ is in the σ-algebra generated by the countable subset $\bigcup \mathcal{C}_i$. Hence $\mathcal{A}'$ is a σ-algebra containing $\mathcal{C}$ and it contains $\mathcal{A}$.

1.5 The axiom of choice and infinite direct products

21. For each $y \in Y$, let $A_y = f^{-1}[\{y\}]$. Consider the collection $\mathfrak{A} = \{A_y : y \in Y\}$. Since f is onto, $A_y \neq \emptyset$ for all y. By the axiom of choice, there is a function F on $\mathfrak{A}$ such that $F(A_y) \in A_y$ for all $y \in Y$.

i.e. $F(A_y) \in f^{-1}[\{y\}]$ so $f(F(A_y)) = y$. Define $g : Y \to X, y \mapsto F(A_y)$. Then $f \circ g = id_Y$.

1.6 Countable sets

22. Let $E = \{x_1, \ldots, x_n\}$ be a finite set and let $A \subset E$. If $A = \emptyset$, then A is finite by definition. If $A \neq \emptyset$, choose $x \in A$. Define a new sequence $\langle y_1, \ldots, y_n \rangle$ by setting $y_i = x_i$ if $x_i \in A$ and $y_i = x$ if $x_i \notin A$. Then A is the range of $\langle y_1, \ldots, y_n \rangle$ and is therefore finite.

23. Consider the mapping $\langle p, q, 1 \rangle \mapsto p/q, \langle p, q, 2 \rangle \mapsto -p/q, \langle 1, 1, 3 \rangle \mapsto 0$. Its domain is a subset of the set of finite sequences from $\mathbb{N}$, which is countable by Propositions 4 and 5. Thus its range, the set of rational numbers, is countable.

24. Let f be a function from $\mathbb{N}$ to E. Then $f(v) = \langle a_{vn} \rangle_{n=1}^{\infty}$ with $a_{vn} = 0$ or 1 for each n. Let $b_v = 1 - a_{vv}$ for each v. Then $\langle b_n \rangle \in E$ but $\langle b_n \rangle \neq \langle a_{vn} \rangle$ for any v. Thus E cannot be the range of any function from $\mathbb{N}$ and E is uncountable.

25. Let $E = \{x : x \notin f(x)\} \subset X$. If E is in the range of f, then $E = f(x_0)$ for some $x_0 \in X$. Now if $x_0 \notin E$, then $x_0 \in f(x_0) = E$. Contradiction. Similarly when $x_0 \in E$. Hence E is not in the range of f.

26. Let X be an infinite set. By the axiom of choice, there is a choice function $F : \mathcal{P}(X) \setminus \{\emptyset\} \to X$. Pick $a \in X$. For each $n \in \mathbb{N}$, let $f_n : X^n \to X$ be defined by $f_n(x_1, \ldots, x_n) = F(X \setminus \{x_1, \ldots, x_k\})$. By the generalised principle of recursive definition, there exists a unique sequence $\langle x_i \rangle$ from X such that $x_1 = a, x_{i+1} = f_i(x_1, \ldots, x_i)$. In particular, $x_{i+1} = F(X \setminus \{x_1, \ldots, x_i\}) \in X \setminus \{x_1, \ldots, x_i\}$ so $x_i \neq x_j$ if $i \neq j$ and the range of the sequence $\langle x_i \rangle$ is a countably infinite subset of X.

1.7 Relations and equivalences

27. Let $F, G \in Q = X/\equiv$. Choose $x_1, x_2 \in F$ and $y_1, y_2 \in G$. Then $x_1 \equiv x_2$ and $y_1 \equiv y_2$ so $x_1 + x_2 \equiv y_1 + y_2$. Thus $y_1 + y_2 \in E_{x_1+x_2}$ and $E_{x_1+x_2} = E_{y_1+y_2}$.

28. Suppose $\equiv$ is compatible with $+$. Then $x \equiv x'$ implies $x + y \equiv x' + y$ since $y \equiv y$. Conversely, suppose $x \equiv x'$ implies $x + y \equiv x' + y$. Now suppose $x \equiv x'$ and $y \equiv y'$. Then $x + y \equiv x + y'$ and $x + y' \equiv x' + y'$ so $x + y \equiv x' + y'$.

Let $E_1, E_2, E_3 \in Q = X/\equiv$. Choose $x_i \in E_i$ for $i = 1, 2, 3$. Then $(E_1 + E_2) + E_3 = E_{x_1+x_2} + E_3 = E_{(x_1+x_2)+x_3}$ and $E_1 + (E_2 + E_2) = E_1 + E_{x_2+x_3} = E_{x_1+(x_2+x_3)}$. Since X is a group under $+$, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$ so $+$ is associative on Q. Let $\mathbf{0}$ be the identity of X. For any $F \in Q$, choose $x \in F$. Then $F + E_{\mathbf{0}} = E_{x+\mathbf{0}} = E_x = F$. Similarly for $E_{\mathbf{0}} + F$. Thus $E_{\mathbf{0}}$ is the identity of Q. Let $F \in Q$ and choose $x \in F$ and let $-x$ be the inverse of x in X. Then $F + E_{-x} = E_{x+(-x)} = E_{\mathbf{0}}$. Similarly for $E_{-x} + F$. Thus E_{-x} is the inverse of F in Q. Hence the induced operation $+$ makes the quotient space Q into a group.

1.8 Partial orderings and the maximal principle

29. Given a partial order $\prec$ on X, define $x < y$ if $x \prec y$ and $x \neq y$. Also define $x \leq y$ if $x \prec y$ or $x = y$. Then $<$ is a strict partial order on X and $\leq$ is a reflexive partial order on X. Furthermore, $x \prec y \Leftrightarrow x < y \Leftrightarrow x \leq y$ for $x \neq y$. Uniqueness follows from the definitions.

30. Consider the set $(0, 1] \cup [2, 3)$ with the ordering $\prec$ given by $x \prec y$ if and only if either $x, y \in (0, 1]$ and $x < y$, or $x, y \in [2, 3)$ and $x < y$. Then $\prec$ is a partial ordering on the set, 2 is the unique minimal element and there is no smallest element.

1.9 Well ordering and the countable ordinals

31a. Let X be a well-ordered set and let $A \subset X$. Any strict linear ordering on X is also a strict linear ordering on A and every nonempty subset of A is also a nonempty subset of X. Thus any nonempty subset of A has a smallest element.

31b. Let $<$ be a partial order on X with the property that every nonempty subset has a least element. For any two elements $x, y \in X$, the set $\{x, y\}$ has a least element. If the least element is x, then $x < y$. If the least element is y, then $y < x$. Thus $<$ is a linear ordering on X with the property that every nonempty subset has a least element and consequently a well ordering.

32. Let $Y = \{x : x < \Omega\}$ and let E be a countable subset of Y. Y is uncountable so $Y \setminus E$ is nonempty. If for every $y \in Y \setminus E$ there exists $x_y \in E$ such that $y < x_y$, then $y \mapsto x_y$ defines a mapping from $Y \setminus E$ into the countable set E so $Y \setminus E$ is countable. Contradiction. Thus there exists $y \in Y \setminus E$ such that $x < y$ for all $x \in E$. i.e. E has an upper bound in Y. Consider the set of upper bounds of E in Y. This is a nonempty subset of Y so it has a least element, which is then a least upper bound of E.

33. Let $\{S_\lambda : \lambda \in \Lambda\}$ be a collection of segments. Suppose $\bigcup S_\lambda \neq X$. Then each S_λ is of the form $\{x \in X : x < y_\lambda\}$ for some $y_\lambda \in X$. $X \setminus \bigcup S_\lambda$ is a nonempty subset of the well-ordered set X so it has a least element y_0. If $y_0 < y_\lambda$ for some λ, then $y_0 \in S_\lambda$. Contradiction. Thus $y_0 \geq y_\lambda$ for all λ. Clearly, $\bigcup S_\lambda \subset \{x \in X : x < y_0\}$. Conversely, if $x < y_0$, then $x \notin X \setminus \bigcup S_\lambda$ so $x \in \bigcup S_\lambda$. Hence $\bigcup S_\lambda = \{x \in X : x < y_0\}$.

34a. Suppose f and g are distinct successor-preserving maps from X into Y. Then $\{x \in X : f(x) \neq g(x)\}$ is a nonempty subset of X and has a least element x_0. Now g_{x_0} is the first element of Y not in $g[\{z : z < x_0\}] = f[\{z : z < x_0\}]$ and so is $f(x_0)$. Thus $f(x_0) = g(x_0)$. Contradiction. Hence there is at most one successor-preserving map from X into Y.

34b. Let f be a successor-preserving map from X into Y. Suppose $f[X] \neq Y$. Then $\{y : y \notin f[X]\}$ is a nonempty subset of Y and has a least element y_0. Clearly, $\{y : y < y_0\} \subset f[X]$. Conversely, if $y \in f[X]$, then $y = f(x)$ is the first element not in $f[\{z : z < x\}]$. Since $y_0 \notin f[X] \supset f[\{z : z < x\}]$, $y < y_0$. Thus $f[X] \subset \{y : y < y_0\}$. Hence $f[X] = \{y : y < y_0\}$.

34c. Suppose f is successor-preserving. Let $x, y \in X$ with $x < y$. $f(y)$ is the first element of Y not in $f[\{z : z < y\}]$ so $f(y) \neq f(x)$. Furthermore, if $f(y) < f(x)$, then $f(y) \in f[\{z : z < x\}]$ and $y < x$. Contradiction. Thus $f(x) < f(y)$. Hence f is a one-to-one order preserving map. Since f is one-to-one, f^{-1} is defined on $f[X]$. If $f(x_1) < f(x_2)$, then $x_1 < x_2$ since f is order preserving. i.e. $f^{-1}(f(x_1)) < f^{-1}(f(x_2))$. Thus f^{-1} is order preserving.

34d. Let $S = \{z : z < x\}$ be a segment of X and let $z \in S$. $f|_S(z) = f(z)$ is the first element of Y not in $f[\{w : w < z\}]$. Thus $f|_S(z) \notin f|_S[\{w : w < z\}]$. Suppose $y < f|_S(z)$. Then $y \in f[\{w : w < z\}]$ so $y = f(w)$ for some $w < z$. Thus $w \in S$ and $y = f|_S(w) \in f|_S[\{w : w < z\}]$. Hence $f|_S(z)$ is the first element of Y not in $f|_S[\{w : w < z\}]$.

34e. Consider the collection of all segments of X on which there is a successor-preserving map into Y. Let S be the union of all such segments S_λ and let f_λ be the corresponding map on S_λ. Given $s \in S$, define $f(s) = f_\lambda(s)$ if $s \in S_\lambda$. f is a well-defined map because if s belongs to two segments S_λ and S_μ, we may assume $S_\lambda \subset S_\mu$ so that $f_\mu|_{S_\lambda} = f_\lambda$ by parts (d) and (a). Also, $f(s) = f_\lambda(s)$ is the first element of Y not in $f_\lambda[\{z : z < s\}] = f[\{z : z < s\}]$. Thus f is a successor-preserving map from S into Y. By Q33, S is a segment so either $S = X$ or $S = \{x : x < x_0\}$ for some $x_0 \in X$. In the second case, suppose $f[S] \neq Y$. Then there is a least $y_0 \notin Y \setminus f[S]$. Consider $S \cup \{x_0\}$. If there is no $x \in X$ such that $x > x_0$, then $S \cup \{x_0\} = X$. Otherwise, $\{x \in X : x > x_0\}$ has a least element x' and $S \cup \{x_0\} = \{x : x < x'\}$. Thus $S \cup \{x_0\}$ is a segment of X. Define $f(x_0) = y_0$. Then $f(x_0)$ is the first element of Y not in $f[\{x : x < x_0\}]$. Thus f is a successor-preserving map on the segment $S \cup \{x_0\}$ so $S \cup \{x_0\} \subset S$. Contradiction. Hence $f[S] = Y$. Now if $S = X$, then by part (b), $f[X] = f[S]$ is a segment of Y. On the other hand, if $f[S] = Y$, then f^{-1} is a successor-preserving map on Y and $f^{-1}[Y] = S$ is a segment of X.

34f. Let X be the well-ordered set in Proposition 8 and let Y be an uncountable set well-ordered by $\prec$ such that there is a last element Ω_Y in Y and if $y \in Y$ and $y \neq \Omega_Y$, then $\{z \in Y : z < y\}$ is countable. By part (e), we may assume there is a successor-preserving map f from X onto a segment of Y. If $f[X] = Y$, then we are done. Otherwise, $f[X] = \{z : z < y_0\}$ for some $y_0 \in Y$. If $y_0 \neq \Omega_Y$, then $f[X]$ is countable and since f is one-to-one, X is countable. Contradiction. If $y_0 = \Omega_Y$, then $f(\Omega_X) < \Omega_Y$ and $f(\Omega_X)$ is the largest element in $f[X]$. i.e. $f[X] = \{z : z \leq f(\Omega_X)\}$, which is countable. Contradiction. Hence f is an order preserving bijection from X onto Y.

2 The Real Number System

2.1 Axioms for the real numbers

1. Suppose $1 \notin P$. Then $-1 \in P$. Now take $x \in P$. Then $-x = (-1)x \in P$ so $0 = x + (-x) \in P$. Contradiction.

2. Let S be a nonempty set of real numbers with a lower bound. Then the set $-S = \{-s : s \in S\}$ has an upper bound and by Axiom C, it has a least upper bound b. $-b$ is a lower bound for S and if a is another lower bound for S, then $-a$ is an upper bound for $-S$ and $b \leq -a$ so $-b \geq a$. Thus $-b$ is a greatest lower bound for S.

3. Let L and U be nonempty subsets of $\mathbb{R}$ with $\mathbb{R} = L \cup U$ and such that for each $l \in L$ and $u \in U$ we have $l < u$. L is bounded above so it has a least upper bound l_0. Similarly, U is bounded below so it has a greatest lower bound u_0. If $l_0 \in L$, then it is the greatest element in L. Otherwise, $l_0 \in U$ and $u_0 \leq l_0$. If $u_0 < l_0$, then there exists $l \in L$ with $u_0 < l$. Thus l is a lower bound for U and is greater than u_0. Contradiction. Hence $u_0 = l_0$ so $u_0 \in U$ and it is the least element in U.

4a.

	$x \leq y \leq z$	$x \leq z \leq y$	$y \leq x \leq z$	$y \leq z \leq x$	$z \leq x \leq y$	$z \leq y \leq x$
$x \wedge y$	x	x	y	y	x	y
$y \wedge z$	y	z	y	y	z	z
$(x \wedge y) \wedge z$	$x \wedge z = x$	$x \wedge z = x$	$y \wedge z = y$	$y \wedge z = y$	$x \wedge z = z$	$y \wedge z = z$
$x \wedge (y \wedge z)$	$x \wedge y = x$	$x \wedge z = x$	$x \wedge y = y$	$x \wedge y = y$	$x \wedge z = z$	$x \wedge z = z$

4b. Suppose $x \leq y$. Then $x \wedge y = x$ and $x \vee y = y$ so $x \wedge y + x \vee y = x + y$. Similarly for $y \leq x$.

4c. Suppose $x \leq y$. Then $-y \leq -x$ so $(-x) \wedge (-y) = -y = -(x \vee y)$. Similarly for $y \leq x$.

4d. Suppose $x \leq y$. Then $x + z \leq y + z$ so $x \vee y + z = y + z = (x + z) \vee (y + z)$. Similarly for $y \leq x$.

4e. Suppose $x \leq y$. Then $zx \leq zy$ if $z \geq 0$ so $z(x \vee y) = zy = (zx) \vee (zy)$. Similarly for $y \leq x$.

5a. If $x, y \geq 0$, then $xy \geq 0$ so $|xy| = xy = |x||y|$. If $x, y < 0$, then $xy > 0$ so $|xy| = xy = (-x)(-y) = |x||y|$. If $x \geq 0$ and $y < 0$, then $xy \leq 0$ so $|xy| = -xy = x(-y) = |x||y|$. Similarly when $x < 0$ and $y \geq 0$.

5b. If $x, y \geq 0$, then $x + y \geq 0$ so $|x + y| = x + y = |x| + |y|$. If $x, y < 0$, then $x + y < 0$ so $|x + y| = -x - y = |x| + |y|$. If $x \geq 0$, $y < 0$ and $x + y \geq 0$, then $|x + y| = x + y \leq x - y = |x| + |y|$. If $x \geq 0$, $y < 0$ and $x + y < 0$, then $|x + y| = -x - y \leq x - y = |x| + |y|$. Similarly when $x < 0$ and $y \geq 0$.

5c. If $x \geq 0$, then $x \geq -x$ so $|x| = x = x \vee (-x)$. Similarly for $x < 0$.

5d. If $x \geq y$, then $x - y \geq 0$ so $x \vee y = x = \frac{1}{2}(x + y + x - y) = \frac{1}{2}(x + y + |x - y|)$. Similarly for $y \geq x$.

5e. Suppose $-y \leq x \leq y$. If $x \geq 0$, then $|x| = x \leq y$. If $x < 0$, then $|x| = -x \leq y$ since $-y \leq x$.

2.2 The natural and rational numbers as subsets of $\mathbb{R}$

No problems

2.3 The extended real numbers

6. If $E = \emptyset$, then $\sup E = -\infty$ and $\inf E = \infty$ so $\sup E < \inf E$. If $E \neq \emptyset$, say $x \in E$, then $\inf E \leq x \leq \sup E$.

2.4 Sequences of real numbers

7. Suppose l_1 and l_2 are (finite) limits of a sequence $\langle x_n \rangle$. Given $\varepsilon > 0$, there exist N_1 and N_2 such that $|x_n - l_1| < \varepsilon$ for $n \geq N_1$ and $|x_n - l_2| < \varepsilon$ for $n \geq N_2$. Then $|l_1 - l_2| < 2\varepsilon$ for $n \geq \max(N_1, N_2)$. Since ε is arbitrary, $|l_1 - l_2| = 0$ and $l_1 = l_2$. The cases where ∞ or $-\infty$ is a limit are clear.

8. Suppose there is a subsequence $\langle x_{n_j} \rangle$ that converges to $l \in \mathbb{R}$. Then given $\varepsilon > 0$, there exists N' such that $|x_{n_j} - l| < \varepsilon$ for $j \geq N'$. Given N, choose $j \geq N'$ such that $n_j \geq N$. Then $|x_{n_j} - l| < \varepsilon$ and l is a cluster point of $\langle x_n \rangle$. Conversely, suppose $l \in \mathbb{R}$ is a cluster point of $\langle x_n \rangle$. Given $\varepsilon > 0$, there exists $n_1 \geq 1$

such that $|x_{n_1} - l| < \varepsilon$. Suppose $x_{n_1}, \ldots, x_{n_k}$ have been chosen. Choose $n_{k+1} \geq 1 + \max(x_{n_1}, \ldots, x_{n_k})$ such that $|x_{n_{k+1}} - l| < \varepsilon$. Then the subsequence $\langle x_{n_j} \rangle$ converges to l. The cases where l is ∞ or $-\infty$ are similarly dealt with.

9a. Let $l = \overline{\lim}\, x_n \in \mathbb{R}$. Given $\varepsilon > 0$, there exists n_1 such that $x_k < l + \varepsilon$ for $k \geq n_1$. Also, there exists $k_1 \geq n_1$ such that $x_{k_1} > l - \varepsilon$. Thus $|x_{k_1} - l| < \varepsilon$. Suppose $n_1, \ldots, n_j$ and $x_{k_1}, \ldots, x_{k_j}$ have been chosen. Choose $n_{j+1} > \max(k_1, \ldots, k_j)$. Then $x_k < l + \varepsilon$ for $k \geq n_{j+1}$. Also, there exists $k_{j+1} \geq n_{j+1}$ such that $x_{k_{j+1}} > l - \varepsilon$. Thus $|x_{k_{j+1}} - l| < \varepsilon$. Then the subsequence $\langle x_{k_j} \rangle$ converges to l and l is a cluster point of $\langle x_n \rangle$. If l is ∞ or $-\infty$, it follows from the definitions that l is a cluster point of $\langle x_n \rangle$. If l' is a cluster point of $\langle x_n \rangle$ and $l' > l$, we may assume that $l \in \mathbb{R}$. By definition, there exists N such that $\sup_{k \geq N} x_k < (l + l')/2$. If $l' \in \mathbb{R}$, then since it is a cluster point, there exists $k \geq N$ such that $|x_k - l'| < (l' - l)/2$ so $x_k > (l + l')/2$. Contradiction. By definition again, there exists N' such that $\sup_{k \geq N'} x_k < l + 1$. If $l' = \infty$, then there exists $k \geq N'$ such that $x_k \geq l + 1$. Contradiction. Hence l is the largest cluster point of $\langle x_n \rangle$. Similarly for $\underline{\lim}\, x_n$.

9b. Let $\langle x_n \rangle$ be a bounded sequence. Then $\overline{\lim}\, x_n \leq \sup x_n < \infty$. Thus $\overline{\lim}\, x_n$ is a finite real number and by part (a), there is a subsequence converging to it.

10. If $\langle x_n \rangle$ converges to l, then l is a cluster point. If $l' \neq l$ and l' is a cluster point, then there is a subsequence of $\langle x_n \rangle$ converging to l'. Contradiction. Thus l is the only cluster point. Conversely, suppose there is exactly one extended real number x that is a cluster point of $\langle x_n \rangle$. If $x \in \mathbb{R}$, then $\overline{\lim}\, x_n = \underline{\lim}\, x_n = x$. Given $\varepsilon > 0$, there exists N such that $\sup_{k \geq N} x_k < x + \varepsilon$ and there exists N' such that $\inf_{k \geq N'} x_k > x - \varepsilon$. Thus $|x_k - x| < \varepsilon$ for $k \geq \max(N, N')$ and $\langle x_n \rangle$ converges to x. If $x = \infty$, then $\underline{\lim}\, x_n = \infty$ so for any Δ there exists N such that $\inf_{k \geq N} x_k > \Delta$. i.e. $x_k > \Delta$ for $k \geq N$. Thus $\lim x_n = \infty$. Similarly when $x = -\infty$.

The statement does not hold when the word "extended" is removed. The sequence $\langle 1, 1, 1, 2, 1, 3, 1, 4, \ldots \rangle$ has exactly one real number 1 that is a cluster point but it does not converge.

11a. Suppose $\langle x_n \rangle$ converges to $l \in \mathbb{R}$. Given $\varepsilon > 0$, there exists N such that $|x_n - l| < \varepsilon/2$ for $n \geq N$. Now if $m, n \geq N$, $|x_n - x_m| \leq |x_n - l| + |x_m - l| < \varepsilon$. Thus $\langle x_n \rangle$ is a Cauchy sequence.

11b. Let $\langle x_n \rangle$ be a Cauchy sequence. Then there exists N such that $||x_n| - |x_m|| \leq |x_n - x_m| < 1$ for $m, n \geq N$. In particular, $||x_n| - |x_N|| < 1$ for $n \geq N$. Thus the sequence $\langle |x_n| \rangle$ is bounded above by $\max(|x_1|, \ldots, |x_{N-1}|, |x_N| + 1)$ so the sequence $\langle x_n \rangle$ is bounded.

11c. Suppose $\langle x_n \rangle$ is a Cauchy sequence with a subsequence $\langle x_{n_k} \rangle$ that converges to l. Given $\varepsilon > 0$, there exists N such that $|x_n - x_m| < \varepsilon/2$ for $m, n \geq N$ and $|x_{n_k} - l| < \varepsilon/2$ for $n_k \geq N$. Now choose k such that $n_k \geq N$. Then $|x_n - l| \leq |x_n - x_{n_k}| + |x_{n_k} - l| < \varepsilon$ for $n \geq N$. Thus $\langle x_n \rangle$ converges to l.

11d. If $\langle x_n \rangle$ is a Cauchy sequence, then it is bounded by part (b) so it has a subsequence that converges to a real number l by Q9b. By part (c), $\langle x_n \rangle$ converges to l. The converse holds by part (a).

12. If $x = \lim x_n$, then every subsequence of $\langle x_n \rangle$ also converges to x. Conversely, suppose every subsequence of $\langle x_n \rangle$ has in turn a subsequence that converges to x. If $x \in \mathbb{R}$ and $\langle x_n \rangle$ does not converge to x, then there exists $\varepsilon > 0$ such that for all N, there exists $n \geq N$ with $|x_n - x| \geq \varepsilon$. This gives rise to a subsequence $\langle x_{n_k} \rangle$ with $|x_{n_k} - x| \geq \varepsilon$ for all k. This subsequence will not have a further subsequence that converges to x. If $x = \infty$ and $\lim x_n \neq \infty$, then there exists Δ such that for all N, there exists $n \geq N$ with $x_n < \Delta$. This gives rise to a subsequence $\langle x_{n_k} \rangle$ with $x_{n_k} < \Delta$ for all k. This subsequence will not have a further subsequence with limit ∞. Similarly when $x = -\infty$.

13. Suppose $l = \overline{\lim}\, x_n$. Given ε, there exists n such that $\sup_{k \geq n} x_k < l + \varepsilon$ so $x_k < l + \varepsilon$ for all $k \geq n$. Furthermore, given ε and n, $\sup_{k \geq n} x_k > l - \varepsilon$ so there exists $k \geq n$ such that $x_k > l - \varepsilon$. Conversely, suppose conditions (i) and (ii) hold. By (ii), for any ε and n, $\sup_{k \geq n} x_k \geq l - \varepsilon$. Thus $\sup_{k \geq n} x_k \geq l$ for all n. Furthermore by (i), if $l' > l$, then there exists n such that $x_k < l'$ for all $k \geq n$. i.e. $\sup_{k \geq n} x_k \leq l'$. Hence $l = \inf_n \sup_{k \geq n} x_k = \overline{\lim}\, x_n$.

14. Suppose $\overline{\lim}\, x_n = \infty$. Then given Δ and n, $\sup_{k \geq n} x_k > \Delta$. Thus there exists $k \geq n$ such that $x_k > \Delta$. Conversely, suppose that given Δ and n, there exists $k \geq n$ such that $x_k > \Delta$. Let $x_{n_1} = x_1$ and let $n_{k+1} > n_k$ be chosen such that $x_{n_{k+1}} > x_{n_k}$. Then $\lim x_{n_k} = \infty$ so $\overline{\lim}\, x_n = \infty$.

15. For all $m < n$, $\inf_{k \geq m} x_k \leq \inf_{k \geq n} x_k \leq \sup_{k \geq n} x_k$. Also, $\inf_{k \geq n} x_k \leq \sup_{k \geq n} x_k \leq \sup_{k \geq m} x_k$. Thus $\inf_{k \geq m} x_k \leq \sup_{k \geq n} x_k$ whenever $m \neq n$. Hence $\underline{\lim}\, x_n = \sup_n \inf_{k \geq n} x_k \leq \inf_n \sup_{k \geq n} x_k = \overline{\lim}\, x_n$.

If $\underline{\lim}\, x_n = \overline{\lim}\, x_n = l$, then the sequence has exactly one extended real number that is a cluster point so it converges to l by Q10. Conversely, if $l = \lim x_n$ and $\underline{\lim}\, x_n < \overline{\lim}\, x_n$, then the sequence has a subsequence converging to $\underline{\lim}\, x_n$ and another subsequence converging to $\overline{\lim}\, x_n$. Contradiction. Thus $\underline{\lim}\, x_n = \overline{\lim}\, x_n = l$.

16. For all n, $x_k + y_k \le \sup_{k\ge n} x_k + \sup_{k\ge n} y_k$ for $k \ge n$. Thus $\sup_{k\ge n}(x_k+y_k) \le \sup_{k\ge n} x_k + \sup_{k\ge n} y_k$ for all n. Then $\inf_n \sup_{k\ge n}(x_k+y_k) \le \inf_n \sup_{k\ge n} x_k + \inf_n \sup_{k\ge n} y_k$. i.e. $\overline{\lim}\,(x_n+y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$. Now $\overline{\lim}\, x_n \le \overline{\lim}\,(x_n+y_n) + \overline{\lim}\,(-y_n) = \overline{\lim}\,(x_n+y_n) - \underline{\lim}\, y_n$. Thus $\overline{\lim}\, x_n + \underline{\lim}\, y_n \le \overline{\lim}\,(x_n+y_n)$.

17. For any n and any $k \ge n$, $x_k y_k \le \sup_{k\ge n} x_k \sup_{k\ge n} y_k$. Thus for all n, $\sup_{k\ge n} x_k > 0$ and $\sup_{k\ge n} x_k y_k \le \sup_{k\ge n} x_k \sup_{k\ge n} y_k$. Now, taking limits, we get $\overline{\lim}\, x_n y_n \le \overline{\lim}\, x_n \overline{\lim}\, y_n$.

18. Since each $x_v \ge 0$, the sequence $\langle s_n\rangle$ is monotone increasing. If the sequence $\langle s_n\rangle$ is bounded, then $\lim s_n = \sup s_n \in \mathbb{R}$ so $\sup s_n = \sum_{v=1}^\infty x_v$. If the sequence $\langle s_n\rangle$ is unbounded, then $\lim s_n = \infty$ so $\infty = \sum_{v=1}^\infty x_v$.

19. For each n, let $t_n = \sum_{v=1}^n |x_v|$. Since $\sum_{v=1}^\infty |x_v| < \infty$, the sequence $\langle t_n\rangle$ is a Cauchy sequence so given ε, there exists N such that $|t_n - t_m| < \varepsilon$ for $n > m \ge N$. Then $|s_n - s_m| = |x_{m+1} + \cdots + x_n| \le |x_{m+1}| + \cdots + |x_n| = |t_n - t_m| < \varepsilon$ for $n > m \ge N$. Thus the sequence $\langle s_n\rangle$ is a Cauchy sequence so it converges and $\langle x_n\rangle$ has a sum.

20. $x_1 + \sum_{v=1}^\infty (x_{v+1} - x_v) = x_1 + \lim_n \sum_{v=1}^n (x_{v+1} - x_v) = \lim_n [x_1 + \sum_{v=1}^n (x_{v+1} - x_v)] = \lim_n x_{n+1} = \lim_n x_n$. Thus $x = \lim_n x_n$ if and only if $x = x_1 + \sum_{v=1}^\infty (x_{v+1} - x_v)$.

21a. Suppose $\sum_{x\in E} x < \infty$. For each n, let $E_n = \{x \in E : x \ge 1/n\}$. Then each E_n is a finite subset of E. Otherwise, if E_{n_0} is an infinite set for some n_0, then letting F_k be a subset of E_{n_0} with kn_0 elements for each $k \in \mathbb{N}$, $s_{F_k} \ge k$. Then $\sum_{x\in E} x \ge s_{F_k} \ge k$ for each k. Contradiction. Now $E = \bigcup E_n$ so E is countable.

21b. Clearly, $\{x_1, \ldots, x_n\} \in \mathcal{F}$ for all n. Thus $\sup s_n \le \sup_{F\in\mathcal{F}} s_F$. On the other hand, given $F \in \mathcal{F}$, there exists n such that $F \subset \{x_1, \ldots, x_n\}$ so $s_F \le s_n$ and $\sup_{F\in\mathcal{F}} s_F \le \sup s_n$. Hence $\sum_{x\in E} x = \sup_{F\in\mathcal{F}} s_F = \sup s_n = \sum_{n=1}^\infty x_n$.

22. Given $x \in \mathbb{R}$, let a_1 be the largest integer such that $0 \le a_1 < p$ and $a_1/p \le x$. Suppose $a_1, \ldots, a_n$ have been chosen. Let a_{n+1} be the largest integer such that $0 \le a_{n+1} < p$ and $a_{n+1}/p^{n+1} \le x - \sum_{k=1}^n a_k/p^k$. This gives rise to a sequence $\langle a_n\rangle$ of integers with $0 \le a_n < p$ and $x - \sum_{k=1}^n a_k/p^k < 1/p^n$ for all n. Now given ε, there exists N such that $1/p^N < \varepsilon$. Then $|x - \sum_{k=1}^n a_n/p^n| < 1/p^N < \varepsilon$ for $n \ge N$. Thus $x = \sum_{n=1}^\infty a_n/p^n$. This sequence is unique by construction. When $x = q/p^n$ with $q \in \{1, \ldots, p-1\}$, the sequence $\langle a_n\rangle$ obtained in this way is such that $a_n = q$ and $a_m = 0$ for $m \ne n$. However the sequence $\langle b_n\rangle$ with $b_m = 0$ for $m < n$, $b_n = q - 1$ and $b_m = p - 1$ for $m > n$ also satisfies $x = \sum_{n=1}^\infty b_n$.

Conversely, if $\langle a_n\rangle$ is a sequence of integers with $0 \le a_n < p$, let $s_n = \sum_{k=1}^n a_k/p^k$. Then $0 \le s_n \le (p-1)\sum_{k=1}^\infty 1/p^k = 1$ for all n. Thus $\langle s_n\rangle$ is a bounded monotone increasing sequence so it converges. Furthermore, since $0 \le s_n \le 1$ for all n, the sequence converges to a real number x with $0 \le x \le 1$.

23. Given a real number x with $0 \le x \le 1$, form its binary expansion (by taking $p = 2$ in Q22), which we may regard as unique by fixing a way of representing those numbers of the form $q/2^n$. By Q22, this gives a bijection from $[0,1]$ to the set of infinite sequences from $\{0,1\}$, which is uncountable by Q1.24. Thus $[0,1]$ is uncountable and since $\mathbb{R} \supset [0,1]$, $\mathbb{R}$ is uncountable.

2.5 Open and closed sets of real numbers

24. The set of rational numbers is neither open nor closed. For each $x \in \mathbb{Q}^c$ and for each $\delta > 0$, there is a rational number r with $x < r < x + \delta$ so $\mathbb{Q}^c$ is not open and $\mathbb{Q}$ is not closed. On the other hand, for each $x \in \mathbb{Q}$ and for each $\delta > 0$, there is an irrational number s with $x < s < x + \delta$ so $\mathbb{Q}$ is not open.

(*) Given $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ such that $x/\sqrt{2} < r < y/\sqrt{2}$. We may assume $0 \notin (x, y)$ by taking $x = 0$ if necessary. Then $r \ne 0$ so $r\sqrt{2}$ is irrational and $x < r\sqrt{2} < y$.

25. $\emptyset$ and $\mathbb{R}$ are both open and closed. Suppose X is a nonempty subset of $\mathbb{R}$ that is both open and closed. Take $x \in X$. Since X is open, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset X$. Thus the set $S = \{y : [x, y) \subset X\}$ is nonempty. Suppose $[x, y) \not\subset X$ for some $y > x$. Then S is bounded above. Let $b = \sup S$. Then $b \in S$ and b is a point of closure of X so $b \in X$ since X is closed. But since X is open, there exists $\delta' > 0$ such that $(b - \delta', b + \delta') \subset X$. Then $[x, b + \delta') = [x, b) \cup [b, b + \delta') \subset X$. Contradiction. Thus $[x, y) \subset X$ for all $y > x$. Similarly, $(z, x] \subset X$ for all $z < x$. Hence $X = \mathbb{R}$.

26. Let $A = (-1, 0)$ and $B = (0, 1)$. Then $A \cap B = \emptyset$ but $\bar{A} \cap \bar{B} = [-1, 0] \cap [0, 1] = \{0\}$.

27. Suppose x is a point of closure of E. Then for every n, there exists $y_n \in E$ with $|y_n - x| < 1/n$. Given $\varepsilon > 0$, choose N such that $1/N < \varepsilon$. Then $|y_n - x| < 1/N < \varepsilon$ for $n \geq N$ so $\langle y_n \rangle$ is a sequence in E with $x = \lim y_n$. Conversely, suppose there is a sequence $\langle y_n \rangle$ in E with $x = \lim y_n$. Then for any $\delta > 0$, there exists N such that $|x - y_n| < \delta$ for $n \geq N$. In particular, $|x - y_N| < \delta$. Thus x is a point of closure of E.

28. Let x be a point of closure of E'. Given $\delta > 0$, there exists $y \in E'$ such that $|x - y| < \delta$. If $y = x$, then $x \in E'$ and we are done. Suppose $y \neq x$. We may assume $y > x$. Let $\delta' = \min(y - x, x + \delta - y)$. Then $x \notin (y - \delta', y + \delta')$ and $(y - \delta', y + \delta') \subset (x - \delta, x + \delta)$. Since $y \in E'$, there exists $z \in E \setminus \{y\}$ such that $z \in (y - \delta', y + \delta')$. In particular, $z \in E \setminus \{x\}$ and $|z - x| < \delta$. Thus $x \in E'$. Hence E' is closed.

29. Clearly $E \subset \bar{E}$ and $E' \subset \bar{E}$ so $E \cup E' \subset \bar{E}$. Conversely, let $x \in \bar{E}$ and suppose $x \notin E$. Then given $\delta > 0$, there exists $y \in E$ such that $|y - x| < \delta$. Since $x \notin E$, $y \in E \setminus \{x\}$. Thus $x \in E'$ and $\bar{E} \subset E \cup E'$.

30. Let E be an isolated set of real numbers. For any $x \in E$, there exists $\delta_x > 0$ such that $|y - x| \geq \delta_x$ for all $y \in E \setminus \{x\}$. We may take each δ_x to be rational and let $I_x = \{y : |y - x| < \delta_x\}$. Then $\{I_x : x \in E\}$ is a countable collection of open intervals, each I_x contains only one element of E, namely x, and $E \subset \bigcup_{x \in E} I_x$. If E is uncountable, then I_{x_0} will contain two elements of E for some x_0. Contradiction. Thus E is countable.

31. Let $x \in \mathbb{R}$. Given $\delta > 0$, there exists $r \in \mathbb{Q}$ such that $x < r < x + \delta$. Thus x is an accumulation point of $\mathbb{Q}$. Hence $\mathbb{Q}' = \mathbb{R}$ and $\bar{\mathbb{Q}} = \mathbb{R}$.

32. Let F_1 and F_2 be closed sets. Then F_1^c and F_2^c are open so $F_1^c \cap F_2^c$ is open. i.e. $(F_1 \cup F_2)^c$ is open. Thus $F_1 \cup F_2$ is closed. Let $\mathcal{C}$ be a collection of closed sets. Then F^c is open for any $F \in \mathcal{C}$ so $\bigcup_{F \in \mathcal{F}} F^c$ is open. i.e. $(\bigcap_{F \in \mathcal{C}} F)^c$ is open. Thus $\bigcap_{F \in \mathcal{C}} F$ is closed.

33. Let O_1 and O_2 be open sets. Then O_1^c and O_2^c are closed so $O_1^c \cup O_2^c$ is closed. i.e. $(O_1 \cap O_2)^c$ is closed. Thus $O_1 \cap O_2$ is open. Let $\mathcal{C}$ be a collection of open sets. Then O^c is closed for any $O \in \mathcal{C}$ so $\bigcap_{O \in \mathcal{C}} O^c$ is closed. i.e. $(\bigcup_{O \in \mathcal{C}} O)^c$ is closed. Thus $\bigcup_{O \in \mathcal{C}} O$ is open.

34a. Clearly $A^\circ \subset A$ for any set A. A is open if and only if for any $x \in A$, there exists $\delta > 0$ such that $\{y : |y - x| < \delta\} \subset A$ if and only if every point of A is an interior point of A. Thus A is open if and only if $A = A^\circ$.

34b. Suppose $x \in A^\circ$. Then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$. In particular, $x \notin A^c$ and $x \notin (A^c)'$. Thus $x \in (\overline{A^c})^c$. Conversely, suppose $x \in (\overline{A^c})^c$. Then $x \in A$ and x is not an accumulation point of A^c. Thus there exists $\delta > 0$ such that $|y - x| \geq \delta$ for all $y \in A^c \setminus \{x\} = A^c$. Hence $(x - \delta, x + \delta) \subset A$ so $x \in A^\circ$.

35. Let $\mathcal{C}$ be a collection of closed sets of real numbers such that every finite subcollection of $\mathcal{C}$ has nonempty intersection and suppose one of the sets $F \in \mathcal{C}$ is bounded. Suppose $\bigcap_{F \in \mathcal{C}} F = \emptyset$. Then $\bigcup_{F \in \mathcal{C}} F^c = \mathbb{R} \supset F$. By the Heine-Borel Theorem, there is a finite subcollection $\{F_1, \ldots, F_n\} \subset \mathcal{C}$ such that $F \subset \bigcup_{i=1}^n F_i^c$. Then $F \cap \bigcap_{i=1}^n F_i = \emptyset$. Contradiction. Hence $\bigcap_{F \in \mathcal{C}} F \neq \emptyset$.

36. Let $\langle F_n \rangle$ be a sequence of nonempty closed sets of real numbers with $F_{n+1} \subset F_n$. Then for any finite subcollection $\{F_{n_1}, \ldots, F_{n_k}\}$ with $n_1 < \cdots < n_k$, $\bigcap_{i=1}^k F_{n_i} = F_{n_k} \neq \emptyset$. If one of the sets F_n is bounded, then by Proposition 16, $\bigcap_{i=1}^\infty F_i \neq \emptyset$.

For each n, let $F_n = [n, \infty)$. Then $\langle F_n \rangle$ is a sequence of nonempty closed sets of real numbers with $F_{n+1} \subset F_n$ but none of the sets F_n is bounded. Now $\bigcap_{n=1}^\infty F_n = \{x : x \geq n \text{ for all } n\} = \emptyset$.

37. Removing the middle third $(1/3, 2/3)$ corresponds to removing all numbers in $[0, 1]$ with unique ternary expansion $\langle a_n \rangle$ such that $a_1 = 1$. Removing the middle third $(1/9, 2/9)$ of $[0, 1/3]$ corresponds to removing all numbers with unique ternary expansion such that $a_1 = 0$ and $a_2 = 1$ and removing the middle third $(7/9, 8/9)$ of $[2/3, 1]$ corresponds to removing all numbers with unique ternary expansion such that $a_1 = 2$ and $a_2 = 1$. Suppose the middle thirds have been removed up to the n-th stage, then removing the middle thirds of the remaining intervals corresponds to removing all numbers with unique ternary expansion such that $a_i = 0$ or 2 for $i \leq n$ and $a_{n+1} = 1$. Each stage of removing middle thirds results in a closed set and C is the intersection of all these closed sets so C is closed.

38. Given an element in the Cantor ternary set with (unique) ternary expansion $\langle a_n \rangle$ such that $a_n \neq 1$ for all n, let $\langle b_n \rangle$ be the sequence obtained by replacing all 2's in the ternary expansion by 1's. Then $\langle b_n \rangle$ may be regarded as the binary expansion of a number in $[0, 1]$. This gives a one-to-one mapping from the Cantor ternary set into $[0,1]$. This mapping is also onto since given a number in $[0,1]$, we can take its binary expansion and replace all 1's by 2's to get a sequence consisting of only 0's and 2's, which we may then regard as the ternary expansion of a number in the Cantor ternary set.

39. Since the Cantor ternary set C is closed, $C' \subset C$. Conversely, given $x \in C$, let $\langle a_n \rangle$ be its ternary expansion with $a_n \neq 1$ for all n. Given $\delta > 0$, choose N such that $1/3^N < \delta$. Now let $\langle b_n \rangle$ be the sequence with $b_{N+1} = |a_{N+1} - 2|$ and $b_n = a_n$ for $n \neq N+1$. Let y be the number with ternary expansion $\langle b_n \rangle$. Then $y \in C \setminus \{x\}$ and $|x-y| = 2/3^{N+1} < 1/3^N < \delta$. Thus $x \in C'$. Hence $C = C'$.

2.6 Continuous functions

40. Since F is closed, F^c is open and it is the union of a countable collection of disjoint open intervals. Take g to be linear on each of these open intervals and take $g(x) = f(x)$ for $x \in F$. Then g is defined and continuous on $\mathbb{R}$ and $g(x) = f(x)$ for all $x \in F$.

41. Suppose f is continuous on E. Let O be an open set and let $x \in f^{-1}[O]$. Then $f(x) \in O$ so there exists $\varepsilon_x > 0$ such that $(f(x) - \varepsilon_x, f(x) + \varepsilon_x) \subset O$. Since f is continuous, there exists $\delta_x > 0$ such that $|f(y) - f(x)| < \varepsilon_x$ when $y \in E$ and $|y - x| < \delta_x$. Hence $(x - \delta_x, x + \delta_x) \cap E \subset f^{-1}[O]$. Let $U = \bigcup_{x \in f^{-1}[O]} (x - \delta_x, x + \delta_x)$. Then U is open and $f^{-1}[O] = E \cap U$. Conversely, suppose that for each open set O, there is an open set U such that $f^{-1}[O] = E \cap U$. Let $x \in E$ and let $\varepsilon > 0$. Then $O = (f(x) - \varepsilon, f(x) + \varepsilon)$ is open so there is an open set U such that $f^{-1}[O] = E \cap U$. Now $x \in U$ and U is open so there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Thus $E \cap (x - \delta, x + \delta) \subset f^{-1}[O]$. i.e. for any $y \in E$ with $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$. Thus f is continuous on E.

42. Suppose $\langle f_n \rangle$ is a sequence of continuous functions on E and that $\langle f_n \rangle$ converges uniformly to f on E. Given $\varepsilon > 0$, there exists N such that for all $x \in E$ and $n \geq N$, $|f_n(x) - f(x)| < \varepsilon/3$. Also, there exists $\delta > 0$ such that $|f_N(y) - f_N(x)| < \varepsilon/3$ if $y \in E$ and $|y - x| < \delta$. Now if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| \leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| < \varepsilon$. Thus f is continuous on E.

***43.** f is discontinuous at the nonzero rationals:

Given a nonzero rational q, let $\varepsilon = |f(q) - q| > 0$. If $q > 0$, given any $\delta > 0$, pick an irrational $x \in (q, q+\delta)$. Then $|f(x) - f(q)| = x - f(q) > q - f(q) = \varepsilon$. If $q < 0$, given any $\delta > 0$, pick an irrational $x \in (q - \delta, q)$. Then $|f(x) - f(q)| = f(q) - x > f(q) - q = \varepsilon$.

f is continuous at 0:

Given $\varepsilon > 0$, let $\delta = \varepsilon$. Then when $|x| < \delta$, $|f(x)| \leq |x| < \varepsilon$.

f is continuous at each irrational:

Let x be irrational. First we show that for any M, there exists $\delta > 0$ such that $q \geq M$ for any rational $p/q \in (x - \delta, x + \delta)$. Otherwise, there exists M such that for any n, there exists a rational $p_n/q_n \in (x - 1/n, x + 1/n)$ with $q_n < M$. Then $|p_n| \leq \max(|x-1|, |x+1|) q_n < M \max(|x-1|, |x+1|)$ for all n. Thus there are only finitely many choices of p_n and q_n for each n. This implies that there exists a rational p/q in $(x - 1/n, x + 1/n)$ for infinitely many n. Contradiction.

Given $\varepsilon > 0$, choose M such that $M^2 > \max(|x+1|, |x-1|)/6\varepsilon$. Then choose $\delta > 0$ such that $\delta < \min(1, \varepsilon)$ and $q \geq M$ for any rational $p/q \in (x - \delta, x + \delta)$. Suppose $|x - y| < \delta$. If y is irrational, then $|f(x) - f(y)| = |x - y| < \delta < \varepsilon$. If $y = p/q$ is rational, then $|f(y) - f(x)| \leq |f(y) - y| + |y - x| = |p||1/q - \sin(1/q)| + |y - x| < |p|/6q^3 + \delta < \max(|x+1|, |x-1|)/6q^2 + \delta \leq \max(|x+1|, |x-1|)/6M^2 + \delta < 2\varepsilon$.

(*) Note that $\sin x < x$ for all $x > 0$ and $x < \sin x$ for all $x < 0$. Also, $|x - \sin x| < x^3/6$ for all $x \neq 0$ (by Taylor's Theorem for example).

44a. Let f and g be continuous functions. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$ whenever $|x - y| < \delta$. Then $|(f+g)(x) - (f+g)(y)| = |f(x) - f(y) + g(x) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon$ whenever $|x - y| < \delta$. Thus $f + g$ is continuous at x. Now choose $\delta' > 0$ such that $|g(x) - g(y)| < \varepsilon/2|f(x)|$ and $|f(x) - f(y)| < \varepsilon/2\max(|g(x) - \varepsilon/2|f(x)||, |g(x) + \varepsilon/2|f(x)||)$ whenever $|x - y| < \delta'$. Then $|(fg)(x) - (fg)(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)| < \varepsilon$ whenever $|x - y| < \delta'$. Thus fg is continuous at x.

44b. Let f and g be continuous functions. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(a) - f(b)| < \varepsilon$ whenever $|a - b| < \delta$. There also exists $\delta' > 0$ such that $|g(x) - g(y)| < \delta$ whenever $|x - y| < \delta'$. Thus $|(f \circ g)(x) - (f \circ g)(y)| = |f(g(x)) - f(g(y))| < \varepsilon$ when $|x - y| < \delta'$. Thus $f \circ g$ is continuous at x.

44c. Let f and g be continuous functions. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$ whenever $|x - y| < \delta$. Now $|(f \vee g)(x) - (f \vee g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon$. Thus $f \vee g$ is continuous at x. Furthermore, $f \wedge g = f + g - (f \vee g)$ so $f \wedge g$ is continuous at x.

44d. Let f be a continuous function. Then $|f| = (f \vee 0) - (f \wedge 0)$ so f is continuous.

45. Let f be a continuous real-valued function on $[a, b]$ and suppose that $f(a) \leq \gamma \leq f(b)$. Let $S = \{x \in [a, b] : f(x) \leq \gamma\}$. Then $S \neq \emptyset$ since $a \in S$ and S is bounded. Let $c = \sup S$. Then $c \in [a, b]$. If $f(c) < \gamma$, there exists $\delta > 0$ such that $\delta < b - c$ and $|f(x) - f(c)| < \gamma - f(c)$ whenever $|x - c| < \delta$. In particular, $|f(c + \delta/2) - f(c)| < \gamma - f(c)$. Then $f(c + \delta/2) < \gamma$ so $c + \delta/2 \in S$. Contradiction. On the other hand, if $f(c) > \gamma$, there exists $\delta' > 0$ such that $\delta' < c - a$ and $|f(x) - f(c)| < f(c) - \gamma$ whenever $|x - c| < \delta'$. Then $|f(x) - f(c)| < f(c) - \gamma$ for all $x \in (c - \delta', c]$ so $f(x) > \gamma$ and $x \notin S$ for all such x. Contradiction. Hence $f(c) = \gamma$.

46. Let f be a continuous function on $[a, b]$. Suppose f is strictly monotone. We may assume f is strictly monotone increasing. Then f is one-to-one. Also, by the Intermediate Value Theorem, f maps $[a, b]$ onto $[f(a), f(b)]$. Let $g = f^{-1} : [f(a), f(b)] \to [a, b]$. Then $g(f(x)) = x$ for all $x \in [a, b]$. Let $y \in [f(a), f(b)]$. Then $y = f(x)$ for some $x \in [a, b]$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < \min(f(x) - f(x - \varepsilon), f(x + \varepsilon) - f(x))$. When $|y - z| < \delta$, $z = f(x')$ for some $x' \in [a, b]$ with $|g(y) - g(z)| = |g(f(x)) - g(f(x'))| = |x' - x| < \varepsilon$. Thus g is continuous on $[f(a), f(b)]$. Conversely, suppose there is a continuous function g such that $g(f(x)) = x$ for all $x \in [a, b]$. If $x, y \in [a, b]$ with $x < y$, then $g(f(x)) < g(f(y))$ so $f(x) \neq f(y)$. We may assume $x \neq a$ and $f(a) < f(b)$. If $f(x) < f(a)$, then by the Intermediate Value Theorem, $f(a) = f(x')$ for some $x' \in [x, b]$ and $a = g(f(a)) = g(f(x')) = x'$. Contradiction. Thus $f(a) < f(x)$. Now if $f(x) \geq f(y)$, then $f(a) < f(y) \leq f(x)$ so $f(y) = f(x'')$ for some $x'' \in [a, x]$ and $y = g(f(y)) = g(f(x'')) = x''$. Contradiction. Thus $f(x) < f(y)$. Hence f is strictly monotone increasing.

47. Let f be a continuous function on $[a, b]$ and ε a positive number. Then f is uniformly continuous so there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$. Choose N such that $(b - a)/N < \delta$ and let $x_i = a + i(b - a)/N$ for $i = 1, \ldots, N$. Now define φ on $[a, b]$ to be linear on each $[x_i, x_{i+1}]$ with $\varphi(x_i) = f(x_i)$ for each i so that φ is polygonal on $[a, b]$. Let $x \in [a, b]$. Then $x \in [x_i, x_{i+1}]$ for some i. We may assume $f(x_i) \leq f(x_{i+1})$. Then $\varphi(x_i) \leq \varphi(x) \leq \varphi(x_{i+1})$ and $|\varphi(x) - f(x)| \leq |\varphi(x) - \varphi(x_i)| + |\varphi(x_i) - f(x)| \leq |\varphi(x_{i+1}) - \varphi(x_i)| + |f(x_i) - f(x)| < \varepsilon$.

48. Suppose $x \in [0, 1]$ is of the form $q/3^{n_0}$ with $q = 1$ or 2. Then x has two ternary expansions $\langle a_n \rangle$ and $\langle a'_n \rangle$ where $a_{n_0} = q$, $a_n = 0$ for $n \neq n_0$, $a'_{n_0} = q - 1$, $a'_n = 0$ for $n < n_0$ and $a'_n = 2$ for $n > n_0$. If $q = 1$, then from the first expansion we get $N = n_0$, $b_N = 1$ and $b_n = 0$ for $n < N$ so $\sum_{n=1}^{N} b_n/2^n = 1/2^{n_0}$. From the second expansion we get $N = \infty$, $b_n = 0$ for $n < n_0$ and $b_n = 1$ for $n > n_0$ so $\sum_{n=1}^{N} b_n/2^n = \sum_{n=n_0+1}^{\infty} 1/2^n = 1/2^{n_0}$. If $q = 2$, then from the first expansion we get $N = \infty$, $b_{n_0} = 1$ and $b_n = 0$ for $n \neq n_0$ so $\sum_{n=1}^{N} b_n/2^n = \sum_{n=1}^{\infty} b_n/2^n = 1/2^{n_0}$. From the second expansion we get $N = n_0$, $b_N = 1$ and $b_n = 0$ for $n < N$ so $\sum_{n=1}^{N} b_n/2^n = 1/2^{n_0}$. Hence the sum is independent of the ternary expansion of x.

Let $f(x) = \sum_{n=1}^{N} b_n/2^n$. Given $x, y \in [0, 1]$ with ternary expansions $\langle a_n \rangle$ and $\langle a'_n \rangle$ respectively, suppose $x < y$ and let n_0 be the smallest value of n such that $a_{n_0} \neq a'_{n_0}$. Then $a_{n_0} < a'_{n_0}$ and $b_{n_0} \leq b'_{n_0}$. Thus $f(x) = \sum_{n=1}^{N_x} b_n/2^n \leq \sum_{n=1}^{N_y} b'_n/2^n = f(y)$. Given $x \in [0, 1]$ and $\varepsilon > 0$, choose M such that $1/2^M < \varepsilon$. Now choose $\delta > 0$ such that $\delta < 1/3^{M+2}$. Then when $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Hence f is monotone and continuous on $[0, 1]$.

Each interval contained in the complement of the Cantor ternary set consists of elements with ternary expansions containing 1. Furthermore, for any x, y in the same interval and having ternary expansions $\langle a_n \rangle$ and $\langle a'_n \rangle$ respectively, the smallest n such that $a_n = 1$ is also the smallest such that $a'_n = 1$. Hence f is constant on each such interval. For any $y \in [0, 1]$, let $\langle b_n \rangle$ be its binary expansion. Take $x \in [0, 1]$ with ternary expansion $\langle a_n \rangle$ such that $a_n = 2b_n$ for all n. Then x is in the Cantor ternary set and $f(x) = y$. Thus f maps the Cantor ternary set onto $[0, 1]$.

49a. Suppose $\overline{\lim_{x \to y}} f(x) \leq A$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{0 < |x-y| < \delta} f(x) < A + \varepsilon$. Thus for all x with $0 < |x - y| < \delta$, $f(x) \leq A + \varepsilon$. Conversely, suppose there exists for any $\varepsilon > 0$ some $\delta > 0$ such that $f(x) \leq A + \varepsilon$ for all x with $0 < |x - y| < \delta$. Then for each n, there exists

$\delta_n > 0$ such that $f(x) \leq A + 1/n$ for all x with $0 < |x-y| < \delta_n$ so $\sup\limits_{0<|x-y|<\delta_n} f(x) \leq A + 1/n$. Thus $\overline{\lim\limits_{x\to y}} f(x) = \inf\limits_{\delta>0} \sup\limits_{0<|x-y|<\delta} f(x) \leq \inf\limits_{n} \sup\limits_{0<|x-y|<\delta_n} f(x) \leq A + 1/n$ for all n so $\overline{\lim\limits_{x\to y}} f(x) \leq A$.

49b. Suppose $\overline{\lim\limits_{x\to y}} f(x) \geq A$. Given $\varepsilon > 0$ and $\delta > 0$, $\sup\limits_{0<|x-y|<\delta} f(x) > A - \varepsilon$ so there exists x such that $0 < |x-y| < \delta$ and $f(x) \geq A - \varepsilon$. Conversely, suppose that given $\varepsilon > 0$ and $\delta > 0$, there exists x such that $0 < |x-y| < \delta$ and $f(x) \geq A - \varepsilon$. Then for each n, there exists x_n such that $0 < |x_n - y| < \delta$ and $f(x_n) \geq A - 1/n$. Thus for each $\delta > 0$, $\sup\limits_{0<|x-y|<\delta} f(x) \geq A - 1/n$ for all n so $\sup\limits_{0<|x-y|<\delta} f(x) \geq A$. Hence $\overline{\lim\limits_{x\to y}} f(x) = \inf\limits_{\delta>0} \sup\limits_{0<|x-y|<\delta} f(x) \geq A$.

49c. For any $\delta_1, \delta_2 > 0$, if $\delta_1 < \delta_2$, then $\inf\limits_{0<|x-y|<\delta_2} f(x) \leq \inf\limits_{0<|x-y|<\delta_1} f(x) \leq \sup\limits_{0<|x-y|<\delta_1} f(x) \leq \sup\limits_{0<|x-y|<\delta_2} f(x)$. In particular, $\inf\limits_{0<|x-y|<\delta_1} f(x) \leq \sup\limits_{0<|x-y|<\delta_2} f(x)$ and $\inf\limits_{0<|x-y|<\delta_2} f(x) \leq \sup\limits_{0<|x-y|<\delta_1} f(x)$. Hence $\sup\limits_{\delta>0} \inf\limits_{0<|x-y|<\delta} f(x) \leq \sup\limits_{0<|x-y|<\delta_0} f(x)$ for any $\delta_0 > 0$ so $\sup\limits_{\delta>0} \inf\limits_{0<|x-y|<\delta} f(x) \leq \inf\limits_{\delta>0} \sup\limits_{0<|x-y|<\delta} f(x)$. i.e. $\underline{\lim\limits_{x\to y}} f(x) \leq \overline{\lim\limits_{x\to y}} f(x)$.

Suppose $\underline{\lim\limits_{x\to y}} f(x) = \overline{\lim\limits_{x\to y}} f(x)$ with $\overline{\lim} f \neq \pm\infty$. Let L be the common value. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $\sup\limits_{0<|x-y|<\delta_1} f(x) < L + \varepsilon$. i.e. $f(x) < L + \varepsilon$ whenever $0 < |x-y| < \delta_1$. There also exists $\delta_2 > 0$ such that $\inf\limits_{0<|x-y|<\delta_2} f(x) > L - \varepsilon$. i.e. $f(x) > L - \varepsilon$ whenever $0 < |x-y| < \delta_2$. Let $\delta = \min(\delta_2, \delta_2)$. Then when $0 < |x-y| < \delta$, $|f(x) - L| < \varepsilon$ so $\lim\limits_{x\to y} f(x)$ exists. Conversely, suppose $\lim\limits_{x\to y} f(x)$ exists and let L be its value. Given ε, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x-y| < \delta$. By part (a), $\overline{\lim\limits_{x\to y}} f(x) \leq L$. Similarly, $\underline{\lim\limits_{x\to y}} f(x) \geq L$. i.e. $\overline{\lim\limits_{x\to y}} f(x) \leq \underline{\lim\limits_{x\to y}} f(x)$. Thus equality holds.

49d. Suppose $\overline{\lim\limits_{x\to y}} f(x) = A$ and $\langle x_n \rangle$ is a sequence with $x_n \neq y$ such that $y = \lim x_n$. For any $\delta > 0$, there exists N_δ such that $0 < |x_n - y| < \delta$ for $n \geq N_\delta$. Thus for any $\delta > 0$, $\inf\limits_N \sup\limits_{n\geq N} f(x_n) \leq \sup\limits_{n\geq N_\delta} f(x_n) \leq \sup\limits_{0<|x-y|<\delta} f(x)$ so $\inf\limits_N \sup\limits_{n\geq N} f(x_n) \leq \inf\limits_{\delta>0} \sup\limits_{0<|x-y|<\delta} f(x)$. i.e. $\overline{\lim} f(x_n) \leq \overline{\lim\limits_{x\to y}} f(x) = A$.

49e. Suppose $\overline{\lim\limits_{x\to y}} f(x) = A$. By part (a), for each n, there exists $\delta_n > 0$ such that $f(x) < A + 1/n$ whenever $0 < |x-y| < \delta_n$. By part (b), there exists x_n such that $0 < |x_n - y| < \min(\delta_n, 1/n)$ and $f(x_n) > A - 1/n$. Thus $0 < |x_n - y| < 1/n$ and $|f(x_n) - A| < 1/n$ for each n. i.e. $\langle x_n \rangle$ is a sequence with $x_n \neq y$ such that $y = \lim x_n$ and $A = \lim f(x_n)$.

49f. Suppose $l = \lim\limits_{x\to y} f(x)$ and let $\langle x_n \rangle$ be a sequence with $x_n \neq y$ and $y = \lim x_n$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x-y| < \delta$. Also there exists N such that $0 < |x_n - y| < \delta$ for $n \geq N$. Thus for $n \geq N$, $|f(x_n) - l| < \varepsilon$. i.e. $l = \lim f(x_n)$. Conversely, suppose $l \neq \lim\limits_{x\to y} f(x)$. Then there exists $\varepsilon > 0$ such that for each n there exists x_n with $0 < |x_n - y| < 1/n$ and $|f(x_n) - l| \geq \varepsilon$. Thus $\langle x_n \rangle$ is a sequence with $x_n \neq y$ and $y = \lim x_n$ but $l \neq \lim f(x_n)$.

50a. Let $f(y)$ be finite. Then f is lower semicontinuous at y if and only if $-f$ is upper semicontinuous at y if and only if $-f(y) \geq \overline{\lim\limits_{x\to y}}(-f(x))$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $-f(x) \leq -f(y) + \varepsilon$ whenever $0 < |x-y| < \delta$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $-f(x) \leq -f(y) + \varepsilon$ whenever $|x-y| < \delta$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(y) \leq f(x) + \varepsilon$ whenever $|x-y| < \delta$.

50b. Suppose f is both upper and lower semicontinuous at y. Then $\overline{\lim\limits_{x\to y}} f(x) \leq f(y) \leq \underline{\lim\limits_{x\to y}} f(x)$. Thus $\lim\limits_{x\to y} f(x)$ exists and equals $f(y)$ so f is continuous at y. Conversely, if f is continuous at y, then $\lim\limits_{x\to y} f(x)$ exists and equals $f(y)$. Thus $\underline{\lim\limits_{x\to y}} f(x) = \overline{\lim\limits_{x\to y}} f(x) = f(y)$ so f is both upper and lower semicontinuous at y. The result for intervals follows from the result for points.

50c. Let f be a real-valued function. Suppose f is lower semicontinuous on $[a,b]$. For $\lambda \in \mathbb{R}$, consider the set $S = \{x \in [a,b] : f(x) \le \lambda\}$. Let y be a point of closure of S. Then there is a sequence $\langle x_n \rangle$ with $x_n \in S$ and $y = \lim x_n$. Thus $f(y) \le \underline{\lim} f(x_n) \le \lambda$ so $y \in S$. Hence S is closed so $\{x \in [a,b] : f(x) > \lambda\}$ is open (in $[a,b]$). Conversely, suppose $\{x \in [a,b] : f(x) > \lambda\}$ is open (in $[a,b]$) for each $\lambda \in \mathbb{R}$. Let $y \in [a,b]$. Given $\varepsilon > 0$, the set $\{x \in [a,b] : f(x) > f(y) - \varepsilon\}$ is open so there exists $\delta > 0$ such that $f(x) > f(y) - \varepsilon$ whenever $|x-y| < \delta$. i.e. $f(y) < f(x) + \varepsilon$ whenever $|x-y| < \delta$. By part (a), f is lower semicontinuous at y. Since y is arbitrarily chosen from $[a,b]$, f is lower semicontinuous on $[a,b]$.

50d. Let f and g be lower semicontinuous functions. Let y be in the domain of f and g. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(y) \le f(x) + \varepsilon/2$ and $g(y) \le g(x) + \varepsilon/2$ whenever $|x-y| < \delta$. Thus $(f \vee g)(y) \le (f \vee g)(x) + \varepsilon$ and $(f+g)(y) \le (f+g)(x) + \varepsilon$ whenever $|x-y| < \delta$. By part (a), $f \vee g$ and $f+g$ are lower semicontinuous.

50e. Let $\langle f_n \rangle$ be a sequence of lower semicontinuous functions and define $f(x) = \sup_n f_n(x)$. Given $\varepsilon > 0$, $f(y) - \varepsilon/2 < f_n(y)$ for some n. There also exists $\delta > 0$ such that $f_n(y) \le f_n(x) + \varepsilon/2$ whenever $|x-y| < \delta$. Thus $f(y) \le f_n(x) + \varepsilon \le f(x) + \varepsilon$ whenever $|x-y| < \delta$. Hence f is lower semicontinuous.

50f. Let $\varphi : [a,b] \to \mathbb{R}$ be a step function. Suppose φ is lower semicontinuous. Let c_i and c_{i+1} be the values assumed by φ in (x_{i-1}, x_i) and (x_i, x_{i+1}) respectively. For each n, there exists $\delta > 0$ such that $\varphi(x_i) \le \varphi(x) + 1/n$ whenever $|x - x_i| < \delta$. In particular, $\varphi(x_i) \le c_i + 1/n$ and $\varphi(x_i) \le c_{i+1} + 1/n$ for each n. Thus $\varphi(x_i) \le \min(c_i, c_{i+1})$. Conversely, suppose $\varphi(x_i) \le \min(c_i, c_{i+1})$ for each i. Let $\varepsilon > 0$ and let $y \in [a,b]$. If $y = x_i$ for some i, let $\delta = \min(x_i - x_{i-1}, x_{i+1} - x_i)$. Then $f(y) = f(x_i) \le f(x) + \varepsilon$ whenever $|x-y| < \delta$. If $y \in (x_i, x_{i+1})$ for some i, let $\delta = \min(y - x_i, x_{i+1} - y)$. Then $f(y) < f(y) + \varepsilon = f(x) + \varepsilon$ whenever $|x-y| < \delta$. Thus φ is lower semicontinuous.

***50g.** Let f be a function defined on $[a,b]$. Suppose there is a monotone increasing sequence $\langle \varphi_n \rangle$ of lower semicontinuous step functions on $[a,b]$ such that for each $x \in [a,b]$ we have $f(x) = \lim \varphi_n(x)$. Since $\langle \varphi_n \rangle$ is monotone increasing, for each $x \in [a,b]$ we have $f(x) = \sup_n \varphi_n(x)$. By part (e), f is lower semicontinuous. Conversely, suppose that f is lower semicontinuous. The sets $\{x \in [a,b] : f(x) > c\}$ with $c \in \mathbb{Z}$ form an open covering of $[a,b]$ so by the Heine-Borel Theorem, there is a finite subcovering. Thus there exists $c \in \mathbb{Z}$ such that $f(x) > c$ for all $x \in [a,b]$. Now for each n, let $x_k^{(n)} = a + k(b-a)/2^n$ and let $I_k^{(n)} = (x_{k-1}^{(n)}, x_k^{(n)})$ for $k = 0, 1, 2, \dots, 2^n$. Define $\varphi_n(x) = \inf_{x \in I_k^{(n)}} f(x)$ if $x \in I_k^{(n)}$ and $\varphi_n(x_k^{(n)}) = \min(c_k^{(n)}, c_{k+1}^{(n)}, f(x_k^{(n)}))$ where $c_k^{(n)} = \inf_{x \in I_k^{(n)}} f(x)$ and $c_{k+1}^{(n)} = \inf_{x \in I_{k+1}^{(n)}} f(x)$. Then each φ_n is a lower semicontinuous step function on $[a,b]$ by part (f) and $f \ge \varphi_{n+1} \ge \varphi_n$ for each n. Let $y \in [a,b]$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(y) \le f(x) + \varepsilon$ whenever $|x-y| < \delta$. Choose N such that $1/2^N < \delta$. For $n \ge N$, if $y \in I_k^{(n)}$ for some k, then $\varphi_n(y) = \inf_{x \in I_k^{(n)}} f(x) \ge f(y) - \varepsilon$ since $I_k^{(n)} \subset (y-\delta, y+\delta)$. Thus $f(y) - \varphi_n(y) \le \varepsilon$. If $y = x_k^{(n)}$ for some k, then $\varphi_n(y) =\ge f(y) - \varepsilon$ since $I_k^{(n)} \cup I_{k+1}^{(n)} \subset (y-\delta, y+\delta)$. Thus $f(y) - \varphi_n(y) \le \varepsilon$. Hence $f(y) = \lim \varphi_n(y)$.

***50h.** Let f be a function defined on $[a,b]$. Suppose there is a monotone increasing sequence $\langle \psi_n \rangle$ of continuous functions on $[a,b]$ such that for each $x \in [a,b]$ we have $f(x) = \lim \psi_n(x)$. Then each ψ_n is lower semicontinuous by part (b) and $f(x) = \sup_n \psi_n(x)$. By part (e), f is lower semicontinuous. Conversely, suppose that f is lower semicontinuous. By part (g), there is a monotone increasing sequence $\langle \varphi_n \rangle$ of lower semicontinuous step functions on $[a,b]$ with $f(x) = \lim \varphi_n(x)$ for each $x \in [a,b]$. For each n, define ψ_n by linearising φ_n at a neighbourhood of each partition point such that $\psi_n \le \psi_{n+1}$ and $0 \le \varphi_n(x) - \psi_n(x) < \varepsilon/2$ for each $x \in [a,b]$. Then $\langle \psi_n \rangle$ is a monotone increasing sequence of continuous functions on $[a,b]$ and $f(x) = \lim \psi_n(x)$ for each $x \in [a,b]$.

(*) More rigorous proof: Define ψ_n by $\psi_n(x) = \inf\{f(t) + n|t-x| : t \in [a,b]\}$. Then $\psi_n(x) \le \inf\{f(t) + n|t-y| + n|y-x| : t \in [a,b]\} = \psi_n(y) + n|y-x|$. Thus ψ_n is (uniformly) continuous on $[a,b]$. Also $\psi_n \le \psi_{n+1} \le f$ for all n. In particular, $f(x)$ is an upper bound for $\{\psi_n(x) : n \in \mathbb{N}\}$. Now if $\alpha < f(x)$, then there exists $\delta > 0$ such that $\alpha \le f(y) \le f(y) + n|y-x|$ whenever $|y-x| < \delta$. On the other hand, when $|y-x| \ge \delta$, we have $\alpha \le f(y) + n\delta \le f(y) + n|y-x|$ for sufficiently large n. Thus $\alpha \le \psi_n(x)$ for sufficiently large n. Hence $f(x) = \sup \psi_n(x) = \lim \psi_n(x)$.

50i. Let f be a lower semicontinuous function on $[a,b]$. The sets $\{x \in [a,b] : f(x) > c\}$ with $c \in \mathbb{Z}$ form an open covering of $[a,b]$ so by the Heine-Borel Theorem, there is a finite subcovering. Thus there exists $c \in \mathbb{Z}$ such that $f(x) > c$ for all $x \in [a,b]$. Hence f is bounded from below. Let $m = \inf_{x \in [a,b]} f(x)$, which is finite since f is bounded from below. Suppose $f(x) > m$ for all $x \in [a,b]$. For each $x \in [a,b]$, there exists $\delta_x > 0$ such that $f(x) \le f(y) + m - f(x)$ for $y \in I_x = (x - \delta_x, x + \delta_x)$. The open intervals $\{I_x : x \in [a,b]\}$ form an open covering of $[a,b]$ so by the Heine-Borel Theorem, there is a finite subcovering $\{I_{x_1}, \dots, I_{x_n}\}$. Let $c = \min(f(x_1), \dots, f(x_n))$. Each $y \in [a,b]$ belongs to some I_{x_k} so $f(y) \ge 2f(x_k) - m \ge 2c - m$. Thus $2c - m$ is a lower bound for f on $[a,b]$ but $2c - m > m$. Contradiction. Hence there exists $x_0 \in [a,b]$ such that $m = f(x_0)$.

51a. Let $x \in [a,b]$. Then $\inf\limits_{|y-x|<\delta} f(y) \le f(x) \le \sup\limits_{|y-x|<\delta} f(y)$ for any $\delta > 0$. Hence we have $g(x) = \sup\limits_{\delta>0} \inf\limits_{|y-x|<\delta} f(y) \le f(x) \le \inf\limits_{\delta>0} \sup\limits_{|y-x|<\delta} f(y) = h(x)$. Suppose $g(x) = f(x)$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) - \varepsilon = g(x) - \varepsilon < \inf\limits_{|y-x|<\delta} f(y)$. Thus $f(x) - \varepsilon < f(y)$ whenever $|y - x| < \delta$ and f is lower semicontinuous at x. Conversely, suppose f is lower semicontinuous at x. $f(x)$ is an upper bound for $\{\inf\limits_{|x-y|<\delta} f(y) : \delta > 0\}$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) - \varepsilon \le f(y)$ whenever $|x - y| < \delta$. Thus $f(x) - \varepsilon \le \inf\limits_{|x-y|<\delta} f(y)$ so $f(x) = \sup\limits_{\delta>0} \inf\limits_{|x-y|<\delta} f(y) = g(x)$. By a similar argument, $f(x) = h(x)$ if and only if f is upper semicontinuous at x. Thus f is continuous at x if and only if f is both upper semicontinuous and lower semicontinuous at x if and only if $f(x) = h(x)$ and $g(x) = f(x)$ if and only if $g(x) = h(x)$.

51b. Let $\lambda \in \mathbb{R}$. Suppose $g(x) > \lambda$. Then there exists $\delta > 0$ such that $f(y) > \lambda$ whenever $|x - y| < \delta$. Hence $\{x : g(x) > \lambda\}$ is open in $[a,b]$ and g is lower semicontinuous. Suppose $h(x) < \lambda$. Then there exists $\delta > 0$ such that $f(y) < \lambda$ whenever $|x - y| < \delta$. Hence $\{x : h(x) < \lambda\}$ is open in $[a,b]$ and h is upper semicontinuous.

51c. Let φ be a lower semicontinuous function such that $\varphi(x) \le f(x)$ for all $x \in [a,b]$. Suppose $\varphi(x) > g(x)$ for some $x \in [a,b]$. Then there exists $\delta > 0$ such that $\varphi(x) \le \varphi(y) + \varphi(x) - g(x)$ whenever $|x - y| < \delta$. i.e. $g(x) \le \varphi(y)$ whenever $|x - y| < \delta$. In particular, $g(x) \le \varphi(x)$. Contradiction. Hence $\varphi(x) \le g(x)$ for all $x \in [a,b]$.

2.7 Borel sets

52. Let f be a lower semicontinuous function on $\mathbb{R}$. Then $\{x : f(x) > \alpha\}$ is open. $\{x : f(x) \ge \alpha\} = \bigcap_n \{x : f(x) > \alpha - 1/n\}$ so it is a G_δ set. $\{x : f(x) \le \alpha\} = \{x : f(x) > \alpha\}^c$ is closed. $\{x : f(x) < \alpha\} = \{x : f(x) \ge \alpha\}^c$ so it is an F_σ set. $\{x : f(x) = \alpha\} = \{x : f(x) \ge \alpha\} \cap \{x : f(x) \le \alpha\}$ is the intersection of a G_δ set with a closed set so it is a G_δ set.

53. Let f be a real-valued function defined on $\mathbb{R}$. Let S be the set of points at which f is continuous. If f is continuous at x, then for each n, there exists $\delta_{n,x} > 0$ such that $|f(x) - f(y)| < 1/n$ whenever $|x - y| < \delta_{n,x}$. Consider $G_n = \bigcup_{x \in S} (x - \delta_{n,x}/2, x + \delta_{n,x}/2)$ and $G = \bigcap_n G_n$. Then $x \in G$ for every $x \in S$. Conversely, suppose $x_0 \in G$ for some $x_0 \notin S$. There exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists y with $|y - x_0| < \delta$ and $|f(x_0) - f(y)| \ge \varepsilon$. Choose N such that $1/N < \varepsilon/2$. There exists y with $|y - x_0| < \delta_{N,x}/2$ and $|f(y) - f(x_0)| \ge \varepsilon$. On the other hand, $x_0 \in G_N = \bigcup_{x \in S} (x - \delta_{N,x}/2, x + \delta_{N,x}/2)$ so $x_0 \in (x - \delta_{N,x}/2, x + \delta_{N,x}/2)$ for some $x \in S$. Thus $|f(x) - f(x_0)| < 1/N < \varepsilon/2$. Also, $|y - x| \le |y - x_0| + |x_0 - x| < \delta_{N,x}$ so $|f(y) - f(x)| < 1/N < \varepsilon/2$. Thus $|f(y) - f(x_0)| \le |f(y) - f(x)| + |f(x) - f(x_0)| < \varepsilon$. Contradiction. Hence $G = S$ and S is a G_δ set.

54. Let $\langle f_n \rangle$ be a sequence of continuous functions on $\mathbb{R}$ and let C be the set of points where the sequence converges. If $x \in C$, then for any m, there exists n such that $|f_k(x) - f_n(x)| \le 1/m$ for all $k \ge n$. Consider $F_{n,m} = \{x : |f_k(x) - f_n(x)| \le 1/m \text{ for } k \ge n\}$. Now $C \subset \bigcap_m \bigcup_n F_{n,m}$. Conversely, if $x \in \bigcap_m \bigcup_n F_{n,m}$, then given $\varepsilon > 0$, choose M such that $1/M < \varepsilon$. Now $x \in \bigcup_n F_{n,M}$ so there exists N such that $|f_k(x) - f_N(x)| \le 1/M < \varepsilon$ for $k \ge N$. Thus $\langle f_n(x) \rangle$ converges so $\bigcap_m \bigcup_n F_{n,m} \subset C$. By continuity of each f_k, each $F_{n,m}$ is closed. Hence C is an $F_{\sigma\delta}$ set.

3 Lebesgue Measure

3.1 Introduction

1. Let A and B be two sets in $\mathfrak{M}$ with $A \subset B$. Then $mA \le mA + m(B \setminus A) = mB$.

2. Let $\langle E_n \rangle$ be a sequence of sets in $\mathfrak{M}$. Let $F_1 = E_1$ and let $F_{n+1} = E_{n+1} \setminus \bigcup_{k=1}^n E_k$. Then $F_m \cap F_n = \emptyset$ for $m \neq n$, $F_n \subset E_n$ for each n and $\bigcup E_n = \bigcup F_n$. Thus $m(\bigcup E_n) = m(\bigcup F_n) = \sum mF_n \leq \sum mE_n$.

3. Suppose there is a set A in $\mathfrak{M}$ with $mA < \infty$. Then $mA = m(A \cup \emptyset) = mA + m\emptyset$ so $m\emptyset = 0$.

4. Clearly n is translation invariant and defined for all sets of real numbers. Let $\langle E_k \rangle$ be a sequence of disjoint sets of real numbers. We may assume $E_k \neq \emptyset$ for all k since $n\emptyset = 0$. If some E_k is an infinite set, then so is $\bigcup E_k$. Thus $n(\bigcup E_k) = \infty = \sum nE_k$. If all E_k's are finite sets and $\{E_k : k \in \mathbb{N}\}$ is a finite set, then $\bigcup E_k$ is a finite set and since the E_k's are disjoint, $n(\bigcup E_k) = \sum nE_k$. On the other hand, if all E_k's are finite sets and $\{E_k : k \in \mathbb{N}\}$ is an infinite set, then $\bigcup E_k$ is a countably infinite set so $n(\bigcup E_k) = \infty$ and since $nE_k \geq 1$ for all k, $\sum nE_k = \infty$.

3.2 Outer measure

5. Let A be the set of rational numbers between 0 and 1. Also let $\{I_n\}$ be a finite collection of open intervals covering A. Then $1 = m^*([0,1]) = m^*\overline{A} \leq m^*(\overline{\bigcup I_n}) = m^*(\bigcup \overline{I_n}) \leq \sum m^*\overline{I_n} = \sum l(\overline{I_n}) = \sum l(I_n)$.

6. Given any set A and any $\varepsilon > 0$, there is a countable collection $\{I_n\}$ of open intervals covering A such that $\sum l(I_n) \leq m^*A + \varepsilon$. Let $O = \bigcup I_n$. Then O is an open set such that $A \subset O$ and $m^*O \leq \sum m^*I_n = \sum l(I_n) \leq m^*A + \varepsilon$. Now for each n, there is an open set O_n such that $A \subset O_n$ and $m^*O_n \leq m^*A + 1/n$. Let $G = \bigcap O_n$. Then G is a G_δ set such that $A \subset G$ and $m^*G = m^*A$.

7. Let E be a set of real numbers and let $y \in \mathbb{R}$. If $\{I_n\}$ is a countable collection of open intervals such that $E \subset \bigcup I_n$, then $E + y \subset \bigcup (I_n + y)$ so $m^*(E+y) \leq \sum l(I_n + y) = \sum l(I_n)$. Thus $m^*(E+y) \leq m^*E$. Conversely, by a similar argument, $m^*E \leq m^*(E+y)$. Hence $m^*(E+y) = m^*E$.

8. Suppose $m^*A = 0$. Then $m^*(A \cup B) \leq m^*A + m^*B = m^*B$. Conversely, since $B \subset A \cup B$, $m^*B \leq m^*(A \cup B)$. Hence $m^*(A \cup B) = m^*B$.

3.3 Measurable sets and Lebesgue measure

9. Let E be a measurable set, let A be any set and let $y \in \mathbb{R}$. Then $m^*A = m^*(A-y) = m^*((A-y) \cap E) + m^*((A-y) \cap E^c) = m^*(((A-y) \cap E) + y) + m^*(((A-y) \cap E^c) + y) = m^*(A \cap (E+y)) + m^*(A \cap (E^c + y)) = m^*(A \cap (E+y)) + m^*(A \cap (E+y)^c)$. Thus $E + y$ is a measurable set.

10. Suppose E_1 and E_2 are measurable. Then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + m(E_2 \setminus E_1) + m(E_1 \cap E_2) = mE_1 + mE_2$.

11. For each n, let $E_n = (n, \infty)$. Then $E_{n+1} \subset E_n$ for each n, $\bigcap E_n = \emptyset$ and $mE_n = \infty$ for each n.

12. Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets and let A be any set. Then $m^*(A \cap \bigcup_{i=1}^\infty E_i) = m^*(\bigcup_{i=1}^\infty (A \cap E_i)) \leq \sum_{i=1}^\infty m^*(A \cap E_i)$ by countable subadditivity. Conversely, $m^*(A \cap \bigcup_{i=1}^\infty E_i) \geq m^*(A \cap \bigcup_{i=1}^n E_i) = \sum_{i=1}^n m^*(A \cap E_i)$ for all n by Lemma 9. Thus $m^*(A \cap \bigcup_{i=1}^\infty E_i) \geq \sum_{i=1}^\infty m^*(A \cap E_i)$. Hence $m^*(A \cap \bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m^*(A \cap E_i)$.

13a. Suppose $m^*E < \infty$.

$(i) \Rightarrow (ii)$. Suppose E is measurable. Given $\varepsilon > 0$, there is a countable collection $\{I_n\}$ of open intervals such that $E \subset \bigcup I_n$ and $\sum l(I_n) < m^*E + \varepsilon$. Let $O = \bigcup I_n$. Then O is an open set, $E \subset O$ and $m^*(O \setminus E) = m(O \setminus E) = m(\bigcup I_n) - mE = m^*(\bigcup I_n) - m^*E \leq \sum l(I_n) - m^*E < \varepsilon$.

$(ii) \Rightarrow (vi)$. Given $\varepsilon > 0$, there is an open set O such that $E \subset O$ and $m^*(O \setminus E) < \varepsilon/2$. O is the union of a countable collection of disjoint open intervals $\{I_n\}$ so $\sum l(I_n) = m(\bigcup I_n) < mE + \varepsilon/2$. Thus there exists N such that $\sum_{n=N+1}^\infty l(I_n) < \varepsilon/2$. Let $U = \bigcup_{n=1}^N I_n$. Then $m^*(U \Delta E) = m^*(U \setminus E) + m^*(E \setminus U) \leq m^*(O \setminus E) + m^*(O \setminus U) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$(vi) \Rightarrow (ii)$. Given $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon/3$. Also there is an open set O such that $E \setminus U \subset O$ and $m^*O \leq m^*(E \setminus U) + \varepsilon/3$. Then $E \subset U \cup O$ and $m^*((U \cup O) \setminus E) = m^*((U \setminus E) \cup (O \setminus E)) \leq m^*((O \setminus (E \setminus U)) \cup (E \setminus U) \cup (U \setminus E)) < \varepsilon$.

13b. $(i) \Rightarrow (ii)$. Suppose E is measurable. The case where $m^*E < \infty$ was proven in part (a). Suppose $m^*E = \infty$. For each n, let $E_n = [-n, n] \cap E$. By part (a), for each n, there exists an open set $O_n \supset E_n$ such that $m^*(O_n \setminus E_n) < \varepsilon/2^n$. Let $O = \bigcup O_n$. Then $E \subset O$ and $m^*(O \setminus E) = m^*(\bigcup O_n \setminus \bigcup E_n) \leq m^*(\bigcup (O_n \setminus E_n)) < \varepsilon$.

$(ii) \Rightarrow (iv)$. For each n, there exists an open set O_n such that $E \subset O_n$ and $m^*(O_n \setminus E) < 1/n$. Let $G = \bigcap O_n$. Then $E \subset G$ and $m^*(G \setminus E) \le m^*(O_n \setminus E) < 1/n$ for all n. Thus $m^*(G \setminus E) = 0$.

$(iv) \Rightarrow (i)$. There exists a G_δ set G such that $E \subset G$ and $m^*(G \setminus E) = 0$. Now G and $G \setminus E$ are measurable sets so $E = G \setminus (G \setminus E)$ is a measurable set.

13c. $(i) \Rightarrow (iii)$. Suppose E is measurable. Then E^c is measurable. By part (b), given $\varepsilon > 0$, there is an open set O such that $E^c \subset O$ and $m^*(O \setminus E^c) < \varepsilon$. i.e. $m^*(O \cap E) < \varepsilon$. Let $F = O^c$. Then F is closed, $F \subset E$ and $m^*(E \setminus F) = m^*(E \setminus O^c) = m^*(E \cap O) < \varepsilon$.

$(iii) \Rightarrow (v)$. For each n, there is a closed set F_n such that $F_n \subset E$ and $m^*(E \setminus F_n) < 1/n$. Let $F = \bigcup F_n$. Then $F \subset E$ and $m^*(E \setminus F) \le m^*(E \setminus F_n) < 1/n$ for all n. Thus $m^*(E \setminus F) = 0$.

$(v) \Rightarrow (i)$. There exists an F_σ set F such that $F \subset E$ and $m^*(E \setminus F) = 0$. Now F and $E \setminus F$ are measurable sets so $E = F \cup (E \setminus F)$ is a measurable set.

14a. For each n, let E_n be the union of the intervals removed in the nth step. Then $mE_n = 2^{n-1}/3^n$ so $m(\bigcup E_n) = \sum mE_n = 1$. Thus $mC = m([0,1]) - m(\bigcup E_n) = 0$.

14b. Each step of removing open intervals results in a closed set and F is the intersection of all these closed sets so F is closed. Let $x \in [0,1] \setminus F^c$. Note that removing intervals in the nth step results in 2^n disjoint intervals, each of length less than $1/2^n$. Given $\delta > 0$, choose N such that $1/2^N < 2\delta$. Then $(x-\delta, x+\delta)$ must intersect one of the intervals removed in step N. i.e. there exists $y \in (x-\delta, x+\delta) \cap F^c$. Thus F^c is dense in $[0,1]$. Finally, $mF = m([0,1]) - \sum \alpha 2^{n-1}/3^n = 1 - \alpha$.

3.4 A nonmeasurable set

15. Let E be a measurable set with $E \subset P$. For each i, let $E_i = E \overset{\circ}{+} r_i$. Since $E_i \subset P_i$ for each i, $\langle E_i \rangle$ is a disjoint sequence of measurable sets with $mE_i = mE$. Thus $\sum mE_i = m(\bigcup E_i) \le m([0,1)) = 1$. Since $\sum mE_i = \sum mE$, we must have $mE = 0$.

16. Let A be any set with $m^*A > 0$. If $A \subset [0,1)$, let $\langle r_i \rangle_{i=0}^\infty$ be an enumeration of $\mathbb{Q} \cap [-1,1)$. For each i, let $P_i = P + r_i$. Then $[0,1) \subset \bigcup P_i$ since for any $x \in [0,1)$, there exists $y \in P$ such that x and y differ by some rational r_i. Also, $P_i \cap P_j = \emptyset$ if $i \neq j$ since if $p_i + r_i = p_j + r_j$, then $p_i \sim p_j$ and since P contains exactly one element from each equivalence class, $p_i = p_j$ and $r_i = r_j$. Now let $E_i = A \cap P_i$ for each i. If each E_i is measurable, then $mE_i = m(A \cap P_i) = m((A - r_i) \cap P) = 0$ for each i by Q15. On the other hand, $\sum m^*E_i \ge m^*(\bigcup E_i) \ge m^*(A \cap [0,1)) = m^*A > 0$. Hence E_{i_0} is nonmeasurable for some i_0 and $E_{i_0} \subset A$. Similarly, if $A \subset [n, n+1)$ where $n \in \mathbb{Z}$, then there is a nonmeasurable set $E \subset A$. In general, $A = A \cap \bigcup_{n \in \mathbb{Z}} [n, n+1)$ and $0 < m^*A \le \sum_{n \in \mathbb{Z}} m^*(A \cap [n, n+1))$ so $m^*(A \cap [n, n+1)) > 0$ for some $n \in \mathbb{Z}$ and there is a nonmeasurable set $E \subset A \cap [n, n+1) \subset A$.

17a. Let $\langle r_i \rangle_{i=0}^\infty$ be an enumeration of $\mathbb{Q} \cap [-1,1)$ and let $P_i = P + r_i$ for each i. Then $\langle P_i \rangle$ is a disjoint sequence of sets with $m^*(\bigcup P_i) \le m^*[-1,2) = 3$ and $\sum m^*P_i = \sum m^*P = \infty$. Thus $m^*(\bigcup P_i) < \sum m^*P_i$.

17b. For each i, let P_i be as defined in part (a) and let $E_i = \bigcup_{n \ge i} P_n$. Then $\langle E_i \rangle$ is a sequence with $E_i \supset E_{i+1}$ for each i and $m^*E_i \le m^*(\bigcup P_n) < \infty$ for each i. Furthermore, $\bigcap E_i = \emptyset$ since if $x \in P_k$, then $x \notin \bigcup_{n \ge k+1} P_n$ so $m^*(\bigcap E_i) = 0$. On the other hand, $P_i \subset E_i$ for each i so $0 < m^*P = m^*P_i \le m^*E_i$ for each i and $\lim m^*E_i \ge m^*P > 0 = m^*(\bigcap E_i)$.

3.5 Measurable functions

18. Let E be the nonmeasurable set defined in Section 3.4. Let f be defined on $[0,1]$ with $f(x) = x + 1$ if $x \in E$ and $f(x) = -x$ if $x \notin E$. Then f takes each value at most once so $\{x : f(x) = \alpha\}$ has at most one element for each $\alpha \in \mathbb{R}$ and each of these sets is measurable. However, $\{x : f(x) > 0\} = E$, which is nonmeasurable.

19. Let D be a dense set of real numbers and let f be an extended real-valued function on $\mathbb{R}$ such that $\{x : f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Let $\beta \in \mathbb{R}$. For each n, there exists $\alpha_n \in D$ such that $\beta < \alpha_n < \beta + 1/n$. Now $\{x : f(x) > \beta\} = \bigcup \{x : f(x) \ge \beta + 1/n\} = \bigcup \{x : f(x) > \alpha_n\}$ so $\{x : f(x) > \beta\}$ is measurable and f is measurable.

20. Let $\varphi_1 = \sum_{i=1}^{n_1} \alpha_i \chi_{A_i}$ and let $\varphi_2 = \sum_{i=1}^{n_2} \beta_i \chi_{B_i}$. Then $\varphi_1 + \varphi_2 = \sum_{i=1}^{n_1} \alpha_i \chi_{A_i} + \sum_{i=1}^{n_2} \beta_i \chi_{B_i}$ is a simple function. Also $\varphi_1\varphi_2 = \sum_{i,j} \alpha_i\beta_j\chi_{A_i}\chi_{B_j}$ is a simple function. $\chi_{A\cap B}(x) = 1$ if and only if $x \in A$ and $x \in B$ if and only if $\chi_A(x) = 1 = \chi_B(x)$. Thus $\chi_{A\cap B} = \chi_A\chi_B$. If $\chi_{A\cup B}(x) = 1$, then $x \in A \cup B$. If $x \in A\cap B$, then $\chi_A(x)+\chi_B(x)+\chi_A\chi_B(x) = 1+1-1 = 1$. If $x \notin A\cap B$, then $x \in A\setminus B$ or $x \in B\setminus A$ so $\chi_A(x)+\chi_B(x) = 1$ and $\chi_A\chi_B(x) = 0$. If $\chi_{A\cup B}(x) = 0$, then $x \notin A\cup B$ so $\chi_A(x) = \chi_B(x) = \chi_A\chi_B(x) = 0$. Hence $\chi_{A\cup B} = \chi_A + \chi_B + \chi_A\chi_B$. If $\chi_{A^c}(x) = 1$, then $x \notin A$ so $\chi_A(x) = 0$. If $\chi_{A^c}(x) = 0$, then $x \in A$ so $\chi_A(x) = 1$. Hence $\chi_{A^c} = 1 - \chi_A$.

21a. Let D and E be measurable sets and f a function with domain $D \cup E$. Suppose f is measurable. Since D and E are measurable subsets of $D\cup E$, $f|_D$ and $f|_E$ are measurable. Conversely, suppose $f|_D$ and $f|_E$ are measurable. Then for any $\alpha \in \mathbb{R}$, $\{x : f(x) > \alpha\} = \{x \in D : f|_D(x) > \alpha\}\cup\{x \in E : f|_E(x) > \alpha\}$. Each set on the right is measurable so $\{x : f(x) > \alpha\}$ is measurable and f is measurable.

21b. Let f be a function with measurable domain D. Let g be defined by $g(x) = f(x)$ if $x \in D$ and $g(x) = 0$ if $x \notin D$. Suppose f is measurable. If $\alpha \geq 0$, then $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\}$, which is measurable. If $\alpha < 0$, then $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup D^c$, which is measurable. Hence g is measurable. Conversely, suppose g is measurable. Then $f = g|_D$ and since D is measurable, f is measurable.

22a. Let f be an extended real-valued function with measurable domain D and let $D_1 = \{x : f(x) = \infty\}, D_2 = \{x : f(x) = -\infty\}$. Suppose f is measurable. Then D_1 and D_2 are measurable by Proposition 18. Now $D\setminus(D_1 \cup D_2)$ is a measurable subset of D so the restriction of f to $D\setminus(D_1\cup D_2)$ is measurable. Conversely, suppose D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable. For $\alpha \in \mathbb{R}$, $\{x : f(x) > \alpha\} = D_1\cup\{x : f|_{D\setminus(D_1\cup D_2)}(x) > \alpha\}$, which is measurable. Hence f is measurable.

22b. Let f and g be measurable extended real-valued functions defined on D. $D_1 = \{fg = \infty\} = \{f = \infty, g > 0\} \cup \{f = -\infty, g < 0\} \cup \{f > 0, g = \infty\} \cup \{f < 0, g = -\infty\}$, which is measurable. $D_2 = \{fg = -\infty\} = \{f = \infty, g < 0\} \cup \{f = -\infty, g > 0\} \cup \{f > 0, g = -\infty\} \cup \{f < 0, g = \infty\}$, which is measurable. Let $h = fg|_{D\setminus(D_1\cup D_2)}$ and let $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then $\{x : h(x) > \alpha\} = \{x : f|_{D\setminus\{x:f(x)=\pm\infty\}}(x) \cdot g|_{D\setminus\{x:g(x)=\pm\infty\}}(x) > \alpha\}$, which is measurable. If $\alpha < 0$, then $\{x : h(x) > \alpha\} = \{x : f(x) = 0\} \cup \{x : g(x) = 0\} \cup \{x : f|_{D\setminus\{f=\pm\infty\}}(x) \cdot g|_{D\setminus\{g=\pm\infty\}}(x) > \alpha\}$, which is measurable. Hence fg is measurable.

22c. Let f and g be measurable extended real-valued functions defined on D and α a fixed number. Define $f+g$ to be α whenever it is of the form $\infty-\infty$ or $-\infty+\infty$. $D_1 = \{f+g = \infty\} = \{f \in \mathbb{R}, g = \infty\}\cup \{f = g = \infty\} \cup \{f = \infty, g \in \mathbb{R}\}$, which is measurable. $D_2 = \{f + g = -\infty\} = \{f \in \mathbb{R}, g = -\infty\} \cup \{f = g = -\infty\} \cup \{f = -\infty, g \in \mathbb{R}\}$, which is measurable. Let $h = (f + g)|_{D\setminus(D_1\cup D_2)}$ and let $\beta \in \mathbb{R}$. If $\beta \geq \alpha$, then $\{x : h(x) > \beta\} = \{x : f|_{D\setminus\{f=\pm\infty\}}(x) + g|_{D\setminus\{g=\pm\infty\}}(x) > \beta\}$, which is measurable. If $\beta < \alpha$, then $\{x : h(x) > \beta\} = \{f = \infty, g = -\infty\} \cup \{f = -\infty, g = \infty\} \cup \{x : f|_{D\setminus\{f=\pm\infty\}}(x) + g|_{D\setminus\{g=\pm\infty\}}(x) > \beta\}$, which is measurable. Hence $f + g$ is measurable.

22d. Let f and g be measurable extended real-valued functions that are finite a.e. Then the sets $D_1, D_2, \{x : h(x) > \beta\}$ can be written as unions of sets as in part (c), possibly with an additional set of measure zero. Thus these sets are measurable and $f + g$ is measurable.

23a. Let f be a measurable function on $[a, b]$ that takes the values $\pm\infty$ only on a set of measure zero and let $\varepsilon > 0$. For each n, let $E_n = \{x \in [a, b] : |f(x)| > n\}$. Each E_n is measurable and $E_{n+1} \subset E_n$ for each n. Also, $mE_1 \leq b - a < \infty$. Thus $\lim mE_n = m(\bigcap E_n) = 0$. Thus there exists M such that $mE_M < \varepsilon/3$. i.e. $|f| \leq M$ except on a set of measure less than $\varepsilon/3$.

23b. Let f be a measurable function on $[a, b]$. Let $\varepsilon > 0$ and M be given. Choose N such that $M/N < \varepsilon$. For each $k \in \{-N, -N + 1, \ldots, N - 1\}$, let $E_k = \{x \in [a, b] : kM/N \leq f(x) < (k + 1)M/N\}$. Each E_k is measurable. Define φ by $\varphi = \sum_{k=-N}^{N-1}(kM/N)\chi_{E_k}$. Then φ is a simple function. If $x \in [a, b]$ such that $|f(x)| < M$, then $x \in E_k$ for some k. i.e. $kM/N \leq f(x) < (k + 1)M/N$ and $\varphi(x) = kM/N$. Thus $|f(x) - \varphi(x)| < M/N < \varepsilon$. If $m \leq f \leq M$, we may take φ so that $m \leq \varphi \leq M$ by replacing M/N by $(M - m)/N < \varepsilon$ in the preceding argument.

23c. Let φ be a simple function on $[a, b]$ and let $\varepsilon > 0$ be given. Let $\varphi = \sum_{i=1}^{n} a_i\chi_{A_i}$. For each i, there is a finite union $U_i = \bigcup_{k=1}^{N_i} I_{i,k}$ of (disjoint) open intervals such that $m(U_i\Delta A_i) < \varepsilon/3n$. Define g by $g = \sum_{i=1}^{n} a_i\chi_{U_i\setminus\bigcup_{m=1}^{i-1} U_m}$. Then g is a step function on $[a, b]$. If $g(x) \neq \varphi(x)$, then either $g(x) = a_i \neq \varphi(x)$, or $g(x) = 0$ and $\varphi(x) = a_i$. In the first case, $x \in U_i \setminus A_i$. In the second case, $x \in A_i \setminus U_i$. Thus $\{x \in [a, b] : \varphi(x) \neq g(x)\} \subset \bigcup_{i=1}^{n}((U_i \setminus A_i) \cup (A_i \setminus U_i)) = \bigcup_{i=1}^{n}(U_i\Delta A_i)$ so it has measure less than $\varepsilon/3$. If $m \leq \varphi \leq M$, we may take g so that $m \leq g \leq M$ since both g and φ take values in $\{a_1, \ldots, a_n\}$.

23d. Let g be a step function on $[a,b]$ and let $\varepsilon > 0$ be given. Let $x_0, \ldots, x_n$ be the partition points corresponding to g. Let $d = \min\{x_i - x_{i-1} : i = 1, \ldots, n\}$. For each i, let I_i be an open interval of length less than $\min(\varepsilon/3(n+1), d/2)$ centred at x_i. Define h by linearising g in each I_i. Then h is continuous and $\{x \in [a,b] : g(x) \neq h(x)\} \subset \bigcup_{i=0}^{n} I_i$, which has measure less than $\varepsilon/3$. If $m \leq g \leq M$, we may take h so that $m \leq h \leq M$ by construction.

24. Let f be measurable and B a Borel set. Let $\mathcal{C}$ be the collection of sets E such that $f^{-1}[E]$ is measurable. Suppose $E \in \mathcal{C}$. Then $f^{-1}[E^c] = (f^{-1}[E])^c$, which is measurable, so $E^c \in \mathcal{C}$. Suppose $\langle E_i \rangle$ is a sequence of sets in $\mathcal{C}$. Then $f^{-1}[\bigcup E_i] = \bigcup f^{-1}[E_i]$, which is measurable, so $\bigcup E_i \in \mathcal{C}$. Thus $\mathcal{C}$ is a σ-algebra. Now for any $a, b \in \mathbb{R}$ with $a < b$, $\{x : f(x) > a\}$ and $\{x : f(x) < b\}$ are measurable. i.e. (a, ∞) and $(-\infty, b)$ are in $\mathcal{C}$. Thus $(a,b) \in \mathcal{C}$ and $\mathcal{C}$ is a σ-algebra containing all the open intervals so it contains all the Borel sets. Hence $f^{-1}[B]$ is measurable.

25. Let f be a measurable real-valued function and g a continuous function defined on $(-\infty, \infty)$. Then g is also measurable. For any $\alpha \in \mathbb{R}$, $\{x : (g \circ f)(x) > \alpha\} = (g \circ f)^{-1}[(\alpha, \infty)] = f^{-1}[g^{-1}[(\alpha, \infty)]]$, which is measurable by Q24. Hence $g \circ f$ is measurable.

26. Propositions 18 and 19 and Theorem 20 follow from arguments similar to those in the original proofs and the fact that the collection of Borel sets is a σ-algebra. If f is a Borel measurable function, then for any $\alpha \in \mathbb{R}$, the set $\{x : f(x) > \alpha\}$ is a Borel set so it is Lebesgue measurable. Thus f is Lebesgue measurable. If f is Borel measurable and B is a Borel set, then consider the collection $\mathcal{C}$ of sets E such that $f^{-1}[E]$ is a Borel set. By a similar argument to that in Q24, $\mathcal{C}$ is a σ-algebra containing all the open intervals. Thus $\mathcal{C}$ contains all the Borel sets. Hence $f^{-1}[B]$ is a Borel set. If f and g are Borel measurable, then for $\alpha \in \mathbb{R}$, $\{x : (f \circ g)(x) > \alpha\} = (f \circ g)^{-1}[(\alpha, \infty)] = g^{-1}[f^{-1}[(\alpha, \infty)]]$, which is a Borel set. Thus $f \circ g$ is Borel measurable. If f is Borel measurable and g is Lebesgue measurable, then for any $\alpha \in \mathbb{R}$, $f^{-1}[(\alpha, \infty)]$ is a Borel set and $g^{-1}[f^{-1}[(\alpha, \infty)]]$ is Lebesgue measurable by Q24. Thus $f \circ g$ is Lebesgue measurable.

27. Call a function $\mathfrak{A}$-measurable if for each $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\}$ is in $\mathfrak{A}$. Propositions 18 and 19 and Theorem 20 still hold. An $\mathfrak{A}$-measurable function need not be Lebesgue measurable. For example, let $\mathfrak{A}$ be the σ-algebra generated by the nonmeasurable set P defined in Section 4. Then χ_P is $\mathfrak{A}$-measurable but not Lebesgue measurable. There exists a Lebesgue measurable function g and a Lebesgue measurable set A such that $g^{-1}[A]$ is nonmeasurable (see Q28). If f and g are Lebesgue measurable, $f \circ g$ may not be Lebesgue measurable. For example, take g and A to be Lebesgue measurable with $g^{-1}[A]$ nonmeasurable. Let $f = \chi_A$ so that f is Lebesgue measurable. Then $\{x : (f \circ g)(x) > 1/2\} = g^{-1}[A]$, which is nonmeasurable. This is also a counterexample for the last statement.

28a. Let f be defined by $f(x) = f_1(x) + x$ for $x \in [0,1]$. By Q2.48, f_1 is continuous and monotone on $[0,1]$ so f is continuous and strictly monotone on $[0,1]$ and f maps $[0,1]$ onto $[0,2]$. By Q2.46, f has a continuous inverse so it is a homeomorphism of $[0,1]$ onto $[0,2]$.

28b. By Q2.48, f_1 is constant on each interval contained in the complement of the Cantor set. Thus f maps each of these intervals onto an interval of the same length. Thus $m(f[[0,1] \setminus C]) = m([0,1] \setminus C) = 1$ and since f is a bijection of $[0,1]$ onto $[0,2]$, $mF = m(f[C]) = m([0,2]) - 1 = 1$.

28c. Let $g = f^{-1} : [0,2] \to [0,1]$. Then g is measurable. Since $mF = 1 > 0$, there is a nonmeasurable set $E \subset F$. Let $A = f^{-1}[E]$. Then $A \subset C$ so it has outer measure zero and is measurable but $g^{-1}[A] = E$ so it is nonmeasurable.

28d. The function $g = f^{-1}$ is continuous and the function $h = \chi_A$ is measurable, where A is as defined in part (c). However the set $\{x : (h \circ g)(x) > 1/2\} = g^{-1}[A]$ is nonmeasurable. Thus $h \circ g$ is not measurable.

28e. The set A in part (c) is measurable but by Q24, it is not a Borel set since $g^{-1}[A]$ is nonmeasurable.

3.6 Littlewood's three principles

29. Let $E = \mathbb{R}$ and let $f_n = \chi_{[n,\infty)}$ for each n. Then $f_n(x) \to 0$ for each $x \in E$. For any measurable set $A \subset E$ with $mA < 1$ and any integer N, pick $x \geq N$ such that $x \notin A$. Then $|f_N(x) - 0| \geq 1$.

30. Egoroff's Theorem: Let $\langle f_n \rangle$ be a sequence of measurable functions that converges to a real-valued function f a.e. on a measurable set E of finite measure. Let $\eta > 0$ be given. For each n, there exists $A_n \subset E$ with $mA_n < \eta/2^n$ and there exists N_n such that for all $x \notin A_n$ and $k \geq N_n$, $|f_k(x) - f(x)| < 1/n$.

Let $A = \bigcup A_n$. Then $A \subset E$ and $mA < \sum \eta/2^n = \eta$. Choose n_0 such that $1/n_0 < \eta$. If $x \notin A$ and $k \geq N_{n_0}$, we have $|f_k(x) - f(x)| < 1/n_0 < \eta$. Thus f_n converges to f uniformly on $E \setminus A$.

31. Lusin's Theorem: Let f be a measurable real-valued function on $[a,b]$ and let $\delta > 0$ be given. For each n, there is a continuous function h_n on $[a,b]$ such that $m\{x : |h_n(x) - f(x)| \geq \delta/2^{n+2}\} < \delta/2^{n+2}$. Let $E_n = \{x : |h_n(x) - f(x)| \geq \delta/2^{n+2}\}$. Then $|h_n(x) - f(x)| < \delta/2^{n+2}$ for $x \in [a,b] \setminus E_n$. Let $E = \bigcup E_n$. Then $mE < \delta/4$ and $\langle h_n \rangle$ is a sequence of continuous, thus measurable, functions that converges to f on $[a,b] \setminus E$. By Egoroff's Theorem, there is a subset $A \subset [a,b] \setminus E$ such that $mA < \delta/4$ and h_n converges uniformly to f on $[a,b] \setminus (E \cup A)$. Thus f is continuous on $[a,b] \setminus (E \cup A)$ with $m(E \cup A) < \delta/2$. Now there is an open set O such that $O \supset (E \cup A)$ and $m(O \setminus (E \cup A)) < \delta/2$. Then f is continuous on $[a,b] \setminus O$, which is closed, and $mO < \delta$. By Q2.40, there is a function φ that is continuous on $(-\infty, \infty)$ such that $f = \varphi$ on $[a,b] \setminus O$. In particular, φ is continuous on $[a,b]$ and $m\{x \in [a,b] : f(x) \neq \varphi(x)\} = mO < \delta$. If f is defined on $(-\infty, \infty)$, let $\delta' = \min(\delta/2^{n+3}, 1/2)$. Then for each n, there is a continuous function φ_n on $[n + \delta', n + 1 - \delta']$ such that $m\{x \in [n + \delta', n + 1 - \delta'] : f(x) \neq \varphi_n(x)\} < \delta/2^{n+2}$. Similarly for $[-n - 1 + \delta', -n - \delta']$. Linearise in each interval $[n - \delta', n + \delta']$. Similarly for intervals $[-n - \delta', -n + \delta']$. Then we have a continuous function φ defined on $(-\infty, \infty)$ with $m\{x : f(x) \neq \varphi(x)\} < 4\sum \delta/2^{n+2} = \delta$.

***32.** For $t \in [0,1)$ with $1/2^{i+1} \leq t < 1/2^i$, define $f_t : [0,1) \to \mathbb{R}$ by $f_t(x) = 1$ if $x \in P_i = P + r_i$ and $x = 2^{i+1}t - 1$, and $f_t(x) = 0$ otherwise. For each t, there is at most one x such that $f_t(x) = 1$ so each f_t is measurable. For each x, $x \in P_{i(x)}$ for some $i(x)$. Let $t(x) = (x+1)/2^{i(x)+1}$. Then $1/2^{i(x)+1} \leq t(x) < 1/2^{i(x)}$ and $f_{t(x)}(x) = 1$. This is the only t such that $f_t(x) = 1$. Thus for each x, $f_t(x) \to 0$ as $t \to 0$. Note that any measurable subset of $[0,1)$ with positive measure intersects infinitely many of the sets P_i. Thus for any measurable set $A \subset [0,1)$ with $mA < 1/2$, $m([0,1) \setminus A) \geq 1/2$ so there exists $x \in [0,1) \setminus A$ with $i(x)$ arbitrarily large and so with $t(x)$ arbitrarily small. i.e. there exist an $x \in [0,1) \setminus A$ and arbitrarily small t such that $f_t(x) \geq 1/2$.

(*) Any measurable set $A \subset [0,1)$ with positive measure intersects infinitely many of the sets P_i:

Suppose A intersects only finitely many of the sets P_i. i.e. $A \subset \bigcup_{i=1}^n P_{q_i}$, where $P_{q_i} = P + q_i$. Choose $r_1 \in \mathbb{Q} \cap [-1,1]$ such that $r_1 \neq q_i - q_j$ for all i, j. Suppose $r_1, \ldots, r_n$ have been chosen. Choose r_{n+1} such that $r_{n+1} \neq q_i - q_j + r_k$ for all i, j and $k \leq n$. Now the measurable sets $A + r_i$ are disjoint by the definition of P and the construction of the sequence $\langle r_i \rangle$. Then $m(\bigcup_{i=1}^n (A + r_i)) = \sum_{i=1}^n m(A + r_i) = nmA$ for each n. Since $\bigcup_{i=1}^n (A + r_i) \subset [-1, 2]$, $nmA \leq 3$ for all n and $mA = 0$.

4 The Lebesgue Integral

4.1 The Riemann integral

1a. Let f be defined by $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational. For any subdivision $a = \xi_0 < \xi_1 < \cdots < \xi_n = b$ of $[a,b]$, $M_i = \sup_{\xi_{i-1} < x \leq \xi_i} f(x) = 1$ and $m_i = \inf_{\xi_{i-1} < x \leq \xi_i} f(x) = 0$ for each i. Thus $S = \sum_{i=1}^n (\xi_i - \xi_{i-1}) M_i = b - a$ and $s = \sum_{i=1}^n (\xi_i - \xi_{i-1}) m_i = 0$ for any subdivision of $[a,b]$. Hence $R\overline{\int}_a^b f(x) = b - a$ and $R\underline{\int}_a^b f(x) = 0$.

1b. Let $\langle r_n \rangle$ be an enumeration of $\mathbb{Q} \cap [a,b]$. For each n, let $f_n = \chi_{\{r_1, \ldots, r_n\}}$. Then $\langle f_n \rangle$ is a sequence of nonnegative Riemann integrable functions increasing monotonically to f. Thus the limit of a sequence of Riemann integrable functions may not be Riemann integrable so in general we cannot interchange the order of integration and the limiting process.

4.2 The Lebesgue integral of a bounded function over a set of finite measure

2a. Let f be a bounded function on $[a,b]$ and let h be the upper envelope of f. i.e. $h(y) = \inf\limits_{\delta > 0} \sup\limits_{|y-x|<\delta} f(x)$. Since f is bounded, h is upper semicontinuous by Q2.51b and h is bounded. For any $\alpha \in \mathbb{R}$, the set $\{x : h(x) < \alpha\}$ is open. Thus h is measurable. Let φ be a step function on $[a,b]$ such that $\varphi \geq f$. Then $\varphi \geq h$ except at a finite number of points (the partition points). Thus $R\overline{\int}_a^b f = \inf_{\varphi \geq f} \int_a^b \varphi \geq \int_a^b h$. Conversely, there exists a monotone decreasing sequence $\langle \varphi_n \rangle$ of step functions such that $h(x) = \lim \varphi_n(x)$ for each $x \in [a,b]$. Thus $\int_a^b h = \lim \int_a^b \varphi_n \geq R\overline{\int}_a^b f$. Hence $R\overline{\int}_a^b f = \int_a^b h$.

2b. Let f be a bounded function on $[a, b]$ and let E be the set of discontinuities of f. Let g be the lower envelope of f. By a similar argument as in part (a), $R\underline{\int}_a^b f = \int_a^b g$. Suppose E has measure zero. Then $g = h$ a.e. so $R\underline{\int}_a^b f = \int_a^b g = \int_a^b h = R\overline{\int}_a^b f$ so f is Riemann integrable. Conversely, suppose f is Riemann integrable. Then $\int_a^b g = \int_a^b h$. Thus $\int_a^b |g - h| = 0$ so $g = h$ a.e. since $\int_a^b |g - h| \geq (1/n)m\{x : |g(x) - h(x)| > 1/n\}$ for all n. Hence f is continuous a.e. and $mE = 0$.

4.3 The integral of a nonnegative function

3. Let f be a nonnegative measurable function and suppose $\inf f = 0$. Let $E = \{x : f(x) > 0\}$. Then $E = \bigcup E_n$ where $E_n = \{x : f(x) \geq 1/n\}$. Now $\int f \geq (1/n)mE_n$ for each n so $mE_n = 0$ for each n and $mE = 0$. Thus $f = 0$ a.e.

4a. Let f be a nonnegative measurable function. For $n = 1, 2, \ldots$, let $E_{n,i} = f^{-1}[(i-1)2^{-n}, i2^{-n})$ where $i = 1, \ldots, n2^n$ and let $E_{n,0} = f^{-1}[n, \infty)$. Define $\varphi_n = \sum_{i=1}^{n2^n}(i-1)2^{-n}\chi_{(E_{n,i}\cap[-n,n])} + n\chi_{(E_{n,0}\cap[-n,n])}$. Then each φ_n is a nonnegative simple function vanishing outside a set of finite measure and $\varphi_n \leq \varphi_{n+1}$ for each n. Furthermore, for sufficiently large n, $f_n(x) - \varphi_n(x) \leq 2^{-n}$ if $x \in [-n, n]$ and $f(x) < n$. Thus $\varphi_n(x) \to f(x)$ when $f(x) < \infty$. Also, for sufficiently large n, $\varphi_n(x) = n \to \infty$ if $f(x) = \infty$.

4b. Let f be a nonnegative measurable function. Then by part (a), there is an increasing sequence $\langle\varphi_n\rangle$ of simple functions such that $f = \lim \varphi_n$. By the Monotone Convergence Theorem, $\int f = \lim \int \varphi_n = \sup \int \varphi_n$. Thus $\int f \leq \sup \int \varphi$ over all simple functions $\varphi \leq f$. On the other hand, $\int f \geq \int \varphi$ for all simple functions $\varphi \leq f$. Thus $\int f \geq \sup \int \varphi$ over all simple functions $\varphi \leq f$. Hence $\int f = \sup \int \varphi$ over all simple functions $\varphi \leq f$.

5. Let f be a nonnegative integrable function and let $F(x) = \int_{-\infty}^x f$. For each n, let $f_n = f\chi_{(-\infty,x-1/n]}$. Then $\langle f_n\rangle$ is an increasing sequence of nonnegative measurable functions with $f\chi_{(-\infty,x]} = \lim f_n$. By the Monotone Convergence Theorem, $\lim F(x - 1/n) = \lim \int f_n = \int f\chi_{(-\infty,x]} = F(x)$. Now for each n, let $g_n = f\chi_{(x+1/n,\infty)}$. Then $\langle g_n\rangle$ is an increasing sequence of nonnegative measurable functions with $f\chi_{(x,\infty)} = \lim g_n$. By the Monotone Convergence Theorem, $\lim \int g_n = \int f\chi_{(x,\infty)}$. i.e. $\lim \int_{x+1/n}^\infty f = \int_x^\infty f$. Since f is integrable, we have $\lim(\int f - \int_{-\infty}^{x+1/n} f) = \int f - \int_{-\infty}^x f$ so $\lim \int_{-\infty}^{x+1/n} f = \int_{-\infty}^x f$. i.e. $\lim F(x + 1/n) = F(x)$. Now given $\varepsilon > 0$, there exists N such that $F(x) - F(x - 1/n) < \varepsilon$ and $F(x+1/n) - F(x) < \varepsilon$ whenever $n \geq N$. Choose $\delta < 1/N$. Then $|F(y) - F(x)| < \varepsilon$ whenever $|x - y| < \delta$. Hence F is continuous.

6. Let $\langle f_n\rangle$ be a sequence of nonnegative measurable functions converging to f and suppose $f_n \leq f$ for each n. By Fatou's Lemma, $\int f \leq \underline{\lim} \int f_n$. On the other hand, $\int f \geq \overline{\lim} \int f_n$ since $f \geq f_n$ for each n. Hence $\int f = \lim \int f_n$.

7a. For each n, let $f_n = \chi_{[n,n+1)}$. Then $\langle f_n\rangle$ is a sequence of nonnegative measurable functions with $\lim f_n = 0$. Now $\int 0 = 0 < 1 = \underline{\lim} \int f_n$ and we have strict inequality in Fatou's Lemma.

7b. For each n, let $f_n = \chi_{[n,\infty)}$. Then $\langle f_n\rangle$ is a decreasing sequence of nonnegative measurable functions with $\lim f_n = 0$. Now $\int 0 = 0 < \infty = \lim \int f_n$ so the Monotone Convergence Theorem does not hold.

8. Let $\langle f_n\rangle$ be a sequence of nonnegative measurable functions. For each n, let $h_n = \inf_{k\geq n} f_k$. Then each h_n is a nonnegative measurable function with $h_n \leq f_n$. By Fatou's Lemma, $\int \underline{\lim} f_n = \int \lim h_n \leq \underline{\lim} \int h_n \leq \underline{\lim} \int f_n$.

9. Let $\langle f_n\rangle$ be a sequence of nonnegative measurable functions such that $f_n \to f$ a.e. and suppose that $\int f_n \to \int f < \infty$. Then for any measurable set E, $\langle f_n\chi_E\rangle$ is a sequence of nonnegative measurable functions with $f_n\chi_E \to f\chi_E$ a.e. By Fatou's Lemma, $\int_E f \leq \underline{\lim} \int_E f_n$. Now $f\chi_E$ is integrable since $f\chi_E \leq f$. Also, f_n is integrable for sufficiently large n so $f_n\chi_E$ is integrable for sufficiently large n. By Fatou's Lemma, $\int(f - f\chi_E) \leq \underline{\lim} \int(f_n - f_n\chi_E)$. i.e. $\int f - \int_E f \leq \underline{\lim} \int f_n - \overline{\lim} \int_E f_n = \int f - \overline{\lim} \int_E f_n$. Thus $\overline{\lim} \int_E f_n \leq \int_E f$ and we have $\int_E f_n \to \int_E f$.

4.4 The general Lebesgue integral

10a. If f is integrable over E, then so are f^+ and f^-. Thus $|f| = f^+ + f^-$ is integrable over E and $|\int_E f| = |\int_E f^+ - \int_E f^-| \leq |\int_E f^+| + |\int_E f^-| = \int_E f^+ + \int_E f^- = \int_E |f|$. Conversely, if $|f|$ is integrable over E, then $\int_E f^+ \le \int_E |f| < \infty$ and $\int_E f^- \le \int_E |f| < \infty$ so f^+ and f^- are integrable over E and f is integrable over E.

10b. $f(x) = \sin x/x$ is not Lebesgue integrable on $[0,\infty]$ although $R\int_0^\infty f(x) = \pi/2$ (by contour integration for example). In general, suppose f is Lebesgue integrable and the Riemann integral $R\int_a^b f$ exists with improper lower limit a. If a is finite, let $f_n = f\chi_{[a+1/n,b]}$. Then $f_n \to f$ on $[a,b]$ and $|f_n| \le |f|$ so $R\int_a^b f = \lim R\int_{a+1/n}^b f = \lim \int f_n = \int_a^b f$. If $a = -\infty$, let $g_n = f\chi_{[-n,b]}$. Then $g_n \to f$ on $[a,b]$ and $|g_n| \le |f|$ so $R\int_a^b f = \lim R\int_{-n}^b f = \lim \int g_n = \int_a^b f$. The cases where the Riemann integral has improper upper limit are similar.

11. Let $\varphi = \sum_{i=1}^n a_i\chi_{A_i}$ be a simple function with canonical representation. Let $S+ = \{i : a_i \ge 0\}$ and let $S- = \{i : a_i < 0\}$. Then $\varphi^+ = \sum_{i\in S+} a_i\chi_{A_i}$ and $\varphi^- = -\sum_{i\in S-} a_i\chi_{A_i}$. Clearly φ^+ and φ^- are simple functions. Then $\int \varphi^+ = \sum_{i\in S+} a_i mA_i$ and $\int \varphi^- = -\sum_{i\in S-} a_i mA_i$ so $\int \varphi = \int \varphi^+ - \int \varphi^- = \sum_{i=1}^n a_i mA_i$.

12. Let g be an integrable function on a set E and suppose that $\langle f_n\rangle$ is a sequence of measurable functions with $|f_n| \le g$ a.e. on E. Then $\langle f_n + g\rangle$ is a sequence of nonnegative measurable functions on E. Thus $\int \underline{\lim} f_n + \int g \le \int \underline{\lim}(f_n + g) \le \underline{\lim}\int(f_n + g) \le \underline{\lim}\int f_n + \int g$ so $\int \underline{\lim} f_n \le \underline{\lim}\int f_n$. Also, $\langle g - f_n\rangle$ is a sequence of nonnegative measurable functions on E. Thus $\int g + \int \underline{\lim}(-f_n) \le \int \underline{\lim}(g - f_n) \le \underline{\lim}\int(g - f_n) \le \int g + \underline{\lim}(-\int f_n)$ so $\int \underline{\lim}(-f_n) \le \underline{\lim}(-\int f_n)$. i.e. $\overline{\lim}\int f_n \le \int \overline{\lim} f_n$. Hence we have $\int \underline{\lim} f_n \le \underline{\lim}\int f_n \le \overline{\lim}\int f_n \le \int \overline{\lim} f_n$.

13. Let h be an integrable function and $\langle f_n\rangle$ a sequence of measurable functions with $f_n \ge -h$ and $\lim f_n = f$. For each n, $f_n + h$ is a nonnegative measurable function. Since h is integrable, $\int f_n = \int(f_n + h) - \int h$. Similarly, $\int f = \int(f + h) - \int h$. Now $\int f + \int h \le \int \underline{\lim}(f_n + h) \le \underline{\lim}\int f_n + \int h$ so $\int f \le \underline{\lim}\int f_n$.

14a. Let $\langle g_n\rangle$ be a sequence of integrable functions which converges a.e. to an integrable function g and let $\langle f_n\rangle$ be a sequence of measurable functions such that $|f_n| \le g_n$ and $\langle f_n\rangle$ converges to f a.e. Suppose $\int g = \lim \int g_n$. Since $|f_n| \le g_n$, $|f| \le g$. Thus $|f_n - f| \le |f_n| + |f| \le g_n + g$ and $\langle g_n + g - |f_n - f|\rangle$ is a sequence of nonnegative measurable functions. By Fatou's Lemma, $\int \lim(g_n + g - |f_n - f|) \le \underline{\lim}\int(g_n + g - |f_n - f|)$. i.e. $\int 2g \le \int 2g + \underline{\lim}(-\int |f_n - f|) = \int 2g - \overline{\lim}\int |f_n - f|$. Hence $\overline{\lim}\int |f_n - f| \le 0 \le \underline{\lim}\int |f_n - f|$ and we have $\int |f_n - f| \to 0$.

14b. Let $\langle f_n\rangle$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. If $\int |f_n - f| \to 0$, then $\left|\int |f_n| - \int |f|\right| \le \int \left||f_n| - |f|\right| \le \int |f_n - f| \to 0$. Thus $\int |f_n| \to \int |f|$. Conversely, suppose $\int |f_n| \to \int |f|$. By part (a), with $|f_n|$ in place of g_n and $|f|$ in place of g, we have $\int |f_n - f| \to 0$.

15a. Let f be integrable over E and let $\varepsilon > 0$ be given. By Q4, there is a simple function $\psi \le f^+$ such that $\int_E f^+ - \varepsilon/2 < \int_E \psi$. Also, there is a simple function $\psi' \le f^-$ such that $\int_E f^- - \varepsilon/2 < \int_E \psi'$. Let $\varphi = \psi - \psi'$. Then φ is a simple function and $\int_E |f - \varphi| = \int_E |f^+ - \psi - f^- + \psi'| \le \int_E |f^+ - \psi| + \int_E |f^- - \psi'| = \int_E(f^+ - \psi) + \int_E(f^- - \psi') < \varepsilon$.

15b. Let $f_n = f\chi_{[-n,n]}$. Then $f_n \to f$ and $|f_n| \le |f|$. By Lebesgue's Dominated Convergence Theorem, $\int_E |f - f\chi_{[-n,n]}| \to 0$. i.e. $\int_{E\cap[-n,n]^c} |f| \to 0$. Thus there exists N such that $\int_{E\cap[-N,N]^c} |f| < \varepsilon/3$. By part (a), there is a simple function φ such that $\int_E |f - \varphi| < \varepsilon/3$. By Proposition 3.22, there is a step function ψ on $[-N,N]$ such that $|\varphi - \psi| < \varepsilon/12NM$ except on a set of measure less than $\varepsilon/12NM$, where $M \ge \max(|\varphi|,|\psi|) + 1$. We may regard ψ as a function on $\mathbb{R}$ taking the value 0 outside $[-N,N]$. Then $\int_{-N}^N |\varphi - \psi| < \varepsilon/3$ so $\int_E |f - \psi| = \int_{E\cap[-N,N]} |f - \psi| + \int_{E\cap[-N,N]^c} |f - \psi| \le \int_{E\cap[-N,N]} |f - \varphi| + \int_{E\cap[-N,N]} |\varphi - \psi| + \int_{E\cap[-N,N]^c} |f - \psi| \le \int_E |f - \varphi| + \int_{-N}^N |\varphi - \psi| + \int_{E\cap[-N,N]^c} |f| < \varepsilon$.

15c. By part (b), there is a step function ψ such that $\int_E |f - \psi| < \varepsilon/2$. Suppose ψ is defined on $[a,b]$. We may regard ψ as a function on $\mathbb{R}$ taking the value 0 outside $[a,b]$. By linearising ψ at each partition point, we get a continuous function g vanishing outside a finite interval such that $\psi = g$ except on a set of measure less than $\varepsilon/4M$, where $M \ge |\psi|$. Then $\int_E |f - g| \le \int_E |f - \psi| + \int_E |\psi - g| < \varepsilon$.

16. Riemann-Lebesgue Theorem: Suppose f is integrable on $(-\infty,\infty)$. By Q15, given $\varepsilon > 0$, there is a step function ψ such that $\int |f - \psi| < \varepsilon/2$. Now $|\int f(x)\cos nx\,dx| \le \int |f(x)\cos nx|\,dx \le \int |(f(x) - \psi(x))\cos nx|\,dx + \int |\psi(x)\cos nx|\,dx < \varepsilon/2 + \int |\psi(x)\cos nx|\,dx$. Integrating $|\psi(x)\cos nx|$ over each interval on which ψ is constant, we see that $\int |\psi(x)\cos nx|\,dx \to 0$ as $n \to \infty$. Thus there exists N such that $\int |\psi(x)\cos nx|\,dx < \varepsilon/2$ for $n \ge N$ so $|\int f(x)\cos nx\,dx| < \varepsilon$ for $n \ge N$. i.e.

$\lim_{n\to\infty} \int f(x)\cos nx\, dx = 0$.

17a. Let f be integrable over $(-\infty,\infty)$. Then f^+ and f^- are nonnegative integrable functions. There exists an increasing sequence $\langle\varphi_n\rangle$ of nonnegative simple functions such that $f^+ = \lim\varphi_n$. Now since $\int \chi_E(x)\,dx = mE = m(E-t) = \int \chi_E(x+t)\,dx$ for any measurable set E, we have $\int\varphi_n(x)\,dx = \int\varphi_n(x+t)\,dx$ for all n. By the Monotone Convergence Theorem, $\int f^+(x)\,dx = \int f^+(x+t)\,dx$. Similarly for f^-. Thus $\int f(x)\,dx = \int f(x+t)\,dx$.

17b. Let g be a bounded measurable function and let M be such that $|g|\le M$. Since f is integrable, given $\varepsilon>0$, there is a continuous function h vanishing outside a finite interval $[a,b]$ such that $\int|f-h|<\varepsilon/4M$. Now $\int|g(x)[f(x)-f(x+t)]| \le \int|g(x)[h(x)-h(x+t)]| + \int|g(x)[(f-h)(x)-(f-h)(x+t)]|$. Now h is uniformly continuous on $[a,b]$ so there exists $\delta>0$ such that $|h(x)-h(x+t)|<\varepsilon/2M(b-a)$ whenever $|t|<\delta$. Then $\int|g(x)[f(x)-f(x+t)]| \le \varepsilon/2 + M\left(\int|(f-h)(x)| + \int|(f-h)(x+t)|\right) = \varepsilon/2 + 2M\int|f-h| < \varepsilon$ whenever $|t|<\delta$. i.e. $\lim_{t\to 0}\int|g(x)[f(x)-f(x+t)]| = 0$.

18. Let f be a function of 2 variables $\langle x,t\rangle$ defined on the square $Q=[0,1]\times[0,1]$ and which is a measurable function of x for each fixed t. Suppose that $\lim_{t\to 0} f(x,t) = f(x)$ and that for all t we have $|f(x,t)|\le g(x)$ where g is an integrable function on $[0,1]$. Let $\langle t_n\rangle$ be a sequence such that $t_n\neq 0$ for all n and $\lim_n t_n = 0$. Then $\lim_n f(x,t_n) = f(x)$. For each n, $h_n(x) = f(x,t_n)$ is measurable and $|h_n|\le g$. By Lebesgue's Dominated Convergence Theorem, $\lim_n\int h_n = \int f$. i.e. $\lim_{t\to 0}\int f(x,t)\,dx = \int f(x)\,dx$. Suppose further that $f(x,t)$ is continuous in t for each x and let $h(t) = \int f(x,t)\,dx$. Let $\langle t_n\rangle$ be a sequence converging to t. Then $\lim f(x,t_n) = f(x,t)$ for each x. By Lebesgue's Dominated Convergence Theorem, $\lim\int f(x,t_n)\,dx = \int f(x,t)\,dx$. i.e. $\lim h(t_n) = h(t)$. Hence h is a continuous function of t.

19. Let f be a function defined and bounded in the square $Q=[0,1]\times[0,1]$ and suppose that for each fixed t the function f is a measurable function of x. For each $\langle x,t\rangle$ in Q, let the partial derivative $\partial f/\partial t$ exist. Suppose that $\partial f/\partial t$ is bounded in Q. Let $\langle s_n\rangle$ be a sequence such that $s_n\neq 0$ for all n and $\lim_n s_n = 0$. Then $\lim_n[f(x,t+s_n)-f(x,t)]/s_n \to \partial f/\partial t$. Since $\partial f/\partial t$ is bounded, there exists M such that $|[f(x,t+s_n)-f(x,t)]/s_n| \le M+1$ for sufficiently large n. For each fixed t, f is a bounded measurable function of x so $[\int_0^1 f(x,t+s_n)\,dx - \int_0^1 f(x,t)\,dx]/s_n = \int_0^1([f(x,t+s_n)-f(x,t)]/s_n)\,dx$. Thus $\frac{d}{dt}\int_0^1 f(x,t)\,dx = \lim_n[\int_0^1 f(x,t+s_n)\,dx - \int_0^1 f(x,t)\,dx]/s_n = \lim_n\int_0^1([f(x,t+s_n)-f(x,t)]/s_n)\,dx = \int_0^1 \frac{\partial f}{\partial t}\,dx$, the last equality following from Lebesgue's Dominated Convergence Theorem.

4.5 Convergence in measure

20. Let $\langle f_n\rangle$ be a sequence that converges to f in measure. Then given $\varepsilon>0$, there exists N such that $m\{x : |f_n(x)-f(x)|\ge\varepsilon\}<\varepsilon$ for $n\ge N$. For any subsequence $\langle f_{n_k}\rangle$, choose M such that $n_k\ge N$ for $k\ge M$. Then $m\{x : |f_{n_k}(x)-f(x)|\ge\varepsilon\}<\varepsilon$ for $k\ge M$. Thus $\langle f_{n_k}\rangle$ converges to f in measure.

21. Fatou's Lemma: Let $\langle f_n\rangle$ be a sequence of nonnegative measurable functions that converges in measure to f on E. Then there is a subsequence $\langle f_{n_k}\rangle$ such that $\lim\int_E f_{n_k} = \underline{\lim}\int_E f_n$. By Q20, $\langle f_{n_k}\rangle$ converges in measure to f on E so it in turn has a subsequence $\langle f_{n_{k_j}}\rangle$ that converges to f a.e. Thus $\int_E f \le \underline{\lim}\int_E f_{n_{k_j}} = \lim\int_E f_{n_k} = \underline{\lim}\int_E f_n$.

Monotone Convergence Theorem: Let $\langle f_n\rangle$ be an increasing sequence of nonnegative measurable functions that converges in measure to f. Any subsequence $\langle f_{n_k}\rangle$ also converges in measure to f so it in turn has a subsequence $\langle f_{n_{k_j}}\rangle$ that converges to f a.e. Thus $\int f = \lim\int f_{n_{k_j}}$. By Q2.12, $\int f = \lim\int f_n$.

Lebesgue's Dominated Convergence Theorem: Let g be integrable over E and let $\langle f_n\rangle$ be a sequence of measurable functions such that $|f_n|\le g$ on E and converges in measure to f on E. Any subsequence $\langle f_{n_k}\rangle$ also converges in measure to f so it in turn has a subsequence $\langle f_{n_{k_j}}\rangle$ that converges to f a.e. Thus $\int f = \lim\int f_{n_{k_j}}$. By Q2.12, $\int f = \lim\int f_n$.

22. Let $\langle f_n\rangle$ be a sequence of measurable functions on a set E of finite measure. If $\langle f_n\rangle$ converges to f in measure, then so does any subsequence $\langle f_{n_k}\rangle$. Thus any subsequence of $\langle f_{n_k}\rangle$ also converges to f in measure. Conversely, if $\langle f_n\rangle$ does not converge in measure to f, then there exists $\varepsilon>0$ such that for any N there exists $n\ge N$ with $m\{x : |f_n(x)-f(x)|\ge\varepsilon\}\ge\varepsilon$. This gives rise to a subsequence $\langle f_{n_k}\rangle$ such that $m\{x : |f_{n_k}(x)-f(x)\ge\varepsilon\}\ge\varepsilon$ for all k. This subsequence will not have a further subsequence that converges in measure to f.

23. Let $\langle f_n\rangle$ be a sequence of measurable functions on a set E of finite measure. If $\langle f_n\rangle$ converges to f

in measure, then so does any subsequence $\langle f_{n_k}\rangle$. Thus $\langle f_{n_k}\rangle$ has in turn a subsequence that converges to f a.e. Conversely, if every subsequence $\langle f_{n_k}\rangle$ has in turn a subsequence $\langle f_{n_{k_j}}\rangle$ that converges to f a.e., then $\langle f_{n_{k_j}}\rangle$ converges to f in measure so by Q22, $\langle f_n\rangle$ converges to f in measure.

24. Suppose that $f_n \to f$ in measure and that there is an integrable function g such that $|f_n| \le g$ for all n. Let $\varepsilon > 0$ be given. Now $|f_n - f|$ is integrable for each n and $|f_n - f|\chi_{[-k,k]}$ converges to $|f_n - f|$. By Lebesgue's Dominated Convergence Theorem, $\int_{-k}^{k} |f_n - f|$ converges to $\int |f_n - f|$. Thus there exists N such that $\int_{|x|\ge N} |f_n - f| < \varepsilon/3$. By Proposition 14, for each n, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any set A with $mA < \delta$, $\int_A |f_n - f| < \varepsilon/3$. We may assume $\delta < \varepsilon/6N$. There also exists N' such that $m\{x : |f_n(x) - f(x)| \ge \delta\} < \delta$ for all $n \ge N'$. Let $A = \{x : |f_n(x) - f(x)| \ge \delta\}$. Then $\int |f_n - f| = \int_{|x|\ge N} |f_n - f| + \int_{A\cap[-N,N]} |f_n - f| + \int_{A^c\cap[-N,N]} |f_n - f| < \varepsilon/3 + \varepsilon/3 + 2N\delta < \varepsilon$ for all $n \ge N'$. i.e. $\int |f_n - f| \to 0$.

25. Let $\langle f_n\rangle$ be a Cauchy sequence in measure. Then we may choose $n_{v+1} > n_v$ such that $m\{x : |f_{n_{v+1}}(x) - f_{n_v}(x)| \ge 1/2^v\} < 1/2^v$. Let $E_v = \{x : |f_{n_{v+1}}(x) - f_{n_v}(x)| \ge 1/2^v\}$ and let $F_k = \bigcup_{v\ge k} E_v$. Then $m(\bigcap_k F_k) \le m(\bigcup_{v\ge k} E_v) \le 1/2^{k-1}$ for all k so $m(\bigcap_k F_k) = 0$. If $x \notin \bigcap_k F_k$, then $x \notin F_k$ for some k so $|f_{n_{v+1}}(x) - f_{n_v}(x)| < 1/2^v$ for all $v \ge k$ and $|f_{n_w}(x) - f_{n_v}(x)| < 1/2^{v-1}$ for $w \ge v \ge k$. Thus the series $\sum(f_{n_{v+1}} - f_{n_v})$ converges a.e. to a function g. Let $f = g + f_{n_1}$. Then $f_{n_v} \to f$ in measure since the partial sums of the series are of the form $f_{n_v} - f_{n_1}$. Given $\varepsilon > 0$, choose N such that $m\{x : |f_n(x) - f_r(x)| \ge \varepsilon/2\} < \varepsilon/2$ for all $n, r \ge N$ and $m\{x : |f_{n_v}(x) - f(x)| \ge \varepsilon/2\} < \varepsilon/2$ for all $v \ge N$. Now $\{x : |f_n(x) - f(x)| \ge \varepsilon\} \subset \{x : |f_n(x) - f_{n_v}(x)| \ge \varepsilon/2\} \cup \{x : |f_{n_v}(x) - f(x)| \ge \varepsilon/2\}$ for all $n, v \ge N$. Thus $m\{x : |f_n(x) - f(x)| \ge \varepsilon\} < \varepsilon$ for all $n \ge N$. i.e. $f_n \to f$ in measure.

5 Differentiation and Integration

5.1 Differentiation of monotone functions

1. Let f be defined by $f(0) = 0$ and $f(x) = x\sin(1/x)$ for $x \ne 0$. Then $D^+f(0) = \overline{\lim}_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \overline{\lim}_{h\to 0^+} \sin(1/h) = 1$. Similarly, $D^-f(0) = 1, D_+f(0) = D_-f(0) = -1$.

2a. $D^+[-f(x)] = \overline{\lim}_{h\to 0^+} \frac{[-f(x+h)]-[-f(x)]}{h} = -\underline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = -D_+f(x)$.

2b. Let $g(x) = f(-x)$. Then $D^+g(x) = \overline{\lim}_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = \overline{\lim}_{h\to 0^+} \frac{f(-x-h)-f(-x)}{h} = \overline{\lim}_{h\to 0^+} -\frac{f(-x)-f(-x-h)}{h} = -\underline{\lim}_{h\to 0^+} \frac{f(-x)-f(-x-h)}{h} = -D_-f(-x)$.

3a. Suppose f is continuous on $[a,b]$ and assumes a local maximum at $c \in (a,b)$. Now there exists $\delta > 0$ such that $f(c+h) < f(c)$ for $0 < h < \delta$. Then $\frac{f(c+h)-f(c)}{h} < 0$ for $0 < h < \delta$. Thus $D^+f(c) = \overline{\lim}_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$. Similarly, there exists $\delta' > 0$ such that $f(c) > f(c-h)$ for $0 < h < \delta'$. Then $\frac{f(c)-f(c-h)}{h} > 0$ for $0 < h < \delta'$. Thus $D_-f(c) = \underline{\lim}_{h\to 0^+} \frac{f(c)-f(c-h)}{h} \ge 0$. Hence $D_+f(c) \le D^+f(c) \le 0 \le D_-f(c) \le D^-f(c)$.

(*) Note error in book.

3b. If f has a local maximum at a, then $D_+f(a) \le D^+f(a) \le 0$. If f has a local maximum at b, then $0 \le D_-f(b) \le D^-f(b)$.

4. Suppose f is continuous on $[a,b]$ and one of its derivates, say D^+f, is everywhere nonnegative on (a,b). First consider a function g such that $D^+g(x) \ge \varepsilon > 0$ for all $x \in (a,b)$. Suppose there exist $x, y \in [a,b]$ with $x < y$ and $g(x) > g(y)$. Since $D^+g(x) > 0$ for all $x \in (a,b)$, g has no local maximum in (a,b) by Q3. Thus g is decreasing on $(a,y]$ and $D^+g(c) \le 0$ for all $c \in (a,y)$. Contradiction. Hence g is nondecreasing on $[a,b]$. Now for any $\varepsilon > 0$, $D^+(f(x)+\varepsilon x) \ge \varepsilon$ on (a,b) so $f(x)+\varepsilon x$ is nondecreasing on $[a,b]$. Let $x < y$. Then $f(x)+\varepsilon x \le f(y)+\varepsilon y$. Suppose $f(x) > f(y)$. Then $0 < f(x)-f(y) \le \varepsilon(y-x)$. In particular, choosing $\varepsilon = (f(x)-f(y))/(2(y-x))$, we have $f(x)-f(y) \le (f(x)-f(y))/2$. Contradiction. Hence f is nondecreasing on $[a,b]$.

The case where D^-f is everywhere nonnegative on (a,b) follows from a similar argument and the cases where D_+f or D_-f is everywhere nonnegative on (a,b) follow from the previous cases.

5a. For any x, $D^+(f+g)(x) = \overline{\lim}_{h\to 0^+} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \overline{\lim}_{h\to 0^+} (\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}) \le \overline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)}{h} + \overline{\lim}_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = D^+f(x) + D^+g(x)$.

5b. For any x, $D_+(f+g)(x) = \underline{\lim}_{h\to 0^+} \frac{(f+g)(x+h)-(f+g)(x)}{h} = \underline{\lim}_{h\to 0^+}(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}) \leq \overline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)}{h} + \underline{\lim}_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = D^+f(x) + D_+g(x)$.
Similarly, $D^-(f+g) \leq D^-f + D^-g$ and $D_-(f+g) \leq D^-f + D_-g$.

5c. Let f and g be nonnegative and continuous at c. Then $D^+(fg)(c) = \overline{\lim}_{h\to 0^+} \frac{(fg)(c+h)-(fg)(c)}{h} = \overline{\lim}_{h\to 0^+} \frac{f(c+h)g(c+h)-f(c)g(c)}{h} = \overline{\lim}_{h\to 0^+} \frac{(f(c+h)-f(c))g(c+h)+f(c)(g(c+h)-g(c))}{h} \leq g(c)\overline{\lim}_{h\to 0^+} \frac{f(c+h)-f(c)}{h} + f(c)\overline{\lim}_{h\to 0^+} \frac{g(c+h)-g(c)}{h} = f(c)D^+g(c) + g(c)D^+f(c)$.

***6a.** Let f be defined on $[a,b]$ and g a continuous function on $[\alpha,\beta]$ that is differentiable at γ with $g(\gamma) = c \in (a,b)$. Suppose $g'(\gamma) > 0$. Note that if $D^+(f\circ g)(\gamma) = \pm\infty$, then $D^+f(c) = \pm\infty$. Now suppose $D^+f(c) < \infty$. Let $\varepsilon > 0$ be given and let $\varepsilon_1 = \min(1, \frac{\varepsilon}{g'(\gamma)+1+D^+f(c)})$. There exists $\delta_1 > 0$ such that $|\frac{g(\gamma+h)-g(\gamma)}{h} - g'(\gamma)| < \varepsilon_1$ for $0 < h < \delta_1$. There exists $\delta_2 > 0$ such that $\frac{f(c+h')-f(c)}{h'} - D^+f(c) < \varepsilon_1$ for $0 < h' < \delta_2$ so that $f(c+h')-f(c)-h'D^+f(c) < \varepsilon_1 h'$. By continuity of g, there exists $\delta_3 > 0$ such that $g(\gamma+h'')-g(\gamma) < \delta_2$ for $0 < h'' < \delta_3$. Now let $\delta = \min(\delta_1,\delta_3)$. When $0 < h < \delta$, $|\frac{g(\gamma+h)-g(\gamma)}{h} - g'(\gamma)| < \varepsilon_1$ and $f(g(\gamma+h)) - f(g(\gamma)) - (g(\gamma+h)-g(\gamma))D^+f(c) < \varepsilon_1(g(\gamma+h)-g(\gamma))$. Hence $\frac{f(g(\gamma+h))-f(g(\gamma))}{h} - D^+f(c)g'(\gamma) = \frac{f(g(\gamma+h))-f(g(\gamma))-D^+f(c)(g(\gamma+h)-g(\gamma))}{h} + \frac{D^+f(c)(g(\gamma+h)-g(\gamma))-hD^+f(c)g'(\gamma)}{h} < \varepsilon_1 \frac{g(\gamma+h)-g(\gamma)}{h} + D^+f(c)(\frac{g(\gamma+h)-g(\gamma)}{h} - g'(\gamma)) < \varepsilon_1(g'(\gamma)+1) + \varepsilon_1 D^+f(c) < \varepsilon$. Thus $D^+(f\circ g)(\gamma) \leq D^+f(c)g'(\gamma)$ and similarly, it can be shown that $D^+(f\circ g)(\gamma) \geq D^+f(c)g'(\gamma)$. Hence $D^+(f\circ g)(\gamma) = D^+f(c)g'(\gamma)$.

***6b.** Suppose $g'(\gamma) < 0$. Note that if $D^+(f\circ g)(\gamma) = \pm\infty$, then $D_-f(c)g'(\gamma) = \mp\infty$. Also note that there exists $\delta > 0$ such that $g(\gamma+h) - g(\gamma) < 0$ for $0 < h < \delta$. By a similar argument to that in part (a), $D^+(f\circ g)(\gamma) = D_-f(c)g'(\gamma)$.

***6c.** Suppose $g'(\gamma) = 0$ and all the derivates of f at c are finite. By a similar argument to that in part (a), $D^+(f\circ g)(\gamma) = 0$.

5.2 Functions of bounded variation

7a. Let f be of bounded variation on $[a,b]$. Then $f = g - h$ where g and h are monotone increasing functions on $[a,b]$. Let $c \in (a,b)$. Also let $A = \sup_{x\in[a,c)} g(x)$ and let $B = \sup_{x\in[a,c)} h(x)$. Note that $A, B < \infty$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $A - \varepsilon/2 < g(c-\delta) \leq A$ and $B - \varepsilon/2 < h(c-\delta) \leq B$. Then for $x \in (c-\delta, c)$, $A - \varepsilon/2 < g(x) \leq A$ and $B - \varepsilon/2 < h(x) \leq B$. i.e. $0 \leq A - g(x) < \varepsilon/2$ and $0 \leq B - h(x) < \varepsilon/2$. Now $0 \leq |A - B - f(x)| \leq (A - g(x)) + (B - h(x)) < \varepsilon$ for $x \in (c-\delta, c)$. Hence $f(c-)$ exists. Similarly $f(c+)$ exists. Let g be a monotone function and let E be the set of discontinuities of g. Now for $c \in E$, $g(c-) < g(c+)$ so there is a rational r_c such that $g(c-) < r_c < g(c+)$. Note that if $x_1 < x_2$, then $g(x_1+) \leq g(x_2-)$ so $r_{x_1} \neq r_{x_2}$. Thus we have a bijection between E and a subset of $\mathbb{Q}$ so E is countable. Since a function f of bounded variation is a difference of two monotone functions, f also has only a countable number of discontinuities.

7b. Let $\langle x_n \rangle$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Define f on $[0,1]$ by $f(x) = \sum_{x_n < x} 2^{-n}$. Then f is monotone. Also, at each x_n, for any $\delta > 0$, there exists $x \in (x_n, x_n+\delta)$ such that $f(x) - f(x_n) > 2^{-n-1}$ so f is discontinuous at each x_n.

8a. Suppose $a \leq c \leq b$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of $[a,b]$. If $c = x_k$ for some k, then $\sum_1^n |f(x_i) - f(x_{i-1})| = \sum_1^k |f(x_i) - f(x_{i-1})| + \sum_{k+1}^n |f(x_i) - f(x_{i-1})| \leq T_a^c(f) + T_c^b(f)$. Thus $T_a^b(f) \leq T_a^c(f) + T_c^b(f)$. The case where $c \in (x_k, x_{k+1})$ for some k is similar. Conversely, let $a = x_0 < x_1 < \cdots < x_n = c$ be a subdivision of $[a,c]$ and let $c = y_0 < y_1 < \cdots < y_m = b$ be a subdivision of $[c,b]$. Then $a = x_0 < x_1 < \cdots < x_n = c < y_1 < \cdots < y_m = b$ is a subdivision of $[a,b]$ and $\sum_1^n |f(x_i) - f(x_{i-1})| + \sum_1^m |f(y_i) - f(y_{i-1})| \leq T_a^b(f)$. It follows that $T_a^c(f) + T_c^b(f) \leq T_a^b(f)$. Hence $T_a^b(f) = T_a^c(f) + T_c^b(f)$ and $T_a^c(f) \leq T_a^b(f)$.

8b. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of $[a,b]$. Then $\sum_1^n |(f+g)(x_i) - (f+g)(x_{i-1})| \leq \sum_1^n |f(x_i) - f(x_{i-1})| + \sum_1^n |g(x_i) - g(x_{i-1})| \leq T_a^b(f) + T_a^b(g)$. Hence $T_a^b(f+g) \leq T_a^b(f) + T_a^b(g)$. Let $c \in \mathbb{R}$. If $c = 0$, then $T_a^b(cf) = 0 = |c|T_a^b(f)$. If $c \neq 0$, then $\sum_1^n |cf(x_i) - cf(x_{i-1})| = |c|\sum_1^n |f(x_i) - f(x_{i-1})| \leq |c|T_a^b(f)$. Thus $T_a^b(cf) \leq |c|T_a^b(f)$. On the other hand, $\sum_1^n |f(x_i) - f(x_{i-1})| = |c|^{-1}\sum_1^n |cf(x_i) - cf(x_{i-1})| \leq |c|^{-1}T_a^b(cf)$. Thus $T_a^b(f) \leq |c|^{-1}T_a^b(cf)$ so $|c|T_a^b(f) \leq T_a^b(cf)$. Hence $T_a^b(cf) = |c|T_a^b(f)$.

9. Let $\langle f_n \rangle$ be a sequence of functions on $[a,b]$ that converges at each point of $[a,b]$ to f. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of $[a,b]$ and let $\varepsilon > 0$. Then there exists N such that

$\sum_1^n |f(x_i)-f(x_{i-1})| \le \sum_1^n |f(x_i)-f_n(x_i)|+\sum_1^n |f(x_{i-1})-f_n(x_{i-1})|+\sum_1^n |f_n(x_i)-f_n(x_{i-1})| < \varepsilon+T_a^b(f_n)$ for $n \ge N$. Thus $T_a^b(f) \le \varepsilon + T_a^b(f_n)$ for $n \ge N$ so $T_a^b(f) \le \varepsilon + \underline{\lim} T_a^b(f_n)$. Since ε is arbitrary, $T_a^b(f) \le \underline{\lim} T_a^b(f_n)$.

10a. Let f be defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x^2)$ for $x \ne 0$. Consider the subdivision $-1 < \sqrt{2/n\pi} < \sqrt{2/(n-1)\pi} < \cdots < \sqrt{2/\pi} < 1$ of $[-1, 1]$. Note that $t_a^b(f) \to \infty$ as $n \to \infty$. Thus f is not of bounded variation on $[-1, 1]$.

10b. Let g be defined by $g(0) = 0$ and $g(x) = x^2 \sin(1/x)$ for $x \ne 0$. Note that g is differentiable on $[-1, 1]$ and $|g'(x)| \le 3$ on $[-1, 1]$. Thus for any subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, $\sum_1^n |g(x_i - g(x_{i-1})| \le 4\sum_1^n |x_i - x_{i-1}| = 3(g(1) - g(-1))$. Hence $T_{-1}^1(g) \le 3(g(-1) - g(1)) < \infty$.

11. Let f be of bounded variation on $[a, b]$. For any $x \in [a, b]$, $f(x) = P_a^x(f) - N_a^x(f) - f(a)$ so $f'(x) = \frac{d}{dx}P_a^x(f) - \frac{d}{dx}N_a^x(f)$ a.e. in $[a, b]$. Thus $|f'| \le \frac{d}{dx}P_a^x(f) + \frac{d}{dx}N_a^x(f) = \frac{d}{dx}T_a^x(f)$ a.e. in $[a, b]$ and $\int_a^b |f'| \le \int_a^b \frac{d}{dx}T_a^x(f) \le T_a^b(f) - T_a^a(f) = T_a^b(f)$.

5.3 Differentiation of an integral

No problems

5.4 Absolute continuity

***12.** The function f defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x^2)$ for $x \ne 0$ is absolutely continuous on $[\varepsilon, 1]$ for $\varepsilon > 0$, continuous at 0 but not of bounded variation on $[0, 1]$, thus not absolutely continuous on $[0, 1]$.

Suppose f is absolutely continuous on $[\eta, 1]$ for $\eta > 0$, continuous at 0 and of bounded variation on $[0, 1]$. For $\eta \in (0, 1]$, let $0 = x_0 < x_1 < \cdots < x_n = \eta$ be a subdivision of $[0, \eta]$. Then since f is continuous at 0, $\sum_1^n |f(x_i)-f(x_{i-1})| \to 0$ as $\eta \to 0^+$. Thus $T_0^\eta(f) \to 0$ as $\eta \to 0^+$. Given $\varepsilon > 0$, there exists $\eta \in (0, 1]$ such that $T_0^\eta(f) < \varepsilon/2$. Since f is absolutely continuous on $[\eta, 1]$, there exists $\delta > 0$ such that for any finite collection $\{(x_i, x_i')\}_1^n$ of disjoint intervals in $[\eta, 1]$ with $\sum_1^n |x_i'-x_i| < \delta$, we have $\sum_1^n |f(x_i')-f(x_i)| < \varepsilon/2$. Now let $\{(y_i, y_i')\}_1^n$ be a finite collection of disjoint intervals in $[0, 1]$ with $\sum_1^n |y_i'-y_i| < \delta$. If $\eta \in [y_k', y_{k+1}]$ for some k, then $\sum_1^n |f(y_i') - f(y_i)| \le \sum_1^k |f(y_i') - f(y_i)| + \sum_{k+1}^n |f(y_i') - f(y_i)| < T_0^\eta(f) + \varepsilon/2 < \varepsilon$. If $\eta \in (y_k, y_k')$ for some k, then $\sum_1^n |f(y_i') - f(y_i)| \le \sum_1^{k-1} |f(y_i') - f(y_i)| + |f(\eta) - f(y_k)| + |f(y_k') - f(\eta)| + \sum_{k+1}^n |f(y_i') - f(y_i)| < T_0^\eta(f) + \varepsilon/2 < \varepsilon$. Hence f is absolutely continuous on $[0, 1]$.

13. Let f be absolutely continuous on $[a, b]$. Then f is of bounded variation on $[a, b]$ so by Q11, $\int_a^b |f'| \le T_a^b(f)$. Conversely, since f is absolutely continuous on $[a, b]$, for any subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, $\sum_1^n |f(x_i) - f(x_{i-1})| = \sum_1^n |\int_{x_{i-1}}^{x_i} f'| \le \sum_1^n \int_{x_{i-1}}^{x_i} |f'| = \int_a^b |f'|$. Thus $T_a^b(f) \le \int_a^b |f'|$. Hence $T_a^b(f) = \int_a^b |f'|$.

For any $x \in [a, b]$, $f(x) = P_a^x(f) - N_a^x(f) - f(a)$ so $f'(x) = \frac{d}{dx}P_a^x(f) - \frac{d}{dx}N_a^x(f)$ a.e. in $[a, b]$. Thus $(f')^+ \le (\frac{d}{dx}P_a^x(f))^+ + (\frac{d}{dx}N_a^x(f))^- = \frac{d}{dx}P_a^x(f)$. Thus $\int_a^b (f')^+ \le \int_a^b \frac{d}{dx}P_a^x(f) \le P_a^b(f) - P_a^a(f) = P_a^b(f)$. Conversely, for any subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, $\sum_1^n (f(x_i) - f(x_{i-1}))^+ = \sum_1^n (\int_{x_{i-1}}^{x_i} f')^+ \le \sum_1^n \int_{x_{i-1}}^{x_i} (f')^+ = \int_a^b (f')^+$. Thus $P_a^b(f) \le \int_a^b (f')^+$. Hence $P_a^b(f) = \int_a^b (f')^+$.

(*) If $\int g < 0$, then $(\int g)^+ = 0 \le \int g^+$. If $\int g \ge 0$, then $(\int g)^+ = \int g \le \int g^+$.

14a. Let f and g be two absolutely continuous functions on $[a, b]$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_1^n |f(x_i') - f(x_i)| < \varepsilon/2$ and $\sum_1^n |g(x_i') - g(x_i)| < \varepsilon/2$ for any finite collection $\{(x_i, x_i')\}_1^n$ of disjoint intervals in $[a, b]$ with $\sum_1^n |x_i' - x_i| < \delta$. Then $\sum_1^n |(f \pm g)(x_i') - (f \pm g)(x_i)| \le \sum_1^n |f(x_i') - f(x_i)| + \sum_1^n |g(x_i') - g(x_i)| < \varepsilon$. Thus $f + g$ and $f - g$ are absolutely continuous.

14b. Let f and g be two absolutely continuous functions on $[a, b]$. There exists M such that $|f(x)| \le M$ and $|g(x)| \le M$ for any $x \in [a, b]$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_1^n |f(x_i') - f(x_i)| < \varepsilon/2M$ and $\sum_1^n |g(x_i') - g(x_i)| < \varepsilon/2M$ for any finite collection $\{(x_i, x_i')\}$ of disjoint intervals in $[a, b]$ with $\sum_1^n |x_i'-x_i| < \delta$. Then $\sum_1^n |(fg)(x_i')-(fg)(x_i)| \le \sum_1^n |f(x_i')||g(x_i')-g(x_i)|+\sum_1^n |g(x_i)||f(x_i')-f(x_i)| < \varepsilon$. Thus fg is absolutely continuous.

14c. Suppose f is absolutely continuous on $[a, b]$ and is never zero there. Let $g = 1/f$. There exists M such that $|f(x)| \ge M$ for $x \in [a, b]$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_1^n |f(x_i') - f(x_i)| < \varepsilon M^2$

for any finite collection $\{(x_i, x_i')\}$ of disjoint intervals in $[a,b]$ with $\sum_1^n |x_i' - x_i| < \delta$. Then $|g(x_i') - g(x_i)| = \frac{|f(x_i') - f(x_i)|}{|f(x_i)f(x_i')|} < \varepsilon$. Thus g is absolutely continuous.

15. Let f be the Cantor ternary function. By Q2.48, f is continuous and monotone on $[0,1]$. Note that $f' = 0$ a.e. on $[0,1]$ since f is constant on each interval in the complement of the Cantor set and the Cantor set has measure zero. If f is absolutely continuous, then $1 = f(1) = \int_0^1 f' + f(0) = 0$. Contradiction. Thus f is not absolutely continuous.

16a. Let f be a monotone increasing function on $[a,b]$. Let g be defined by $g(x) = \int_a^x f'$ and let $h = f - g$. Then g is absolutely continuous, $h'(x) = f'(x) - \frac{d}{dx}\int_a^x f' = 0$ a.e. so h is singular, and $f = g + h$.

16b. Let f be a nondecreasing singular function on $[a,b]$. Let $\varepsilon, \delta > 0$ be given. Since f is singular on $[a,b]$, for each $x \in [a,b]$, there is an arbitrarily small interval $[x, x+h] \subset [a,b]$ such that $|f(x+h) - f(x)| < \varepsilon h/(b-a)$. Then there exists a finite collection $\{[x_k, y_k]\}$ of nonoverlapping intervals of this sort which cover all of $[a,b]$ except for a set of measure less than δ. Labelling x_k such that $x_k \leq x_{k+1}$, we have $y_0 = a \leq x_1 < y_1 \leq x_2 < \cdots \leq y_n \leq b = x_{n+1}$. Then $\sum_0^n |x_{k+1} - y_k| < \delta$ and $\sum_1^n |f(y_k) - f(x_k)| < \varepsilon$. Since f is nondecreasing, $\sum_0^n |f(x_{k+1}) - f(y_k)| > f(b) - f(a) - \varepsilon$.

16c. Let f be a nondecreasing function on $[a,b]$ with property (S). i.e. Given $\varepsilon, \delta > 0$, there is a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals in $[a,b]$ such that $\sum_1^n |x_k - y_k| < \delta$ and $\sum_1^n (f(x_k) - f(y_k)) > f(b) - f(a) - \varepsilon$. By part (a), $f = g + h$ where $g = \int_a^x f'$ and h is singular. It suffices to show that $g = 0$ a.e. Letting $x_0 = a$ and $y_{n+1} = b$, we have $\sum_1^n (f(y_{k+1}) - f(x_k)) < \varepsilon$. We may choose δ such that $\int_{\bigcup_1^n [y_k, x_k]} f' < \varepsilon$. Then $\int_a^b f' < 2\varepsilon$ so $\int_a^b f' = 0$ and $g = 0$.

16d. Let $\langle f_n \rangle$ be a sequence of nondecreasing singular functions on $[a,b]$ such that the function $f(x) = \sum f_n(x)$ is everywhere finite. Let $\varepsilon, \delta > 0$ be given. Now $f(b) - f(a) = \sum (f_n(b) - f_n(a)) < \infty$ so there exists N such that $\sum_{N+1}^\infty (f_n(b) - f_n(a)) < \varepsilon/2$. Let $F(x) = \sum_1^N f_n(x)$. Then F is nondecreasing and singular. By part (b), there exists a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals such that $\sum |x_k - y_k| < \delta$ and $\sum (F(y_k) - F(x_k)) > F(b) - F(a) - \varepsilon/2$. Now $\sum (f(y_k) - f(x_k)) \geq \sum (F(y_k) - F(x_k)) > F(b) - F(a) - \varepsilon/2 = f(b) - f(a) - \sum_{N+1}^\infty (f_n(b) - f_n(a)) - \varepsilon/2 > f(b) - f(a) - \varepsilon$. By part (c), f is singular.

***16e.** Let C be the Cantor ternary function on $[0,1]$. Extend C to $\mathbb{R}$ by defining $C(x) = 0$ for $x < 0$ and $C(x) = 1$ for $x > 1$. For each n, define f_n by $f_n(x) = 2^{-n} C(\frac{x - a_n}{b_n - a_n})$ where $\{[a_n, b_n]\}$ is an enumeration of the intervals with rational endpoints in $[0,1]$. Then each f_n is a nondecreasing singular function on $[0,1]$. Define $f(x) = \sum f_n(x)$. Then f is everywhere finite, strictly increasing and by part (d), f is singular.

17a. Let F be absolutely continuous on $[c,d]$. Let g be monotone and absolutely continuous on $[a,b]$ with $c \leq g \leq d$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite collection $\{(y_i, y_i')\}$ of disjoint intervals with $\sum_1^n |y_i' - y_i| < \delta$, we have $\sum_1^n |F(y_i') - F(y_i)| < \varepsilon$. Now there exists $\delta' > 0$ such that for any finite collection $\{(x_i, x_i')\}$ of disjoint intervals with $\sum_1^n |x_i' - x_i| < \delta'$, we have $\sum_1^n |g(x_i') - g(x_i)| < \delta$. Now $\{(g(x_i), g(x_i'))\}$ is a finite collection of disjoint intervals so $\sum_1^n |F(g(x_i')) - F(g(x_i))| < \varepsilon$. Hence $F \circ g$ is absolutely continuous.

(*) Additional assumption that g is monotone. Counterexample: Consider $f(x) = \sqrt{x}$ for $x \in [0,1]$ and $g(0) = 0$, $g(x) = (x \sin x^{-1})^2$ for $x \in (0,1]$. Then $(f \circ g)(0) = 0$ and $(f \circ g)(x) = x \sin x^{-1}$ for $x \in (0,1]$. f and g are absolutely continuous but not $f \circ g$.

***17b.** Let $E = \{x : g'(x) = 0\}$. Note that $|g(x) - g(a)| = |\int_a^x g'| \leq \int_a^x |g'|$ for all $x \in [a,b]$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\int_A |g'| < \varepsilon/2$ whenever $mA < \delta$. Let $\eta = \varepsilon/4(b-a)$. For any $x \in E$, there exists $h_x > 0$ such that $|g(x+h) - g(x)| < \eta h$ for $0 < h \leq h_x$. Define $\mathfrak{V} = \{[x, x+h_x] : x \in E, |g(y) - g(x)| < \eta(y-x) \text{ for } y \in (x, x+h_x]\}$. Then $\mathfrak{V}$ is a Vitali covering for E so there exists a finite disjoint collection $\{I_1, \ldots, I_N\}$ of intervals in $\mathfrak{V}$ such that $m(E \setminus \bigcup_{n=1}^N I_n) < \delta$. Now let O be an open set such that $O \supset E \setminus \bigcup_{n=1}^N I_n$ and $mO < \delta$. Then O is a countable union of disjoint open intervals J_m and $g[E \setminus \bigcup_{n=1}^N I_n] \subset \bigcup g[J_m]$. Thus $m(g[E \setminus \bigcup_{n=1}^N I_n)] \leq \sum m(g[J_m]) \leq \sum \int_{J_m} |g'| = \int_O |g'| < \varepsilon/2$. Also, $g[E \cap \bigcup_{n=1}^N I_n] \subset \bigcup_{n=1}^N g[I_n]$ so $m(g[E \cap \bigcup_{n=1}^N I_n)] \leq \sum 2h_{x_n}\eta < 2(b-a)\eta = \varepsilon/2$. Hence $m(g[E]) < \varepsilon$. Since ε is arbitrary, $m(g[E]) = 0$.

18. Let g be an absolutely continuous monotone function on $[0,1]$ and E a set of measure zero. Let $\varepsilon > 0$. There is an open set $O \supset E$ such that $mO = m(O \setminus E) < \delta$ where δ is given by absolute continuity of g. Now O is a countable union $\bigcup I_n$ of disjoint open intervals so $\sum l(I_n) < \delta$ and $\sum l(g[I_n \cap [0,1]]) < \varepsilon$. Now $g[E] \subset \bigcup g[I_n \cap [0,1]]$ so $m(g[E]) < \varepsilon$. Since ε is arbitrary, $m(g[E]) = 0$.

19a. Let G be the complement of a generalised Cantor set of positive measure and let $g = \int_0^x \chi_G$. Then g is absolutely continuous and strictly monotone on $[0,1]$. Also, $g' = \chi_G = 0$ on G^c.

19b. Since $\{x : g'(x) = 0\}$ has positive measure, it has a nonmeasurable subset F. By Q17(b), $m(g[F]) = 0$. Also, $g^{-1}[g[F]] = F$ is nonmeasurable.

20a. Suppose f is Lipschitz. There exists M such that $|f(x) - f(y)| \le M|x - y|$ for all x, y. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. For any finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with $\sum_1^n |x_i' - x_i| < \delta$, we have $\sum_1^n |f(x_i') - f(x_i)| \le M \sum_1^n |x_i' - x_i| < \varepsilon$. Thus f is absolutely continuous.

20b. Let f be absolutely continuous. Suppose f is Lipschitz. Now $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$ so $|f'(x)| = \lim_{y\to x} |\frac{f(y)-f(x)}{y-x}| \le M$ for all x. Conversely, if f is not Lipschitz, then for any M, there exist x and y such that $|f(x) - f(y)| > M|x - y|$. Then $|f'(c)| > M$ for some $c \in (x, y)$ by the Mean Value Theorem. Thus for any M, there exists c such that $|f'(c)| > M$ so $|f'|$ is unbounded.

***20c.**

21a. Let O be an open set in $[c,d]$. Then O is a countable union $\bigcup I_n$ of disjoint open intervals. Now for each n, $I_n = (g(c_n), g(d_n))$ for some $c_n, d_n \in [c,d]$. Also, $g^{-1}[O] = \bigcup g^{-1}[I_n] = \bigcup(c_n, d_n)$. Thus $mO = \sum l(I_n) = \sum(g(d_n) - g(c_n)) = \sum \int_{c_n}^{d_n} g' = \int_{g^{-1}[O]} g'$.

21b. Let $H = \{x : g'(x) \ne 0\}$. Let $E \subset [c,d]$ with $mE = 0$ and let $\delta > 0$. Then there exists an open set $O \supset E$ with $mO < \delta$. By part (a), $\int_{g^{-1}[O]} g' < \delta$. Thus $\int_{g^{-1}[E]\cap H} g' = \int_{g^{-1}[E]} g' < \delta$. Since $\delta > 0$ is arbitrary, $\int_{g^{-1}[E]\cap H} g' = 0$. Since $g' > 0$ on $g^{-1}[E] \cap H$, the set $g^{-1}[E] \cap H$ has measure zero.

21c. Let E be a measurable subset of $[c,d]$ and let $F = g^{-1}[E] \cap H$. Since g is absolutely continuous, it is continuous and thus measurable so $g^{-1}[E]$ is measurable. Also, g' is measurable so H is measurable. Thus $F = g^{-1}[E] \cap H$ is measurable.

There exists a G_δ set $G \supset E$ with $m(G \setminus E) = 0$. We may assume $G \subset [c,d]$. By part (b), $m((g^{-1}[G] \setminus g^{-1}[E]) \cap H) = 0$ so $\int_{g^{-1}[G]\cap H} g' = \int_{g^{-1}[E]\cap H} g'$. Now G is a countable intersection $\bigcap O_n$ of open sets. Let $G_k = \bigcap_{n=1}^k O_n$. Then $G_1 \supset G_2 \supset \cdots$ so $\lim mG_k = m(\bigcap G_k) = mG$. Now $mE = mG = \lim mG_k = \lim \int_{g^{-1}[G_k]\cap H} g' = \lim \int_{\bigcap_{n=1}^k g^{-1}[O_n]\cap H} g' = \int_{\bigcap g^{-1}[O_n]\cap H} g' = \int_{g^{-1}[G]\cap H} g' = \int_{g^{-1}[E]\cap H} g' = \int_F g'$. Also, $\int_F g' = \int_{g^{-1}[E]} g' = \int_a^b \chi_E(g(x))g'(x)\,dx$.

21d. Let f be a nonnegative measurable function on $[c,d]$. Then there is an increasing sequence $\langle\varphi_n\rangle$ of simple functions on $[c,d]$ with $\lim \varphi_n = f$ so $\lim \varphi_n(g(x))g'(x) = f(g(x))g'(x)$. Since each $(\varphi_n \circ g)g'$ is measurable, $(f \circ g)g'$ is measurable. Now $\int_a^b \varphi_n(g(x))g'(x)\,dx = \sum \int_a^b c_k \chi_{E_k}(g(x))g'(x)\,dx = \sum c_k mE_k = \int_c^d \varphi_n(y)\,dy$. By the Monotone Convergence Theorem, $\lim \int \varphi_n = \int f$. Thus $\int_c^d f(y)\,dy = \lim \int_c^d \varphi_n(y)\,dy = \lim \int_a^b \varphi_n(g(x))g'(x)\,dx = \int_a^b f(g(x))g'(x)\,dx$.

22a. F is absolutely continuous on $[c,d]$, g is monotone and absolutely continuous on $[a,b]$ with $c \le g \le d$. By Q17(a), $H = F \circ g$ is absolutely continuous. Whenever H' and g' exist with $g'(x) \ne 0$, we have $D^+F(g(x)) = D_+F(g(x)) = D^-F(g(x)) = D_-F(g(x)) = H'(x)/g'(x)$ by Q6a so $F'(g(x))$ exists. Now H' and g' exist a.e. so $H'(x) = F'(g(x))g'(x)$ a.e. except on $E = \{x : g'(x) = 0\}$.

22b. Let f_0 be defined by $f_0(y) = f(y)$ if $y \notin g[E]$ and $f_0(y) = 0$ if $y \in g[E]$. By Q17b, $m(g[E]) = 0$ so $f_0 = f$ a.e. Hence $H'(x) = f(g(x))g'(x) = f_0(g(x))g'(x)$ a.e.

***22c.**

***22d.**

5.5 Convex functions

23a. Let φ be convex on a finite interval $[a,b)$. Let $x_0 \in (a,b)$ and let $f(x) = m(x - x_0) + \varphi(x_0)$ be the equation of a supporting line at x_0. Then $\varphi(x) \ge f(x)$ for all $x \in (a,b)$. Since φ is continuous at a, we have $\varphi(x) \ge f(x) \ge \min(f(a), f(b))$ for all $x \in [a,b)$. Hence φ is bounded from below.

23b. Suppose φ is convex on (a,b). If φ is monotone on (a,b), then $\varphi(x)$ has limits (possibly infinite) as it approaches a and b respectively from within (a,b). If φ is not monotone, then there exists $c \in (a,b)$

such that $D^+\varphi(x) \le 0$ on $(a,c]$ and $D^+\varphi(x) \ge 0$ on $[c,b)$ since the right-hand derivative of φ is increasing on (a,b). Thus φ is monotone on $(a,c]$ and on $[c,b)$ and it follows that the right-hand and left-hand limits exist at a and b respectively. If a (or b) is finite, then by part (a), the limits at a (or b) cannot be $-\infty$.

23c. Let φ be continuous on an interval I (open, closed, half-open) and convex in the interior of I. Then $\varphi(tx+(1-t)y) \le t\varphi(x)+(1-t)\varphi(y)$ for all x,y in the interior of I and all $t \in [0,1]$. Since φ is continuous, the inequality also holds at the included endpoints.

24. Let φ have a second derivative at each point of (a,b). If $\varphi''(x) \ge 0$ for all $x \in (a,b)$, then φ' is increasing on (a,b). Also, φ is continuous on (a,b). Hence φ is convex on (a,b). Conversely, if φ is convex on (a,b), then its left- and right-hand derivatives are monotone increasing on (a,b) so φ' is monotone increasing on (a,b). Hence $\varphi''(x) \ge 0$ for all $x \in (a,b)$.

25a. Suppose $a \ge 0$ and $b > 0$. Let $\varphi(t) = (a+bt)^p$. Then φ is continuous on $[0,\infty)$ for all p. For $p=1$, $\varphi(t) = a+bt$ so $\varphi''(t) = 0$. For $1 < p < \infty$, $\varphi''(t) = b^2p(p-1)(a+bt)^{p-2} > 0$. For $0 < p < 1$, $-\varphi''(t) = -b^2p(p-1)(a+bt)^{p-2} > 0$. Hence, by Q24, φ is convex for $1 \le p < \infty$ and concave for $0 < p \le 1$.

25b. For $p > 1$, $\varphi''(t) > 0$ for all $t \in (0,\infty)$ so φ' is strictly increasing on $(0,\infty)$. Now for $x < y$, $\frac{\varphi(\lambda x+(1-\lambda)y)-\varphi(x)}{(1-\lambda)(y-x)} = \varphi'(\xi_1)$ for some $\xi_1 \in (x, \lambda x+(1-\lambda)y)$. Also, $\frac{\varphi(y)-\varphi(\lambda(x)+(1-\lambda)(y))}{\lambda(y-x)} = \varphi'(\xi_2)$ for some $\xi_2 \in (\lambda x+(1-\lambda)y, y)$. Since $\varphi'(\xi_1) < \varphi'(\xi_2)$, $\frac{\varphi(\lambda x+(1-\lambda)y)-\varphi(x)}{(1-\lambda)(y-x)} < \frac{\varphi(y)-\varphi(\lambda(x)+(1-\lambda)(y))}{\lambda(y-x)}$. Equivalently, $\varphi(\lambda x+(1-\lambda)y) < \lambda\varphi(x)+(1-\lambda)\varphi(y)$. Hence φ is strictly convex for $p > 1$. Similarly, φ is strictly concave for $0 < p < 1$.

***26.** Let $\alpha = \int f(t)\,dt$ and let $g(x) = m(x-\alpha)+\exp(\alpha)$ be the equation of a supporting line at α. Equality holds when $\int \exp(f(t))\,dt = \exp(\alpha)$. Now $\int \exp(f(t))\,dt - \exp(\alpha) = \int \exp(f(t))\,dt - g(\alpha) = \int \exp(f(t))\,dt - g(\int f(t)\,dt) = \int(\exp(f(t)) - g(f(t)))\,dt$. Since $\exp(f(t)) - g(f(t)) \ge 0$, the integral is zero only when $\exp(f(t)) - g(f(t)) = 0$ a.e. and this can happen only when $f(t) = \alpha$ a.e.

27. Let $\langle\alpha_n\rangle$ be a sequence of nonnegative numbers whose sum is 1 and let $\langle\xi_n\rangle$ be a sequence of positive numbers. Define f on $[0,1]$ by $f(x) = \log\xi_n$ if $x \in [\sum_{n=1}^{k-1}\alpha_n, \sum_{n=1}^{k}\alpha_n)$. For each k, $\prod_{n=1}^{k}\xi_n^{\alpha_n} = \exp(\sum_{n=1}^{k}\alpha_n\log\xi_n) = \exp(\int_0^{\sum_{n=1}^{k}\alpha_n} f(t)\,dt)$. Thus $\prod_{n=1}^{k}\xi_n^{\alpha_n} \le \int_0^{\sum_{n=1}^{k}\alpha_n}\exp(f(t))\,dt = \sum_{n=1}^{k}\alpha_n\xi_n$. Letting $k \to \infty$, we have $\prod_{n=1}^{\infty}\xi_n^{\alpha_n} \le \sum_{n=1}^{\infty}\alpha_n\xi_n$.

28. Let g be a nonnegative measurable function on $[0,1]$. Since log is concave, $\int -\log(g(t))\,dt \ge -\log(\int g(t)\,dt)$ by Jensen's inequality. Hence $\log(\int g(t)\,dt) \ge \int \log(g(t))\,dt$.

6 The Classical Banach Spaces

6.1 The L^p spaces

1. If $|f(t)| \le M_1$ a.e. and $|g(t)| \le M_2$ a.e., then $|f(t)+g(t)| \le M_1+M_2$ a.e. so $||f+g||_\infty \le M_1+M_2$. Note that $|f(t)| \le ||f||_\infty$ a.e. and $|g(t)| \le ||g||_\infty$ a.e. Thus $||f+g||_\infty \le ||f||_\infty + ||g||_\infty$.

2. Let f be a bounded measurable function on $[0,1]$. Now $||f||_p = (\int_0^1 |f|^p)^{1/p} \le (\int_0^1 ||f||_\infty^p)^{1/p} = ||f||_\infty$. Thus $\overline{\lim}_{p\to\infty}||f||_p \le ||f||_\infty$. Let $\varepsilon > 0$ and let $E = \{x \in [0,1] : |f(x)| > ||f||_\infty - \varepsilon\}$. Then $||f||_p = (\int_0^1 |f|^p)^{1/p} \ge (\int_E |f|^p)^{1/p} \ge (||f||_\infty - \varepsilon)(mE)^{1/p}$. If $mE = 0$, then $||f||_\infty \le ||f||_\infty - \varepsilon$. Contradiction. Thus $mE > 0$ and $\underline{\lim}_{p\to\infty}||f||_p \ge ||f||_\infty - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\underline{\lim}_{p\to\infty}||f||_p \ge ||f||_\infty$. Hence $\lim_{p\to 0}||f||_p = ||f||_\infty$.

3. $||f+g||_1 = \int_0^1 |f+g| \le \int_0^1(|f|+|g|) = \int_0^1 |f| + \int_0^1 |g| = ||f||_1 + ||g||_1$.

4. Suppose $f \in L^1$ and $g \in L^\infty$. Then $\int |fg| \le \int |f| ||g||_\infty = ||g||_\infty \int |f| = ||f||_1 ||g||_\infty$.

6.2 The Minkowski and Hölder inequalities

5a. Let f and g be two nonnegative functions in L^p with $0 < p < 1$. We may assume $||f||_p > 0$ and $||g||_p > 0$. Let $\alpha = ||f||_p$ and $\beta = ||g||_p$ so $f = \alpha f_0$ and $g = \beta g_0$ where $||f_0||_p = ||g_0||_p = 1$. Set $\lambda = \alpha/(\alpha+\beta)$. Then $1-\lambda = \beta/(\alpha+\beta)$ and $|f+g|^p = (f+g)^p = (\alpha f_0+\beta g_0)^p = (\alpha+\beta)^p(\lambda f_0 + (1-\lambda)g_0)^p \ge (\alpha+\beta)^p(\lambda f_0^p + (1-\lambda)g_0^p)$ by concavity of the function $\varphi(t) = t^p$ for $0 < p < 1$. Thus

$||f+g||_p^p \geq (\alpha+\beta)^p(\lambda||f_0||_p^p+(1-\lambda)||g_0||_p^p) = (\alpha+\beta)^p = (||f||_p+||g||_p)^p$. Hence $||f+g||_p \geq ||f||_p+||g||_p$.

5b. Suppose $f \in L^p$ and $g \in L^p$. For $1 \leq p \leq \infty$, $f+g \in L^p$ by the Minkowski inequality. For $0 < p < 1$, $||f+g||_p^p \leq ||2\max(f,g)||_p^p = 2^p||\max(f,g)||_p^p \leq 2^p(||f||_p^p + ||g||_p^p)$. Thus $f+g \in L^p$.

***6.** Suppose $0 < p < 1$. Let $p' = 1/p$ so $p' > 1$. Let q be such that $1/p + 1/q = 1$. Note that $q = pp'/(1-p')$. Let $u = (fg)^p$ and let $v = g^{-p}$. Then $fg = u^{p'}$, $f^p = uv$ and $g^q = v^{p'/(p'-1)}$. Let q' be such that $1/p' + 1/q' = 1$. By the Hölder inequality, $\int |uv| \leq ||u||_{p'}||v||_{q'}$. i.e. $\int |f|^p \leq (\int |fg|^{pp'})^{1/p'}(\int |g|^{pp'/(1-p')})^{(p'-1)/p'} = (\int |fg|)^p(\int |g|^q)^{1-p}$. Hence $\int |fg| \geq (\int |f|^p)^{1/p}(\int |g|^q)^{1/q} = ||f||_p||g||_q$.

7a. For $p = \infty$, $||\langle \xi_v+\eta_v\rangle||_\infty = \sup|\xi_v+\eta_v| \leq \sup(|\xi_v|+|\eta_v|) \leq \sup|\xi_v|+\sup|\eta_v| = ||\langle\xi_v\rangle||_\infty+||\langle\eta_v\rangle||_\infty$.
For $1 \leq p < \infty$, let $\alpha = ||\langle\xi_v\rangle||_p$ and $\beta = ||\langle\eta_v\rangle||_p$ so $\langle\xi_v\rangle = \alpha\langle\xi_v'\rangle$ and $\langle\eta_v\rangle = \beta\langle\eta_v'\rangle$ where $||\xi_v'||_p = ||\eta_v'||_p = 1$. Set $\lambda = \alpha/(\alpha+\beta)$. Then $1-\lambda = \beta/(\alpha+\beta)$ and $||\langle\xi_v+\eta_v\rangle||_p = (\sum|\xi_v+\eta_v|^p)^{1/p} \leq (\sum(|\xi_v|+|\eta_v|)^p)^{1/p} = (\sum(\alpha|\xi_v'|+\beta|\eta_v'|)^p)^{1/p} = (\sum(\alpha+\beta)^p[\lambda|\xi_v'|+(1-\lambda)|\eta_v'|]^p)^{1/p} \leq (\sum(\alpha+\beta)^p[\lambda|\xi_v'|^p+(1-\lambda)|\eta_v'|^p])^{1/p} = \alpha+\beta = ||\langle\xi_v\rangle||_p+||\langle\eta_v\rangle||_p$.

7b. For $p = 1, q = \infty$, $\sum|\xi_v\eta_v| \leq \sup|\eta_v|\sum|\xi_v| = ||\langle\xi_v\rangle||_1||\langle\eta_v\rangle||_\infty$.
For $1 < p < \infty$, let $\alpha_v = |\eta_v|^{q/p}$. Then $|\eta_v| = \alpha_v^{p-1}$ and $pt|\xi_v||\eta_v| = pt|\xi_v|\alpha_v^{p-1} \leq (\alpha_v+t|\xi_v|)^p-\alpha_v^p$. Thus $\sum pt|\xi_v||\eta_v| \leq \sum(\alpha_v+t|\xi_v|)^p - \sum\alpha_v^p = ||\langle\alpha_v+t|\xi_v|\rangle||_p^p - ||\langle\alpha_v\rangle||_p^p \leq (||\langle\alpha_v\rangle||_p + t||\langle\xi_v\rangle||_p)^p - ||\langle\alpha_v\rangle||_p^p$. Differentiating with respect to t at $t = 0$, we get $p\sum|\xi_v||\eta_v| \leq p||\langle\xi_v\rangle||_p||\langle\alpha_v\rangle||_p^{p-1} = p||\langle\xi_v\rangle||_p||\langle\eta_v\rangle||_q$. Hence $\sum|\xi_v||\eta_v| \leq ||\langle\xi_v\rangle||_p||\langle\eta_v\rangle||_q$.

8a. Let a, b be nonnegative, $1 < p < \infty$, $1/p+1/q = 1$. Consider the graph of the function $f(x) = x^{p-1}$. The area of the rectangle bounded by the x-axis, the y-axis, $x = a$ and $y = b$ is ab. The area of the region bounded by $f(x)$, the x-axis and $x = a$ is $\int_0^a x^{p-1}\,dx = \frac{a^p}{p}$. The area of the region bounded by $f(x)$, the y-axis and $y = b$ is $\int_0^b y^{1/(p-1)}\,dy = \frac{b^{p/(p-q)}}{p/(p-q)} = \frac{b^q}{q}$. By comparing these areas, we see that $ab \leq \frac{a^p}{p}+\frac{b^q}{q}$.

8b. We may assume that $||f||_p > 0$ and $||g||_q > 0$. By part (a), $\int \frac{|f|}{||f||_p}\frac{|g|}{||g||_q} \leq \int \frac{|f|^p}{p||f||_p^p}+\frac{|g|^q}{q||g||_q^q} = \frac{1}{p}+\frac{1}{q} = 1$. Hence $\int|fg| \leq ||f||_p||g||_q$. Equality holds in part (a) if and only if $b = a^{p-1}$. Thus equality holds here if and only if $||f||_p^{p-1}|g| = ||g||_q|f|^{p-1}$. Equivalently, $||f||_p^p\,|g|^q = ||g||_q^q\,|f|^p$.

8c. Suppose $0 < p < 1$ and $1/p+1/q = 1$. Let $p' = 1/p$ and $q' = -q/p$. Then $p' > 1$, $q' > 1$ and $1/p'+1/q' = 1$. Thus $(ab)^pb^{-p} \leq \frac{(ab)^{pp'}}{p'}+\frac{b^{-pq'}}{q'} = pab - \frac{pb^q}{q}$ so $a^p \leq pab - \frac{pb^q}{q}$ and $ab \geq \frac{a^p}{p}+\frac{b^q}{q}$.

8d. By a similar argument as part (b), $\int|fg| \geq ||f||_p||g||_q$.

6.3 Convergence and completeness

9. Let $\langle f_n\rangle$ be a convergent sequence in L^p. There exists $f \in L^p$ such that for any $\varepsilon > 0$, there exists N such that $||f_n - f||_p < \varepsilon/2$ for $n \geq N$. Now for $n, m \geq N$, $||f_n - f_m||_p \leq ||f_n - f||_p + ||f_m - f||_p < \varepsilon$. Thus $\langle f_n\rangle$ is a Cauchy sequence.

10. Let $\langle f_n\rangle$ be a sequence of functions in L^∞. Suppose $||f_n - f||_\infty \to 0$. Given $\varepsilon > 0$, there exists N such that $\inf\{M : m\{t : |f_n(t)-f(t)| > M\} = 0\} < \varepsilon$ for $n \geq N$. Thus $m\{t : |f_n(t)-f(t)| \geq \varepsilon\} = 0$ for $n \geq N$. Let $E = \{t : |f_n(t)-f(t)| \geq \varepsilon\}$. Then $mE = 0$ and $\langle f_n\rangle$ converges uniformly to f on E^c. Conversely, suppose there exists a set E with $mE = 0$ and $\langle f_n\rangle$ converges uniformly to f on E^c. Given $\varepsilon > 0$, there exists N such that $|f_n(t)-f(t)| < \varepsilon/2$ for $n \geq N$ and $t \in E^c$. Thus $\{t : |f_n(t)-f(t)| > \varepsilon/2\} \subset E$ for $n \geq N$. Hence $\inf\{M : m\{t : |f_n(t)-f(t)| > M\} = 0\} < \varepsilon$ for $n \geq N$. i.e. $||f_n - f||_\infty \to 0$.

11. Let $\langle f_n\rangle$ be a Cauchy sequence in L^∞. Given $\varepsilon > 0$, there exists N such that $\inf\{M : m\{t : |f_n(t)-f_m(t)| > M\} = 0\} = ||f_n - f_m||_\infty < \varepsilon/2$ for $n, m \geq N$. Thus for $n, m \geq N$, there exists $M < \varepsilon/2$ such that $m\{t : |f_n(t)-f_m(t)| > M\} = 0$ so $m\{t : |f_n(t)-f_m(t)| > \varepsilon/2\} = 0$. Then $\langle f_n\rangle$ converges a.e. to a function f and $|f_n - f| < \varepsilon/2$ a.e. for $n \geq N$ so $|f| \leq |f_N| + \varepsilon/2$ a.e. and $f \in L^\infty$. Furthermore, $\inf\{M : m\{t : |f_n(t)-f(t)| > M\} = 0\} < \varepsilon$ for $n \geq N$. i.e. $||f_n - f||_\infty \to 0$.

12. Let $1 \leq p < \infty$ and let $\langle\xi_v^{(n)}\rangle$ be a Cauchy sequence in ℓ^p. Given $\varepsilon > 0$, there exists N such that $\sum|\xi_v^{(n)} - \xi_v^{(m)}|^p < \varepsilon^p$ for $n, m \geq N$. In particular, $|\xi_v^{(n)} - \xi_v^{(m)}|^p < \varepsilon^p$ for $n, m \geq N$ and each v. Thus for each v, $\langle\xi_v^{(n)}\rangle$ is Cauchy in $\mathbb{R}$ so it converges to some ξ_v. Consider $\langle\xi_v\rangle$. Then $\sum_{v=1}^k|\xi_v^{(n)} - \xi_v|^p < \varepsilon^p$ for each k and each $n \geq N$ so $\sum|\xi_v^{(n)} - \xi_v|^p < \varepsilon^p$ for $n \geq N$. Thus $\langle\xi_v^{(n)} - \xi_v\rangle \in \ell^p$ for $n \geq N$ so $\langle\xi_v\rangle \in \ell^p$

and $||\langle\xi_v^{(n)}\rangle - \langle\xi_v\rangle||_p \to 0$.

13. Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ and define $||f|| = \max|f(x)|$ for $f \in C$. It is straightforward to check that $||\cdot||$ is a norm on C. Let $\langle f_n\rangle$ be a Cauchy sequence in C. Given $\varepsilon > 0$, there exists N such that $\max|f_n(x) - f_m(x)| < \varepsilon$ for $n, m \geq N$ so $|f_n(x) - f_m(x)| < \varepsilon$ for $n, m \geq N$ and $x \in [0,1]$. Thus $\langle f_n(x)\rangle$ converges to some $f(x)$ for each $x \in [0,1]$. Furthermore, the convergence is uniform. Thus $f \in C$. Also, $\max|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$. i.e. $||f_n - f|| \to 0$.

14. It is straightforward to check that $||\cdot||_\infty$ is a norm on ℓ^∞. Let $\langle\xi_v^{(n)}\rangle$ be a Cauchy sequence in ℓ^∞. Given $\varepsilon > 0$, there exists N such that $\sup|\xi_v^{(n)} - \xi_v^{(m)}| < \varepsilon$ for $n, m \geq N$. Then $|\xi_v^{(n)} - \xi_v^{(m)}| < \varepsilon$ for each v and $n, m \geq N$. Thus $\langle\xi_v^{(n)}\rangle$ converges to some ξ_v for each v and $|\xi_v^{(n)} - \xi_v| < \varepsilon$ for $n \geq N$. Then $|\xi_v| \leq |\xi_v^{(N)}| + \varepsilon$ for each v and $\langle\xi_v\rangle \in \ell^\infty$. Also, $\sup|\xi_v^{(n)} - \xi_v| < \varepsilon$ for $n \geq N$. i.e. $||\langle\xi_v^{(n)}\rangle - \langle\xi_v\rangle||_\infty \to 0$.

15. Let c be the space of all convergent sequences of real numbers and let c_0 be the space of all sequences which converge to 0. It is straightforward to check that $||\cdot||_\infty$ is a norm on c and c_0. Let $\langle\xi_v^{(n)}\rangle$ be a Cauchy sequence in c. Given $\varepsilon > 0$, there exists N such that $\sup|\xi_v^{(n)} - \xi_v^{(m)}| < \varepsilon$ for $n, m \geq N$. Then $|\xi_v^{(n)} - \xi_v^{(m)}| < \varepsilon$ for each v and $n, m \geq N$. Thus $\langle\xi_v^{(n)}\rangle$ converges to some ξ_v for each v. Now for each v and v', there exists N' such that $|\xi_v - \xi_{v'}| \leq |\xi_v - \xi_v^{(N')}| + |\xi_v^{(N')} - \xi_{v'}^{(N')}| + |\xi_{v'} - \xi_{v'}^{(N')}| < \varepsilon$. Thus $\langle\xi_v\rangle$ is Cauchy in $\mathbb{R}$. Hence $\langle\xi_v\rangle \in c$ and $\sup|\xi_v^{(n)} - \xi_v| < \varepsilon$. i.e. $||\langle\xi_v^{(n)}\rangle - \langle\xi_v\rangle||_\infty \to 0$. If $\langle\xi_v^{(n)}\rangle$ is a Cauchy sequence in c_0, then $\langle\xi_v\rangle$ converges to 0 since $|\xi_v| \leq |\xi_v^{(n)} - \xi_v| + |\xi_v^{(n)}|$.

16. Let $\langle f_n\rangle$ be a sequence in L^p, $1 \leq p < \infty$, which converges a.e. to a function f in L^p. Suppose $||f_n - f||_p \to 0$. Then $||f_n||_p \to ||f||_p$ since $|\ ||f_n||_p - ||f||_p\ | \leq ||f_n - f||_p$. Conversely, suppose $||f_n||_p \to ||f||_p$. Now $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0$ for each n so by Fatou's Lemma, $\int 2^{p+1}|f|^p \leq \underline{\lim} \int 2^p(|f_n|^p + |f|^p) - |f_n - f|^p = \int 2^{p+1}|f|^p - \overline{\lim} \int |f_n - f|^p$. Thus $\overline{\lim} \int |f_n - f|^p \leq 0 \leq \underline{\lim} \int |f_n - f|^p$. Hence $||f_n - f||_p \to 0$.

17. Let $\langle f_n\rangle$ be a sequence in L^p, $1 < p < \infty$, which converges a.e. to a function f in L^p. Suppose there is a constant M such that $||f_n||_p \leq M$ for all n. Let $g \in L^q$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E |g|^q < (\varepsilon/4M)^q$ whenever $mE < \delta$. By Egoroff's Theorem, there exists E such that $mE < \delta$ and $\langle f_n\rangle$ converges uniformly to f on E^c. Thus there exists N such that $|f_n(x) - f(x)| < \varepsilon/(2(mE^c)^{1/p}||g||_q)$ for $n \geq N$ and $x \in E^c$. Now $|\int f_n g - \int fg| \leq \int |f_n - f||g| \leq (\int_E |f_n - f|^p)^{1/p}(\int_E |g|^q)^{1/q} + (\int_{E^c} |f_n - f|^p)^{1/p}(\int_{E^c} |g|^q)^{1/q} \leq 2M(\varepsilon/4m) + (\varepsilon/(2(mE^c)^{1/p}||g||_q))(mE^c)^{1/p}||g||_q = \varepsilon$ for $n \geq N$. i.e. $\int fg = \lim \int f_n g$.

For $p = 1$, it is not true. Let $f_n = n\chi_{[0,1/n]}$ for each n. Then $f_n \to 0$ and $||f_n||_1 = 1$ for each n. Let $g = \chi_{[0,1]} \in L^\infty$. Then $\int fg = 0$ but $\int f_n g = \int f_n = 1$ for each n.

18. Let $f_n \to f$ in L^p, $1 \leq p < \infty$ and let $\langle g_n\rangle$ be a sequence of measurable functions such that $|g_n| \leq M$ for all n and $g_n \to g$ a.e. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E |f|^p < (\varepsilon/8M)^p$ whenever $mE < \delta$. By Egoroff's Theorem, there exists E such that $mE < \delta$ and $\langle g_n\rangle$ converges uniformly to g on E^c. Thus there exists N such that $||f_n - f||_p < \varepsilon/2M$ and $|g_n(x) - g(x)| < \varepsilon/(4(mE^c)^{1/p}||f||_p)$ for $n \geq N$ and $x \in E^c$. Now $||g_n f_n - gf||_p \leq ||g_n f_n - g_n f||_p + ||g_n f - gf||_p = (\int |g_n|^p |f_n - f|^p)^{1/p} + (\int |g_n - g|^p |f|^p)^{1/p} \leq M||f_n - f||_p + (\int_E |g_n - g|^p |f|^p)^{1/p} + (\int_{E^c} |g_n - g|^p |f|^p)^{1/p} < \varepsilon/2 + (\varepsilon/8M)(2M) + \varepsilon/(4(mE^c)^{1/p}||f||_p)(mE^c)^{1/p}||f||_p = \varepsilon$ for $n \geq N$. Thus $||g_n f_n - gf||_p \to 0$.

6.4 Approximation in L^p

***19.** $||T_\Delta(f)||_p^p = ||\sum_{k=0}^{m-1} \frac{1}{\xi_{k+1} - \xi_k}(\int_{\xi_k}^{\xi_{k+1}} f)\chi_{[\xi_k, \xi_{k+1})}||_p^p \leq \sum_{k=0}^{m-1} ||\frac{1}{\xi_{k+1} - \xi_k}(\int_{\xi_k}^{\xi_{k+1}} f)\chi_{[\xi_k, \xi_{k+1})}||_p^p$. Now $||\frac{1}{\xi_{k+1} - \xi_k}(\int_{\xi_k}^{\xi_{k+1}} f)\chi_{[\xi_k, \xi_{k+1})}||_p^p = \int_a^b |\frac{1}{\xi_{k+1} - \xi_k}(\int_{\xi_k}^{\xi_{k+1}} f)\chi_{[\xi_k, \xi_{k+1})}|^p = \int_{\xi_k}^{\xi_{k+1}} \frac{1}{(\xi_{k+1} - \xi_k)^p} |\int_{\xi_k}^{\xi_{k+1}} f|^p$. By the Hölder inequality, $|\int_{\xi_k}^{\xi_{k+1}} f|^p \leq \int_{\xi_k}^{\xi_{k+1}} |f|^p (\int_{\xi_k}^{\xi_{k+1}} 1)^{p/q} = \int_{\xi_k}^{\xi_{k+1}} |f|^p (\int_{\xi_k}^{\xi_{k+1}} 1)^{p-1}$. Thus $||T_\Delta(f)||_p^p \leq \sum_{k=0}^{m-1} \int_{\xi_k}^{\xi_{k+1}} \left(\frac{1}{(\xi_{k+1} - \xi_k)^p} \int_{\xi_k}^{\xi_{k+1}} |f|^p (\int_{\xi_k}^{\xi_{k+1}} 1)^{p-1}\right) = \sum_{k=0}^{m-1} \frac{1}{(\xi_{k+1} - \xi_k)^{p-1}} \int_{\xi_k}^{\xi_{k+1}} |f|^p (\xi_{k+1} - \xi_k)^{p-1} = \int_a^b |f|^p$. Hence $||T_\Delta(f)||_p^p \leq ||f||_p^p$ and $||T_\Delta(f)||_p \leq ||f||_p$.

***20.** By Chebyshev's inequality, for any $\varepsilon > 0$, $\int |\varphi_\Delta - f|^p \geq \varepsilon^p m\{x : |\varphi_\Delta(x) - f(x)|^p > \varepsilon^p\}$. Thus $m\{x : |\varphi_\Delta(x) - f(x)| > \varepsilon\} = m\{x : |\varphi_\Delta(x) - f(x)|^p > \varepsilon^p\} \leq \varepsilon^{-p}||\varphi_\Delta - f||_p^p < \varepsilon$ for sufficiently small δ.

6.5 Bounded linear functionals on the L^p spaces

21a. Let g be an integrable function on $[0,1]$. If $||g||_1 \neq 0$, let $f = sgn(g)$. Then f is a bounded measurable function, $||f||_\infty = 1$ and $\int fg = \int |g| = ||g||_1 ||f||_\infty$. If $||g||_1 = 0$, then $g = 0$ a.e. Let $f = 1$. Then f is a bounded measurable function, $||f||_\infty = 1$ and $\int fg = \int g = 0 = ||g||_1 ||f||_\infty$.

21b. Let g be a bounded measurable function. Given $\varepsilon > 0$, let $E = \{x : g(x) > ||g||_\infty - \varepsilon\}$ and let $f = \chi_E$. Then $\int fg = \int_E g \geq (||g||_\infty - \varepsilon)mE = (||g||_\infty - \varepsilon)||f||_1$.

22. Let F be a bounded linear functional on ℓ^p. For each v, let e_v be the sequence with 1 in the v-th entry and 0 elsewhere. For $p = 1$, note that $||\langle \xi_v \rangle - \sum_{v=1}^n \xi_v e_v||_1 \to 0$ for each $\langle \xi_v \rangle \in \ell^1$ so $F(\langle \xi_v \rangle) = \sum \xi_v F(e_v)$ by linearity and continuity of F. Now $|F(e_v)| = |F(e_v)|/||e_v||_1 \leq ||F||$ for all v so $\langle F(e_v) \rangle \in \ell^\infty$ and $||\langle F(e_v) \rangle||_\infty \leq ||F||$. Conversely, $|F(\langle \xi_v \rangle)| = |\sum \xi_v F(e_v)| \leq ||\langle \xi_v \rangle||_1 ||\langle F(e_v) \rangle||_\infty$ so $||\langle F(e_v) \rangle||_\infty \geq |F(\langle \xi_v \rangle)|/||\langle \xi_v \rangle||_1$ for all nonzero $\langle \xi_v \rangle \in \ell^1$. Thus $||\langle F(e_v) \rangle||_\infty \geq ||F||$.

For $1 < p < \infty$, note that $||\langle \xi_v \rangle - \sum_{v=1}^n \xi_v e_v||_p \to 0$ for each $\langle \xi_v \rangle \in \ell^p$ so $F(\langle \xi_v \rangle) = \sum \xi_v F(e_v)$ by linearity and continuity of F. For each v, let $x_v = |F(e_v)|^q / F(e_v) = |F(e_v)|^{q-1} sgn(F(e_v))$ and let $x = \langle x_1, \ldots, x_n, 0, 0, \ldots \rangle$. Then $F(x) = \sum_{v=1}^n x_v F(e_v) = \sum_{v=1}^n |F(e_v)|^q$. Now $||x||_p = (\sum_{v=1}^n |x_v|^p)^{1/p} = (\sum_{v=1}^n |F(e_v)|^{p(q-1)})^{1/p} = (\sum_{v=1}^n |F(e_v)|^q)^{1/p}$ so $(\sum_{v=1}^n |F(e_v)|^q)^{1/q} = \frac{\sum_{v=1}^n |F(e_v)|^q}{(\sum_{v=1}^n |F(e_v)|^q)^{1/p}} = \frac{|F(x)|}{||x||_p} \leq ||F||$ for all n. Thus $\langle F(e_v) \rangle \in \ell^q$ and $||\langle F(e_v) \rangle||_q \leq ||F||$. Conversely, $|F(\langle \xi_v \rangle)| = |\sum \xi_v F(e_v)| \leq ||\langle \xi_v \rangle||_p ||\langle F(e_v) \rangle||_q$ so $||\langle F(e_v) \rangle||_q \geq |F(\langle \xi_v \rangle)|/||\langle \xi_v \rangle||_p$ for all nonzero $\langle \xi_v \rangle \in \ell^p$. Thus $||\langle F(e_v) \rangle||_q \geq ||F||$.

23. Note that $||\langle \xi_v \rangle - \sum_{v=1}^n \xi_v e_v - \sum_{v=n+1}^\infty \xi e_v||_\infty \to 0$ for each $\langle \xi_v \rangle \in c$ with $\lim \xi_v = \xi$ so $F(\langle \xi_v \rangle) = \sum \xi_v F(e_v)$ by linearity and continuity of F. For each v, let $x_v = sgn(F(e_v))$ and let x be defined as before. Then $F(x) = \sum_{v=1}^n x_v F(e_v) = \sum_{v=1}^n |F(e_v)|$ so $\sum_{v=1}^n |F(e_v)| = |F(x)|/||x||_\infty \leq ||F||$ for all n. Thus $\langle F(e_v) \rangle \in \ell^1$ and $||\langle F(e_v) \rangle||_1 \leq ||F||$. Conversely, $|F(\langle \xi_v \rangle)| = |\sum \xi_v F(e_v)| \leq ||\langle \xi_v \rangle||_\infty ||\langle F(e_v) \rangle||_1$ so $||\langle F(e_v) \rangle||_1 \geq |F(\langle \xi_v \rangle)|/||\langle \xi_v \rangle||_\infty$ for all nonzero $\langle \xi_v \rangle \in c$. Thus $||\langle F(e_v) \rangle||_1 \geq ||F||$. Similarly for c_0.

***24.** Let F be a bounded linear functional on L^p, $1 \leq p < \infty$, and suppose there exist functions g and h in L^q such that $F(f) = \int fg = \int fh$ for all $f \in L^p$. For $p > 1$, choose $f = |g-h|^{q-2}(g-h)$. Then $|f|^p = |g-h|^{p(q-1)} = |g-h|^q$ so $f \in L^p$. Now $\int |g-h|^{q-2}(g-h)g = \int |g-h|^{q-2}(g-h)h$ so $\int |g-h|^q = 0$. Thus $g = h$ a.e. For $p = 1$, choose $f = sgn(g-h)$. Then $f \in L^1$ and $f(g-h) = |g-h|$. Now $\int fg = \int fh$ so $\int |g-h| = \int f(g-h) = 0$. Thus $g = h$ a.e.

7 Metric Spaces

7.1 Introduction

1a. Clearly, $\rho^*(x,y) \geq 0$ for all x, y. Now $\rho^*(x,y) = 0$ if and only if $|x_i - y_i| = 0$ for all i if and only if $x_i = y_i$ for all i if and only if $x = y$. Since $|x_i - y_i| = |y_i - x_i|$ for each i, $\rho^*(x,y) = \rho^*(y,x)$. Finally, $\rho^*(x,y) = \sum_{i=1}^n |x_i - y_i| \leq \sum_{i=1}^n (|x_i - z_i| + |z_i - y_i|) = \sum_{i=1}^n |x_i - z_i| + \sum_{i=1}^n |z_i - y_i| = \rho^*(x,z) + \rho^*(z,y)$. The argument for ρ^+ is similar except for the last property. For any x_j, y_j, z_j, $|x_j - y_j| \leq |x_j - z_j| + |z_j - y_j| \leq \max_i |x_i - z_i| + \max_i |z_i - y_i| = \rho^+(x,z) + \rho^+(z,y)$. Thus $\rho^+(x,y) \leq \rho^+(x,z) + \rho^+(z,y)$.

1b. For $n = 2$ (resp. $n = 3$), $\{x : \rho(x,y) < 1\}$ is the interior of the circle (resp. sphere) with center y and radius 1. $\{x : \rho^*(x,y) < 1\}$ is the interior of the diamond (resp. bi-pyramid) with center y with height and width 2. $\{x : \rho^+(x,y) < 1\}$ is the interior of the square (resp. cube) with center y and sides of length 1.

2. Suppose $0 < \varepsilon < \delta - \rho(x,z)$ and $y \in S_{z,\varepsilon}$. Then $\rho(z,y) < \varepsilon < \delta - \rho(x,z)$. Hence $\rho(x,y) \leq \rho(x,z) + \rho(z,y) < \delta$ so $y \in S_{x,\delta}$.

3a. For any x, $\rho(x,x) = 0$. If $\rho(x,y) = 0$, then $\rho(y,x) = \rho(x,y) = 0$. If $\rho(x,z) = 0$ and $\rho(z,y) = 0$, then $0 \leq \rho(x,y) \leq \rho(x,z) + \rho(z,y) = 0$ so $\rho(x,y) = 0$. Thus $\rho(x,y) = 0$ is an equivalence relation. Let X^* be the set of equivalence classes under this relation. Suppose x and x' are in the same equivalence class. Also suppose that y and y' are in the same equivalence class. Then $\rho(x,y) \leq \rho(x,x') + \rho(x',y) = \rho(x',y) \leq \rho(x',y') + \rho(y',y) = \rho(x',y')$. If $\rho(x,y) = 0$, then x and y are in the same equivalence class. Thus ρ defines a metric on X^*.

3b. Let ρ be an extended metric on X. For any x, $\rho(x,x) = 0 < \infty$. If $\rho(x,y) < \infty$, then $\rho(y,x) = \rho(x,y) < \infty$. If $\rho(x,z) < \infty$ and $\rho(z,y) < \infty$, then $\rho(x,y) \leq \rho(x,z) + \rho(z,y) < \infty$. Thus $\rho(x,y) < \infty$ is an equivalence relation. Let X_{α_0} be a part of the extended metric space (X, ρ) and let x be a representative of X_{α_0}. If $y \in X_{\alpha_0}$, then for any $z \in S_{y,\delta}$ with $\delta > 0$, $\rho(x,z) \le \rho(x,y) + \rho(y,z) < \infty$ so $z \in X_{\alpha_0}$. i.e. $S_{y,\delta} \subset X_{\alpha_0}$. Thus X_{α_0} is open. Now $X = \bigcup X_\alpha$ so $X_{\alpha_0} = X \setminus \bigcup_{\alpha \ne \alpha_0} X_\alpha$. Since the union is open, X_{α_0} is closed.

7.2 Open and closed sets

4a. Since continuous functions on $[0,1]$ are bounded, $C \subset L^\infty$. By Q6.3.13, C is complete. Let g be a point of closure of C. For every n, there exists $f_n \in C$ such that $||f_n - g||_\infty < 1/n$. Then $\langle f_n \rangle$ is a Cauchy sequence in C and it converges to g so $g \in C$. Thus C is a closed subset of L^∞.

4b. Let g be a point of closure of the set of integrable functions that vanish for $0 \le t < 1/2$. For each n, there exists f_n in the set such that $||f_n - g||_1 < 1/n$. Then $\int_0^{1/2} |g| \le \int_0^{1/2} |f_n - g| + \int_0^{1/2} |f_n| \le \int_0^1 |f_n - g| < 1/n$ for all n. Hence g vanishes a.e. and the set of integrable functions that vanish for $0 \le t < 1/2$ is closed in L^1.

4c. Let $x(t)$ be a measurable function with $\int x < 1$. Let $\delta = 1 - \int x$. For any $y \in S_{x,\delta}$, $\int |y - x| < \delta$ so $\int y = \int (y - x) + \int x < \delta + \int x < 1$. Hence the set of measurable functions $x(t)$ with $\int x < 1$ is open in L^1.

5. If $E \subset F$ and F is closed, then $\bar{E} \subset \bar{F} = F$. Thus $\bar{E} \subset \bigcap_{E \subset F} F$ for closed sets F. Conversely, since $\bar{E}$ is a closed set containing E, $\bigcap_{E \subset F} F \subset \bar{E}$.

5a. From the definition, E° is an open subset of E so $E^\circ \subset \bigcup_{O \subset E} O$. Conversely, for any open subset $O \subset E$ and $y \in O$, there exists $\delta > 0$ such that $x \in O \subset E$ for all x with $\rho(x,y) < \delta$. Thus $O \subset E^\circ$ for any open subset $O \subset E$ so $\bigcup_{O \subset E} \subset E^\circ$.

5b. $(\bar{E})^c = (\bigcap_{E \subset F} F)^c = \bigcup_{E \subset F} F^c = \bigcup_{F^c \subset E^c} F^c = (E^c)^\circ$.

6a. Consider the ball $S_{y,\delta} = \{x : \rho(x,y) < \delta\}$. For any $x \in S_{y,\delta}$, let $0 < \varepsilon < \delta - \rho(x,y)$. By Q2, $S_{x,\varepsilon} \subset S_{y,\delta}$ so $S_{y,\delta}$ is open.

6b. Consider the set $S = \{x : \rho(x,y) \le \delta\}$. Take $x \in S^c$. Then $\rho(x,y) > \delta$. Let $\delta' = \rho(x,y) - \delta$. For any $z \in S_{x,\delta'}$, $\rho(z,y) \ge \rho(x,y) - \rho(x,z) > \rho(x,y) - \delta' = \delta$ so $z \in S^c$. Thus S^c is open and S is closed.

6c. The set in part (b) is not always the closure of the ball $\{x : \rho(x,y) < \delta\}$. For example, let X be any metric space with $|X| > 1$ and ρ being the discrete metric. i.e. $\rho(x,y) = 1$ if $x \ne y$ and $\rho(x,y) = 0$ if $x = y$. Then $\{x : \rho(x,y) < 1\} = \{y\}$, $\overline{\{x : \rho(x,y) < 1\}} = \{y\}$ and $\{x : \rho(x,y) \le 1\} = X$.

***7.** $\mathbb{R}^n$ is separable with the set of n-tuples of rational numbers being a countable dense subset. C is separable with the set of polynomials on $[0,1]$ with rational coefficients being a countable dense subset (Weierstrass approximation theorem).

L^∞ is not separable. Consider $\chi_{[0,x]}$ and $\chi_{[0,y]}$ where $x, y \in [0,1]$. Then $||\chi_{[0,x]} - \chi_{[0,y]}||_\infty = 1$ if $x \ne y$. If there exists a countable dense subset D, then there exists $d \in D$ and $x, y \in [0,1]$ with $x \ne y$ such that $||\chi_{[0,x]} - d||_\infty < 1/2$ and $||\chi_{[0,y]} - d||_\infty < 1/2$. Then $||\chi_{[0,x]} - \chi_{[0,y]}||_\infty < 1$. Contradiction.

L^1 is separable. By Proposition 6.8, given $f \in L^1$ and $\varepsilon > 0$, there exists a step function φ such that $||f - \varphi||_1 < \varepsilon$. We may further approximate φ by a step function where the partition intervals have rational endpoints and the coefficients are rational to get a countable dense subset.

7.3 Continuous functions and homeomorphisms

8. Let h be the function on $[0,1)$ given by $h(x) = x/(1-x)$. The function h, being a rational function, is continuous on $[0,1)$. If $h(x) = h(y)$, then $x(1-y) = y(1-x)$ so $x = y$. Thus h is one-to-one. For any $y \in [0,\infty)$, let $x = y/(1+y)$. Then $x \in [0,1)$ and $h(x) = y$. Thus h is onto. The inverse function h^{-1} is given by $h^{-1}(x) = x/(1+x)$, which is continuous. Hence h is a homeomorphism between $[0,1)$ and $[0,\infty)$.

9a. For a fixed set E, let $f(x) = \rho(x,E) = \inf_{y \in E} \rho(x,y)$. Given $\varepsilon > 0$, let $\delta = \varepsilon$. When $\rho(x,z) < \delta$, take any $y \in E$. Then $f(x) = \rho(x,E) \le \rho(x,y) \le \rho(x,z) + \rho(z,y) < \delta + \rho(z,E) \le \varepsilon + f(z)$. Thus $f(x) - f(z) < \varepsilon$. Similarly, by interchanging x and z, $f(z) - f(x) < \varepsilon$. Thus $|f(x) - f(z)| < \varepsilon$ and f is continuous.

9b. If $\rho(x,E) = 0$, then for any $\delta > 0$, there exists $y \in E$ such that $\rho(x,y) < \delta$ so $x \in \bar{E}$. Conversely, if $\rho(x,E) > 0$, say $\rho(x,E) = \alpha$, then $\rho(x,y) > \alpha/2$ for all $y \in E$ so $x \notin \bar{E}$. Hence $\{x : \rho(x,E) = 0\} = \bar{E}$.

10a. Suppose ρ and σ are equivalent metrics on X. The identity mapping is a homeomorphism between (X,ρ) and (X,σ). Thus given $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\rho(x,y) < \delta$, then $\sigma(x,y) < \varepsilon$. By considering the inverse function, we see that if $\sigma(x,y) < \delta$, then $\rho(x,y) < \varepsilon$. Conversely, the two implications show that the identity mapping is continuous from (X,ρ) to (X,σ) as well as from (X,σ) to (X,ρ). Since the identity mapping is clearly bijective, it is a homeomorphism so ρ and σ are equivalent metrics.

10b. Given $\varepsilon > 0$, let $\delta = \varepsilon/n$. When $\rho^+(x,y) < \delta$, $\rho^*(x,y) < n\delta = \varepsilon$. When $\rho^*(x,y) < \delta$, $\rho^+(x,y) < \delta < \varepsilon$. Thus ρ^+ and ρ^* are equivalent metrics. When $\rho^+(x,y) < \delta$, $\rho(x,y) < (n\delta^2)^{1/2} = (\varepsilon^2/n)^{1/2} < \varepsilon$. When $\rho(x,y) < \delta$, $(\rho^+(x,y))^2 < \delta^2$ so $\rho^+(x,y) < \delta < \varepsilon$. Thus ρ^+ and ρ are equivalent metrics. There exists $\delta' > 0$ with $\delta' < \varepsilon/\sqrt{n}$ such that $\rho^*(x,y) < \delta'$ implies $\rho^+(x,y) < \delta$. Then when $\rho^*(x,y) < \delta'$, $\rho(x,y) < \varepsilon$. When $\rho(x,y) < \delta'$, $\rho^*(x,y) \leq \rho(x,y)\sqrt{n} < \varepsilon$. Thus ρ and ρ^* are equivalent metrics.

10c. Consider the discrete metric ψ. Let $x = (0,\dots,0)$. For any $\delta > 0$, we can choose $y \neq x$ such that $\rho(x,y) < \delta$ but $\psi(x,y) = 1$. Similarly for ρ^* and ρ^+. Thus ψ is not equivalent to the metrics in part (b).

11a. Let ρ be a metric on a set X and let $\sigma = \rho/(1+\rho)$. Clearly, $\sigma(x,y) \geq 0$ with $\sigma(x,y) = 0$ if and only if $x = y$. Also, $\sigma(x,y) = \sigma(y,x)$. Now $\sigma(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} = 1 - \frac{1}{1+\rho(x,y)} \leq 1 - \frac{1}{1+\rho(x,z)+\rho(z,y)} = \frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \leq \sigma(x,z) + \sigma(z,y)$. Hence σ is a metric on X. Note that $\rho(x,y) = \frac{\sigma(x,y)}{1-\sigma(x,y)} = h(\sigma(x,y))$ where h is the function in Q8. Since h is continuous at 0, given $\varepsilon > 0$, there exists $\delta' > 0$ such that $h(\sigma(x,y)) < \varepsilon$ when $\sigma(x,y) < \delta'$. Now given $\varepsilon > 0$, let $\delta < \min(\delta',\varepsilon)$. When $\rho(x,y) < \delta$, $\sigma(x,y) < \rho(x,y) < \delta < \varepsilon$. When $\sigma(x,y) < \delta$, $\rho(x,y) = h(\sigma(x,y)) < \varepsilon$. Hence σ and ρ are equivalent metrics for X. Furthermore, $\sigma(x,y) \leq 1$ for all $x,y \in X$ so (X,σ) is a bounded metric space.

11b. If ρ is an extended metric (resp. pseudometric), then σ is an extended metric (resp. pseudometric). The rest of the argument in part (a) follows.

7.4 Convergence and completeness

12. Suppose the sequence $\langle x_n \rangle$ has x as a cluster point. There exists $n_1 \geq 1$ such that $\rho(x, x_{n_1}) < 1$. Suppose $x_{n_1},\dots,x_{n_k}$ have been chosen. There exists $n_{k+1} \geq n_k$ such that $\rho(x, x_{n_{k+1}}) < 1/(k+1)$. The subsequence $\langle x_{n_k} \rangle$ converges to x. Conversely, suppose there is a subsequence $\langle x_{n_k} \rangle$ that converges to x. Given $\varepsilon > 0$ and given N, there exists N' such that $\rho(x, x_{n_k}) < \varepsilon$ for $k \geq N'$. Pick $k \geq \max(N, N')$. Then $n_k \geq k \geq N$ and $\rho(x, x_{n_k}) < \varepsilon$. Thus x is a cluster point of the sequence $\langle x_n \rangle$.

13. Suppose the sequence $\langle x_n \rangle$ converges to x. Then every subsequence of $\langle x_n \rangle$ also converges to x and so has x as a cluster point. Conversely, suppose $\langle x_n \rangle$ does not converge to x. There exists $\varepsilon > 0$ such that for each N, there exists $n \geq N$ with $\rho(x, x_n) \geq \varepsilon$. Pick n_1 such that $\rho(x, x_{n_1}) \geq \varepsilon$. Suppose $x_{n_1},\dots,x_{n_k}$ have been chosen. Then pick $n_{k+1} \geq n_k$ such that $\rho(x, x_{n_{k+1}}) \geq \varepsilon$. The subsequence $\langle x_{n_k} \rangle$ does not have x as a cluster point.

If every subsequence of $\langle x_n \rangle$ has in turn a subsequence that converges to x, then every subsequence of $\langle x_n \rangle$ has x as a cluster point by Q12. Hence the sequence $\langle x_n \rangle$ converges to x.

14. Let E be a set in a metric space X. If x is a cluster point of a sequence from E, then given $\varepsilon > 0$, there exists n such that $\rho(x, x_n) < \varepsilon$. Since $x_n \in E$, $x \in \bar{E}$. On the other hand, if $x \in \bar{E}$, then for each n, there exists $x_n \in E$ with $\rho(x, x_n) < 1/n$. The sequence $\langle x_n \rangle$ converges to x.

15. Suppose a Cauchy sequence $\langle x_n \rangle$ in a metric space has a cluster point x. By Q12, there is a subsequence $\langle x_{n_k} \rangle$ that converges to x. Given $\varepsilon > 0$, there exists N such that $\rho(x, x_{n_k}) < \varepsilon/2$ for $k \geq N$ and $\rho(x_n, x_m) < \varepsilon/2$ for $n, m \geq N$. Now for $k \geq N$, $\rho(x, x_k) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_k) < \varepsilon$. Thus $\langle x_n \rangle$ converges to x.

16. Let X and Y be metric spaces and f a mapping from X to Y. Suppose f is continuous at x and let $\langle x_n \rangle$ be a sequence in X that converges to x. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma(f(x), f(y)) < \varepsilon$ if $\rho(x,y) < \delta$. There also exists N such that $\rho(x, x_n) < \delta$ for $n \geq N$. Thus $\sigma(f(x), f(x_n)) < \varepsilon$ for $n \geq N$ so the sequence $\langle f(x_n) \rangle$ converges to $f(x)$ in Y. Conversely, suppose f is not continuous at x. Then there exists $\varepsilon > 0$ such that for every n, there exists x_n with $\rho(x, x_n) < 1/n$ but $\sigma(f(x), f(x_n)) \geq \varepsilon$. The sequence $\langle x_n \rangle$ converges to x but $\langle f(x_n) \rangle$ does not converge to $f(x)$.

17a. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be Cauchy sequences from a metric space X. Given $\varepsilon > 0$, there exists N such that $\rho(x_n, x_m) < \varepsilon/2$ and $\rho(y_n, y_m) < \varepsilon/2$ for $n, m \geq N$. Now $\rho(x_n, y_n) \leq \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n)$ so

$\rho(x_n, y_n) - \rho(x_m, y_m) \leq \rho(x_n, x_m) + \rho(y_n, y_m)$. Similarly, $\rho(x_m, y_m) - \rho(x_n, y_n) \leq \rho(x_n, x_m) + \rho(y_n, y_m)$. Thus $|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_n, x_m) + \rho(y_n, y_m) < \varepsilon$ for $n, m \geq N$. Hence the sequence $\langle \rho(x_n, y_n) \rangle$ is Cauchy in $\mathbb{R}$ and thus converges.

17b. Define $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = \lim \rho(x_n, y_n)$ for Cauchy sequences $\langle x_n \rangle$ and $\langle y_n \rangle$. Then $\rho^*(\langle x_n \rangle, \langle y_n \rangle) \geq 0$ since $\rho(x_n, y_n) \geq 0$ for each n. Also, $\rho^*(\langle x_n \rangle, \langle x_n \rangle) = \lim \rho(x_n, x_n) = 0$. Furthermore, $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = \lim \rho(x_n, y_n) = \lim \rho(y_n, x_n) = \rho^*(\langle y_n \rangle, \langle x_n \rangle)$. Finally, $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = \lim \rho(x_n, y_n) \leq \lim \rho(x_n, z_n) + \lim \rho(z_n, y_n) = \rho^*(\langle x_n \rangle, \langle z_n \rangle) + \rho^*(\langle z_n \rangle, \langle y_n \rangle)$. Hence the set of all Cauchy sequences from a metric space becomes a pseudometric space under ρ^*.

17c. Define $\langle x_n \rangle$ to be equivalent to $\langle y_n \rangle$ (written as $\langle x_n \rangle \sim \langle y_n \rangle$) if $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = 0$. If $\langle x_n \rangle \sim \langle x_n' \rangle$ and $\langle y_n \rangle \sim \langle y_n' \rangle$, then $|\rho^*(\langle x_n \rangle, \langle y_n \rangle) - \rho^*(\langle x_n' \rangle, \langle y_n' \rangle)| \leq \rho^*(\langle x_n \rangle, \langle x_n' \rangle) + \rho^*(\langle y_n \rangle, \langle y_n' \rangle) = 0$ so $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = \rho^*(\langle x_n' \rangle, \langle y_n' \rangle)$. If $\rho^*(\langle x_n \rangle, \langle y_n \rangle) = 0$, then $\langle x_n \rangle \sim \langle y_n \rangle$ so they are equal in X^*. Thus the pseudometric space becomes a metric space.

Associate each $x \in X$ with the equivalence class in X^* containing the constant sequence $\langle x, x, \ldots \rangle$. This defines a mapping T from X onto $T[X]$. Since $\rho^*(Tx, Ty) = \lim \rho(x, y) = \rho(x, y)$, if $Tx = Ty$, then $\rho(x, y) = 0$ so $x = y$. Thus T is one-to-one. Also, T is continuous on X and its inverse is continuous on $T[X]$. Hence T is an isometry between X and $T[X] \subset X^*$. Furthermore, $T[X]$ is dense in X^*.

17d. If $\langle x_n \rangle$ is a Cauchy sequence from X, we may assume (by taking a subsequence) that $\rho(x_n, x_{n+1}) < 2^{-n}$. Let $\langle \langle x_{n,m} \rangle_{n=1}^{\infty} \rangle_{m=1}^{\infty}$ be a sequence of such Cauchy sequences which represents a Cauchy sequence in X^*. Given $\varepsilon > 0$, there exists N such that for $m, m' \geq N$, $\rho^*(\langle x_{n,m} \rangle, \langle x_{n,m'} \rangle) < \varepsilon/2$. i.e. $\lim_n \rho(x_{n,m}, x_{n,m'}) < \varepsilon/2$. We may assume that for $n \geq N$, $\rho(x_{n,m}, x_{n,m'}) < \varepsilon/2$. In particular, $\rho(x_{m,m}, x_{m,m'}) < \varepsilon/2$. We may also assume that $\rho(x_{m,m'}, x_{m',m'}) < \varepsilon/2$ since the sequence $\langle x_{n,m'} \rangle_{n=1}^{\infty}$ is Cauchy in X. Thus $\rho(x_{m,m}, x_{m',m'}) < \varepsilon$ for $m, m' \geq N$ so the sequence $\langle x_{n,n} \rangle$ is Cauchy in X and represents the limit of the Cauchy sequence in X^*.

***17e.** T is an isometry from X onto $T[X]$ and T^{-1} is an isometry from $T[X]$ onto X. Thus there is a unique isometry T' from $\bar{X} \cap Y$ onto X^* that extends T. Similarly, there is a unique isometry T'' from X^* onto $\bar{X} \cap Y$ that extends T^{-1}. Then $T'|_X = T$ and $T''|_{T[X]} = T^{-1}$ so $(T' \circ T'')|_{T[X]} = id_{T[X]}$ and $(T'' \circ T')|_X = id_X$. Since $T[X]$ is dense in X^* and X is dense in $\bar{X} \cap Y$, we have $T' \circ T'' = id_{X^*}$ and $T'' \circ T' = id_{\bar{X} \cap Y}$ so $(T' = T'')^{-1}$. Hence X^* is isometric with the closure of X in Y.

18. Let (X, ρ) and (Y, σ) be two complete metric spaces. Let $\langle (x_n, y_n) \rangle$ be a Cauchy sequence in $X \times Y$. Since $\rho(x_n, x_m) \leq (\rho(x_n, x_m)^2 + \sigma(y_n, y_m)^2)^{1/2} = \tau((x_n, y_n), (x_m, y_m))$, the sequence $\langle x_n \rangle$ is Cauchy in X. Similarly, the sequence $\langle y_n \rangle$ is Cauchy in Y. Since X is complete, $\langle x_n \rangle$ converges to some $x \in X$. Similarly, $\langle y_n \rangle$ converges to some $y \in Y$. Given $\varepsilon > 0$, there exists N such that $\rho(x_n, x) < \varepsilon/2$ and $\sigma(y_n, y) < \varepsilon/2$ for $n \geq N$. Then $\tau((x_n, y_n), (x, y)) = (\rho(x_n, x)^2 + \sigma(y_n, y)^2)^{1/2} < \varepsilon$ for $n \geq N$. Hence the sequence $\langle (x_n, y_n) \rangle$ converges to $(x, y) \in X \times Y$ and $X \times Y$ is complete.

7.5 Uniform continuity and uniformity

19. $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) = \rho(x, x') + \sigma(y, y') \geq 0$. $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) = 0$ if and only if $\rho(x, x') = 0$ and $\sigma(y, y') = 0$ if and only if $x = x'$ and $y = y'$ if and only if $\langle x, y \rangle = \langle x', y' \rangle$. $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) = \rho(x, x') + \sigma(y, y') = \rho(x', x) + \sigma(y', y) = \rho_1(\langle x', y' \rangle, \langle x, y \rangle)$. $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) = \rho(x, x') + \sigma(y, y') \leq \rho(x, x'') + \rho(x'', x') + \sigma(y, y'') + \sigma(y'', y') = \rho_1(\langle x, y \rangle, \langle x'', y'' \rangle) + \rho_1(\langle x'', y'' \rangle, \langle x', y' \rangle)$. Hence ρ_1 is a metric. Similarly, ρ_∞ is a metric.

Given $\varepsilon > 0$, let $\delta = \varepsilon/2$. When $\tau(\langle x, y \rangle, \langle x', y' \rangle) < \delta$, $\rho(x, x')^2 < \varepsilon^2/4$ and $\sigma(y, y')^2 < \varepsilon^2/4$ so $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) = \rho(x, x') + \sigma(y, y') < \varepsilon$. Also, $\rho_\infty(\langle x, y \rangle, \langle x', y' \rangle) = \max(\rho(x, x'), \sigma(y, y')) < \varepsilon$. When $\rho_1(\langle x, y \rangle, \langle x', y' \rangle) < \delta$, $\rho(x, x') < \varepsilon/2$ and $\sigma(y, y') < \varepsilon/2$ so $\tau(\langle x, y \rangle, \langle x', y' \rangle) < \sqrt{\varepsilon^2/4 + \varepsilon^2/4} < \varepsilon$. When $\rho_\infty < \delta$, $\rho(x, x') < \varepsilon/2$ and $\sigma(y, y') < \varepsilon/2$ so $\tau(\langle x, y \rangle, \langle x', y' \rangle) < \sqrt{\varepsilon^2/4 + \varepsilon^2/4} < \varepsilon$. Hence ρ_1 and ρ_∞ are uniformly equivalent to the usual product metric τ.

20. Let f be a uniformly continuous mapping of a metric space X into a metric space Y and let $\langle x_n \rangle$ be a Cauchy sequence in X. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\sigma(f(x), f(y)) < \varepsilon$. There exists N such that $\rho(x_n, x_m) < \delta$ for $n, m \geq N$. Thus $\sigma(f(x_n), f(x_m)) < \varepsilon$ for $n, m \geq N$. Hence $\langle f(x_n) \rangle$ is a Cauchy sequence in Y.

21a. Let $\langle x_n \rangle$ be a sequence from E that converges to a point $x \in \bar{E}$. Then $\langle x_n \rangle$ is Cauchy in X so $\langle f(x_n) \rangle$ is Cauchy in Y. Since Y is complete, $\langle f(x_n) \rangle$ converges to some $y \in Y$.

21b. Suppose $\langle x_n \rangle$ and $\langle x'_n \rangle$ both converge to x. Suppose $\langle f(x_n) \rangle$ converges to y and $\langle f(x'_n) \rangle$ converges to y' with $y \neq y'$. Let $\varepsilon = \sigma(y, y')/4 > 0$. There exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\sigma(f(x), f(x')) < \varepsilon$. There also exists N such that $\rho(x_n, x) < \delta/2$ and $\rho(x'_n, x) < \delta/2$ for $n \geq N$. Thus $\rho(x_n, x'_m) < \delta$ for $n, m \geq N$ so $\sigma(f(x_n), f(x'_m)) < \varepsilon$ for $n, m \geq N$. We may assume that $\sigma(f(x_n), y) < \varepsilon$ and $\sigma(f(x'_n), y') < \varepsilon$ for $n \geq N$. Then $\sigma(y, y') \leq \sigma(y, f(x_n)) + \sigma(f(x_n), f(x'_n)) + \sigma(f(x'_n), y') < 3\varepsilon = 3\sigma(y, y')/4$. Contradiction. Hence $\langle f(x_n) \rangle$ and $\langle f(x'_n) \rangle$ converge to the same point. By defining $y = g(x)$, we get a function on $\bar{E}$ extending f.

21c. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\sigma(f(x), f(x')) < \varepsilon/3$ for $x, x' \in E$. Suppose $\rho(\bar{x}, \bar{x}') < \delta/3$ with $\bar{x}, \bar{x}' \in \bar{E}$. Let $\langle x_n \rangle$ be a sequence in E converging to $\bar{x}$ and let $\langle x'_n \rangle$ be a sequence in E converging to $\bar{x}'$. There exists N such that $\rho(x_n, \bar{x}) < \delta/3$ and $\rho(x'_n, \bar{x}') < \delta/3$ for $n \geq N$. Then $\rho(x_n, x'_n) < \delta$ for $n \geq N$ so $\sigma(f(x_n), f(x'_n)) < \varepsilon/3$ for $n \geq N$. Also, $\langle f(x_n) \rangle$ converges to some $y = g(\bar{x}) \in Y$ and $\langle f(x'_n) \rangle$ converges to some $y' = g(\bar{x}') \in Y$. We may assume $\sigma(f(x_n), g(\bar{x})) < \varepsilon/3$ and $\sigma(f(x'_n), g(\bar{x}')) < \varepsilon/3$ for $n \geq N$. Thus $\sigma(g(\bar{x}), g(\bar{x}')) < \varepsilon$. Hence g is uniformly continuous on $\bar{E}$.

21d. Let h be a continuous function from $\bar{E}$ to Y that agrees with f on E. Let $x \in \bar{E}$ and let $\langle x_n \rangle$ be a sequence in E converging to x. Then $\langle h(x_n) \rangle$ converges to $h(x)$ and $\langle g(x_n) \rangle$ converges to $g(x)$. Since $h(x_n) = g(x_n)$ for all n, $g(x) = h(x)$. Hence $h \equiv g$.

22a. Given $\varepsilon > 0$, let $\delta = \varepsilon/n$. When $\rho^+(x, y) < \delta$, $\rho^*(x, y) < n\delta = \varepsilon$. When $\rho^*(x, y) < \delta$, $\rho^+(x, y) < \delta < \varepsilon$. Thus ρ^+ and ρ^* are uniformly equivalent metrics. When $\rho^+(x, y) < \delta$, $\rho(x, y) < (n\delta^2)^{1/2} = (\varepsilon^2/n)^{1/2} < \varepsilon$. When $\rho(x, y) < \delta$, $(\rho^+(x, y))^2 < \delta^2$ so $\rho^+(x, y) < \delta < \varepsilon$. Thus ρ^+ and ρ are uniformly equivalent metrics. There exists $\delta' > 0$ with $\delta' < \varepsilon/sqrtn$ such that $\rho^*(x, y) < \delta'$ implies $\rho^+(x, y) < \delta$. Then when $\rho^*(x, y) < \delta'$, $\rho(x, y) < \varepsilon$. When $\rho(x, y) < \delta'$, $\rho^*(x, y) \leq \rho(x, y)\sqrt{n} < \varepsilon$. Thus ρ and ρ^* are uniformly equivalent metrics.

***22b.** Define $\rho'(x, y) = |x_1^3 - y_1^3| + \sum_{i=2}^n |x_i - y_i|$. Then ρ' is a metric on the set of n-tuples of real numbers. Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, choose $\delta < \min(1, |x_1|, \varepsilon/3n|x_1 + 1|^2, \varepsilon/3n|x_1 - 1|^2, \varepsilon/n, 3\varepsilon|x_1 - 1|^2/n, 3\varepsilon|x_1 + 1|^2/n)$. When $\rho^*(x, y) < \delta$, $|x_i - y_i| < \delta$ for each i and $|x_1^3 - y_1^3| = |3\xi^2||x_1 - y_1|$ for some $\xi \in (x_1 - \delta, x_1 + \delta)$. Thus $|x_1^3 - y_1^3| < 3\max(|x_1 + 1|, |x_1 - 1|)^2\delta < \varepsilon/n$ and $\rho'(x, y) < \varepsilon$. When $\rho'(x, y) < \delta$, $|x_i - y_i| < \varepsilon/n$ for $i = 2, \ldots, n$ and $|x_1^3 - y_1^3| < \delta$. Then $|x_1 - y_1| = |x_1^3 - y_1^3|/|3\xi^2| < \varepsilon/n$. Thus $\rho^*(x, y) < \varepsilon$. Hence ρ' and ρ^* are equivalent metrics and since ρ^* and ρ are equivalent metrics, ρ' and ρ are equivalent metrics.

However ρ and ρ' are not uniformly equivalent metrics. Let $\varepsilon = 1$. Given $\delta > 0$, choose n large enough so that $3n^2\delta > 1$. Let $x = \langle n, 0, \ldots, 0 \rangle$ and let $y = \langle n + \delta, 0, \ldots, 0 \rangle$. Then $\rho(x, y) = \delta$ and $\rho'(x, y) = (n + \delta)^3 - n^3 > 3n^2\delta > 1$.

22c. Note that $\rho(x, y) = \frac{\sigma(x,y)}{1-\sigma(x,y)} = h(\sigma(x, y))$ where h is the function in Q8. Since h is continuous at 0, given $\varepsilon > 0$, there exists $\delta' > 0$ such that $h(\sigma(x, y)) < \varepsilon$ when $\sigma(x, y) < \delta'$. Now given $\varepsilon > 0$, let $\delta < \min(\delta', \varepsilon)$. When $\rho(x, y) < \delta$, $\sigma(x, y) < \rho(x, y) < \delta < \varepsilon$. When $\sigma(x, y) < \delta$, $\rho(x, y) = h(\sigma(x, y)) < \varepsilon$. Hence σ and ρ are uniformly equivalent metrics for X. (c.f. Q11a)

23a. $[0, \infty)$ with the usual metric ρ is an unbounded metric space. By Q22c, the metric $\sigma = \rho/(1 + \rho)$ is uniformly equivalent to ρ so there is a uniform homeomorphism between $([0, \infty), \rho)$ and $([0, \infty), \sigma)$. By Q11a, $([0, \infty), \sigma)$ is a bounded metric space. Hence boundedness is not a uniform property.

23b. Let (X, ρ) and (Y, σ) be metric spaces with X totally bounded. Let $f : (X, \rho) \to (Y, \sigma)$ be a uniform homeomorphism. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\sigma(f(x), f(x')) < \varepsilon$. There exist finitely many balls $S_{x_n,\delta}$ that cover X. i.e. $X = \bigcup_{n=1}^k S_{x_n,\delta}$. Take $y \in Y$. Then $y = f(x)$ for some $x \in X$. Now $x \in S_{x_n,\delta}$ for some n so $\rho(x, x_n) < \delta$ and $\sigma(f(x), f(x_n)) < \varepsilon$. Hence $Y = \bigcup_{n=1}^k S_{f(x_n),\varepsilon}$ so Y is totally bounded. Thus total boundedness is a uniform property.

23c. By Q8, the function $h(x) = x/(1 - x)$ is a homeomorphism between $[0, 1)$ and $[0, \infty)$. Let $\varepsilon > 0$ be given. Choose N such that $N > 2/\varepsilon$ and let $x_n = (n - 1)/N$ for $n = 1, \ldots, N$. The intervals $(x_n - 1/N, x_n + 1/N) \cap [0, 1)$ are balls of radius ε that cover $[0, 1)$. Thus $[0, 1)$ is totally bounded. Suppose $[0, \infty)$ is totally bounded. Then there are a finite number of balls of radius 1 that cover $[0, \infty)$, say $[0, \infty) = \bigcup_{n=1}^k S_{x_n,1}$. We may assume that $x_1, \ldots, x_k$ are arranged in increasing order. But then $x_k + 2$ is not in any of the balls $S_{x_n,1}$. Contradiction. Hence $[0, \infty)$ is not totally bounded. Thus total boundedness is not a topological property.

23d. Let (X, ρ) be a totally bounded metric space. For each n, there are a finite number of balls of radius $1/n$ that cover X. Let S_n be the set of the centres of these balls. Then each S_n is a finite set and $S = \bigcup S_n$ is a countable set. Given $\varepsilon > 0$, choose N such that $N > 1/\varepsilon$. For any $x \in X$, $\rho(x, x') < 1/N < \varepsilon$ for some $x' \in S_N \subset S$. Thus S is a dense subset of X. Hence X is separable.

24a. Let $\langle X_k, \rho_k \rangle$ be a sequence of metric spaces and define their direct product $Z = \prod_{k=1}^{\infty} X_k$. Define $\tau(x, y) = \sum_{k=1}^{\infty} 2^{-k} \rho_k^*(x_k, y_k)$ where $\rho_k^* = \rho_k/(1 + \rho_k)$. Then $\tau(x, y) \geq 0$ since $\rho_k^*(x_k, y_k) \geq 0$ for all k. Also, $\tau(x, y) = 0$ if and only if $\rho_k^*(x_k, y_k) = 0$ for all k if and only if $x_k = y_k$ for all k if and only if $x = y$. Furthermore, $\tau(x, y) = \tau(y, x)$ since $\rho_k^*(x_k, y_k) = \rho_k^*(y_k, x_k)$ for all k. Now for each n, $\tau(x, y) = \sum_{k=1}^{n} 2^{-k} \rho_k^*(x, y) \leq \sum_{k=1}^{n} 2^{-k} \rho_k^*(x, z) + \sum_{k=1}^{n} 2^{-k} \rho_k^*(z, y) \leq \sum_{k=1}^{\infty} 2^{-k} \rho_k^*(x, z) + \sum_{k=1}^{\infty} 2^{-k} \rho_k^*(z, y) = \tau(x, z) + \tau(z, y)$. Hence $\tau(x, y) \leq \tau(x, z) + \tau(z, y)$. Thus τ is a metric on Z.

Suppose a sequence $\langle x^{(n)} \rangle$ in Z converges to $x \in Z$. Given $\varepsilon > 0$, let $\varepsilon' = \min(2^{-k-1}\varepsilon, 2^{-k-2})$. There exists N such that $\tau(x^{(n)}, x) < \varepsilon'$ for $n \geq N$. i.e. $\sum_{k=1}^{\infty} 2^{-k} \rho_k^*(x_k^{(n)}, x_k) < \varepsilon'$ for $n \geq N$. Then $\rho_{k_0}(x_{k_0}^{(n)}, x_{k_0}) < 2^{k_0}\varepsilon'(1 + \rho_{k_0}(x_{k_0}^{(n)}, x_{k_0}))$ for $n \geq N$ so $(1 - 2^{k_0}\varepsilon')\rho_{k_0}(x_{k_0}^{(n)}, x_{k_0}) < 2^{k_0}\varepsilon'$ for $n \geq N$. Since $2^{k_0}\varepsilon' = \min(\varepsilon/2, 1/4)$, we have $3\rho_{k_0}(x_{k_0}^{(n)}, x_{k_0})/4 < 2^{k_0}\varepsilon' < \varepsilon/2$ for $n \geq N$. Hence $\rho_{k_0}(x_{k_0}^{(n)}, x_{k_0}) < \varepsilon$ for $n \geq N$ and $\langle x_k^{(n)} \rangle$ converges to x_k for each k.

Conversely, suppose $\langle x_k^{(n)} \rangle$ converges to x_k for each k. Then given $\varepsilon > 0$, there exists N such that $\sum_{k=N+1}^{\infty} 2^{-k} \rho_k^*(x_k, x_k) < \varepsilon/2$. For $k = 1, \ldots, N$, there exists M such that $\rho_k(x_k^{(n)}, x_k) < \varepsilon/2$ for $n \geq M$. Thus $\tau(x^{(n)}, x) = \sum_{k=1}^{N} 2^{-k} \rho_k^*(x_k, x_k) + \sum_{k=N+1}^{\infty} 2^{-k} \rho_k^*(x_k, x_k) < \sum_{k=1}^{N} \varepsilon/2^{k+1} + \varepsilon/2 < \varepsilon$ for $n \geq M$. Hence $\langle x^{(n)} \rangle$ converges to x.

24b. Suppose each (X_k, ρ_k) is complete. Let $\langle x^{(n)} \rangle$ be a Cauchy sequence in (Z, τ). Then for each k, $\langle x_k^{(n)} \rangle$ is a Cauchy sequence in (X_k, ρ_k) so it converges to $x_k \in X_k$. By part (a), $\langle x^{(n)} \rangle$ converges to $\langle x_k \rangle \in Z$. Hence (Z, τ) is complete.

24c. Suppose that for each k, the spaces (X_k, ρ_k) and (Y_k, σ_k) are homeomorphic with $f_k : X_k \to Y_k$ a homeomorphism. Define $f : \prod_{k=1}^{\infty} X_k \to \prod_{k=1}^{\infty} Y_k$ by $f(\langle x_k \rangle) = \langle f(x_k) \rangle$. Then f is bijective since each f_k is. Also, note that f is continuous if and only if $p_{Y_k} \circ f$ is continuous for each k, where $p_{Y_k} : \prod_{k=1}^{\infty} \to Y_k$ is the projection map. Now $p_{Y_k} \circ f = f_k \circ p_{X_k}$ so it is continuous. Thus f is continuous. Similarly, f^{-1} is continuous since $p_{X_k} \circ f^{-1} = f_k^{-1} \circ p_{Y_k}$ is continuous for each k. Hence f is a homeomorphism between $\prod_{k=1}^{\infty} X_k$ and $\prod_{k=1}^{\infty} Y_k$.

24d. If for each k, the spaces (X_k, ρ_k) and (Y_k, σ_k) are uniformly homeomorphic, then by a similar argument as part (c), the spaces $\prod_{k=1}^{\infty} X_k$ and $\prod_{k=1}^{\infty} Y_k$ are uniformly homeomorphic.

7.6 Subspaces

25. Let A be a complete subset of a metric space X. Let $x \in \bar{A}$. Then by Q14, there is a sequence from A that converges to x. Since A is complete, $x \in A$. Thus A is closed. Now suppose B is a closed subset of a complete metric space Y. Let $\langle y_n \rangle$ be a Cauchy sequence in B. Then it is a Cauchy sequence in Y so it converges to some $y \in Y$. Now $y \in \bar{B} = B$ so B is complete.

***26.** Let O be an open subset of a complete metric space (X, ρ). Let $\varphi(x) = \rho(x, O^c)$ for each $x \in O$. Then $S = \{\langle x, y \rangle : x \in O, y = \varphi(x)\}$ is closed in $X \times \mathbb{R}$. Since $X \times \mathbb{R}$ is complete, S is complete. Consider $f : (O, \rho) \to (S, \rho_\infty)$ with $f(x) = \langle x, \varphi(x) \rangle$. Since ρ_∞ is uniformly equivalent to the usual product metric by Q19, S is complete under the metric ρ_∞. Now f is bijective. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon/2$ when $\rho(x, y) < \delta$. Let $\delta' = \min(\delta, \varepsilon/2)$. When $\rho(x, y) < \delta'$, $\rho_\infty(f(x), f(y)) = \rho(x, y) + |\varphi(x) - \varphi(y)| < \varepsilon$. When $\rho_\infty(\langle x, \varphi(x) \rangle, \langle y, \varphi(y) \rangle) < \delta'$, $\rho(x, y) < \delta' < \varepsilon$. Thus f is a uniform homeomorphism and (O, ρ) is complete. Let $\sigma = \rho/(1 + \rho)$. By Q22c, σ is uniformly equivalent to ρ. Hence σ is a bounded metric for which (O, σ) is a complete metric space.

7.7 Compact metric spaces

27. Let X be a metric space, K a compact subset and F a closed subset. Consider the function $f(x) = \rho(x, F) = \inf_{y \in F} \rho(x, y)$. By Q9b, $\{x : \rho(x, F) = 0\} = \bar{F} = F$. Thus if $F \cap K = \emptyset$, then $\rho(x, F) > 0$ for all $x \in K$. The function $f|_K$ is continuous on a compact set so it attains a minimum $\delta > 0$. Thus $\rho(x, y) > \delta$ for all $x \in K$ and all $y \in F$. Conversely, if $F \cap K \neq \emptyset$, then there exists $x \in K$ and $y \in F$ such that $\rho(x, y) = 0$ so $\rho(F, K) = 0$.

28a. Let X be a totally bounded metric space and let $f : X \to Y$ be a uniformly continuous map onto Y. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\sigma(f(x), f(x')) < \varepsilon$. There exist finitely many balls $\{B_{x_n,\delta}\}_{n=2}^k$ that cover X. Take $y \in Y$. Then $y = f(x)$ for some $x \in X$. Now $x \in B_{x_n,\delta}$ for some n so $\rho(x, x_n) < \delta$ and $\sigma(f(x), f(x_n)) < \varepsilon$. Hence the balls $\{B_{f(x_n),\varepsilon}\}_{n=1}^k$ cover Y and Y is totally bounded.

28b. The function $h(x) = x/(1-x)$ is a continuous map from $[0, 1)$ onto $[0, \infty)$. $[0, 1)$ is totally bounded but $[0, \infty)$ is not.

29a. We may assume $X \notin U$. Set $\varphi(x) = \sup\{r : \exists O \in U \text{ with } B_{x,r} \subset O\}$. For each $x \in X$, there exists $O \in U$ such that $x \in O$. Since O is open, there exists r such that $B_{x,r} \subset O$. Thus $\varphi(x) > 0$. Since X is compact, it is bounded so $\varphi(x) < \infty$.

29b. Suppose $B_{x,r} \subset O$ for some $O \in U$. If $0 < r' < r - \rho(x, y)$, then $B_{y,r'} \subset B_{x,r} \subset O$. If $\varphi(y) < \varphi(x) - \rho(x, y)$, then there exists $r > \varphi(y) + \rho(x, y)$ such that $B_{x,r} \subset O$ for some $O \in U$. Now $\varphi(y) < r - \rho(x, y)$ so taking $r' = \varphi(y) + (r - \rho(x, y) - \varphi(y))/2$, we have $\varphi(y) < r' < r - \rho(x, y)$ and $B_{y,r'} \subset O$. Contradiction. Hence $\varphi(y) \geq \varphi(x) - \rho(x, y)$.

29c. Given $\varepsilon > 0$, let $\delta = \varepsilon$. When $\rho(x, y) < \delta$, $|\varphi(x) - \varphi(y)| \leq \rho(x, y) < \varepsilon$ by part (b). Thus φ is continuous on X.

29d. If X is sequentially compact, φ attains its minimum on X. Let $\varepsilon = \inf \varphi$. Then $\varepsilon > 0$ since $\varphi(x) > 0$ for all x.

29e. Let ε be as defined in part (d). For any $x \in X$ and $\delta < \varepsilon$, $\delta < \varphi(x)$ so there exists $r > \delta$ such that $B_{x,r} \subset O$ for some $O \in U$. Then $B_{x,\delta} \subset B_{x,r} \subset O$.

***30a.** Let $Z = \prod_{k=1}^{\infty} X_k$. Suppose each X_k is totally bounded. Given $\varepsilon > 0$, choose N such that $2^N > 2/\varepsilon$. For $k > N$, pick $p_k \in X_k$. For each $k \leq N$, there exists $A_k = \{x_1^{(k)}, \dots, x_{M_k}^{(k)}\}$ such that for any $x \in X_k$, there exists $x_j^{(k)} \in A_k$ with $\rho_k(x, x_j^{(k)}) < \varepsilon/2$. Let $A = \{\langle x_n \rangle : x_k \in A_k \text{ for } k \leq N, x_k = p_k \text{ for } k > N\}$. Then $|A| = M_1 \cdots M_N$. If $\langle x_n \rangle \in Z$, for each $k \leq N$, there exists $x_j^{(k)} \in A_k$ such that $\rho_k(x_k, x_j^{(k)}) < \varepsilon/2$. Let $y_k = x_j^{(k)}$ for $k \leq N$ and let $y_k = p_k$ for $k > N$. Then $\langle y_n \rangle \in A$ and $\tau(x, y) = \sum_{k=1}^{n} 2^{-k} \frac{\rho_k(x_k, y_k)}{1+\rho_k(x_k, y_k)} < \sum_{k=1}^{N} 2^{-k} \frac{\varepsilon}{2} + \sum_{k=N+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon$.

30b. Suppose each X_k is compact. Then each X_k is complete and totally bounded. By part (a), Z is totally bounded and by Q24b, Z is complete. Hence Z is compact.

7.8 Baire category

31a. Suppose a closed set F is nowhere dense. Then F^c is dense. Thus any open set contains a point in F^c so any open set cannot be contained in F. Conversely, suppose a closed set contains no open set. Then any open set contains a point in F^c so F^c is dense and F is nowhere dense.

31b. Suppose E is nowhere dense and let O be a nonempty open set. Then O contains a point x in $\bar{E}^c$. Now $O \setminus \bar{E}$ is open so there is a ball centred at x and contained in $O \setminus \bar{E} \subset O \setminus E$. Conversely, suppose that for any nonempty open set O there is a ball contained in $O \setminus E$. Let the centre of the ball be x. If $x \in \bar{E}$, then the ball contains a point $y \in E$. Contradiction. Thus $x \in \bar{E}^c$. Hence any nonempty open set contains a point in $\bar{E}^c$ and E is nowhere dense.

32a. Suppose that E is of the first category and $A \subset E$. Then $E = \bigcup E_n$ where each E_n is nowhere dense and $A = \bigcup(A \cap E_n)$. Now $(\overline{A \cap E_n})^c \supset (\bar{A} \cap \bar{E}_n)^c = \bar{A}^c \cup \bar{E}_n{}^c$ for each n. Since $\bar{E}_n{}^c$ is dense for each n, $(\overline{A \cap E_n})^c$ is dense for each n. Hence A is of the first category.

32b. Suppose that $\langle E_n \rangle$ is a sequence of sets of the first category. Then $E_n = \bigcup_k E_{n,k}$ for each n, where each $E_{n,k}$ is nowhere dense. Now $\bigcup E_n = \bigcup_n \bigcup_k E_{n,k} = \bigcup_{n,k} E_{n,k}$, which is a countable union of nowhere dense sets. Hence $\bigcup E_n$ is of the first category.

33a. By Q3.14b, the generalised Cantor set constructed from $[0, 1]$ by removing intervals of length $1/(n3^n)$ at the n-th step is closed, nowhere dense, and has Lebesgue measure $1 - 1/n$.

33b. By part (a), for each n, there is a nowhere dense closed set $F_n \subset [0, 1]$ with $mF_n = 1 - 1/n$. Let $E = \bigcup F_n \subset [0, 1]$. Then E is of the first category and $1 - 1/n \leq mE \leq 1$ for each n so $mE = 1$.

34. Suppose O is open and F is closed. If $\bar{O} \setminus O$ contains an open set O', then O' contains a point in O. Contradiction. Thus $\bar{O} \setminus O$ is a closed set containing no open sets. By Q31a, $\bar{O} \setminus O$ is nowhere dense. If

$F \setminus F^\circ$ contains an open set O'', then for any point $x \in O''$, there is a ball centred at x and contained in O'' so $x \in F^\circ$. Contradiction. Thus $F \setminus F^\circ$ is a closed set containing no open sets. By Q31a, $F \setminus F^\circ$ is nowhere dense.

If F is closed and of the first category in a complete metric space, then F° is of the first category. By the Baire category theorem, F^{circ} is empty. Thus $F = F \setminus F^\circ$ is nowhere dense.

35. Let E be a subset of a complete metric space. If E is residual, then E^c is of the first category so $E^c = \bigcup E_n$ where each E_n is nowhere dense. Thus $E = \bigcap E_n^c \supset \bigcap (\bar{E_n})^c$. Since each $(\bar{E_n})^c$ is dense and the metric space is complete, $\bigcap (\bar{E_n})^c$ is dense so E contains a dense G_δ. Conversely, suppose E contains a dense G_δ, say $E \supset \bigcap O_n$. Then $E^c \subset \bigcup O_n^c$. Since $\bigcap O_n$ is dense, each O_n is dense so O_n^c is nowhere dense and E^c is of the first category. Hence E is residual.

A subset E of a complete metric space is of the first category if and only if E^c is a residual subset of a complete metric space if and only if E^c contains a dense G_δ if and only if E is contained in an F_σ whose complement is dense.

36a. Let X be a complete metric space without isolated points. Suppose X has a countable number of points $x_1, x_2, \ldots$. Then $X = \bigcup \{x_n\}$ where each $\{x_n\}$ is nowhere dense and X is of the first category, contradicting the Baire category theorem. Hence X has an uncountable number of points.

36b. $[0,1]$ is complete since it is a closed subspace of the complete metric space $\mathbb{R}$. Also, $[0,1]$ has no isolated points. By part (a), $[0,1]$ is uncountable.

37a. Let E be a subset of a complete metric space. Suppose E^c is dense and F is a closed set contained in E. Then $F^c \supset E^c$ so F^c is dense and F is nowhere dense.

37b. Suppose E and E^c are both dense in a complete metric space X. Also suppose that E and E^c are both F_σ's, say $E = \bigcup F_n$ and $E^c = \bigcup F_n'$ where each F_n and F_n' is closed. By part (a), each F_n and F_n' is nowhere dense. Then $X = \bigcup F_n \cup \bigcup F_n'$ so X is of the first category, contradicting the Baire category theorem. Hence at most one of the sets E and E^c is an F_σ.

37c. The set of rational numbers in $[0,1]$ is an F_σ. Since the set of rational numbers in $[0,1]$ and the set of irrational numbers in $[0,1]$ are both dense in the complete metric space $[0,1]$, the set of irrational numbers in $[0,1]$ is not an F_σ by part (b). Hence its complement, the set of rational numbers in $[0,1]$, is not a G_δ.

37d. If f is a real-valued function on $[0,1]$ which is continuous on the rationals and discontinuous on the irrationals, then the set of points of continuity of f is a G_δ. i.e. the set of rationals in $[0,1]$ is a G_δ, contradicting part (c). Hence there is no such function.

***38a.** Let $C = C[0,1]$ and set $F_n = \{f : \exists x_0 \text{ with } 0 \le x_0 \le 1 - 1/n \text{ and } |f(x) - f(x_0)| \le n(x - x_0) \text{ for all } x, x_0 \le x < 1\}$. Suppose $||f_k - f|| \to 0$ where $f_k \in F_n$. For each k, there exists x_k with $0 \le x_k \le 1 - 1/n$ and $|f_k(x) - f_k(x_k)| \le n(x - x_k)$ for $x_k \le x < 1$. Since $x_k \in [0, 1-1/n]$ for all k and $[0, 1-1/n]$ is compact, we may assume that $\langle x_k \rangle$ converges to some $x_0 \in [0, 1-1/n]$. Then $|f(x) - f(x_0)| \le n(x - x_0)$ so $f \in F_n$.

***38b.** By Q2.47, for any $g \in C$ and any $\varepsilon > 0$, there exists a polygonal function φ such that $|g(x) - \varphi(x)| < \varepsilon/2$ for all $x \in [0,1]$. There also exists a polygonal function ψ whose right-hand derivative is everywhere greater than n in absolute value and $|\varphi(x) - \psi(x)| < \varepsilon/2$ for all $x \in [0,1]$. Then $\psi \in F_n^c$ and $|g(x) - \psi(x)| < \varepsilon$ for all $x \in [0,1]$. Hence F_n is nowhere dense.

38c. The set D of continuous functions which have a finite derivative on the right for at least one point of $[0,1]$ is the union of the F_n's so D is of the first category in C.

38d. Since D is of the first category in the complete metric space C, $D \ne C$ so there is a function in $C \setminus D$, that is, a nowhere differentiable continuous function on $[0,1]$.

39. Let $\mathfrak{F}$ be a family of real-valued continuous functions on a complete metric space X, and suppose that for each $x \in X$ there is a number M_x such that $|f(x)| \le M_x$ for all $f \in \mathfrak{F}$. For each m, let $E_{m,f} = \{x : |f(x)| \le m\}$, and let $E_m = \bigcap_{\mathfrak{F}} E_{m,f}$. Since each f is continuous, $E_{m,f}$ is closed and so E_m is closed. For each $x \in X$, there exists m such that $|f(x)| \le m$ for all $f \in \mathfrak{F}$. Hence $X = \bigcup E_m$. Then $O = \bigcup E_m^\circ \subset X$ is a dense open set and for each $x \in O$, $x \in E_m^\circ$ for some m so there is a neighbourhood U of x such that $U \subset E_m^\circ$. In particular, $\mathfrak{F}$ is uniformly bounded on U.

40a. Suppose that given $\varepsilon > 0$, there exist N and a neighbourhood U of x such that $\sigma(f_n(x'), f(x)) < \varepsilon$ for $n \ge N$ and all $x' \in U$. Let $\langle x_n \rangle$ be a sequence with $x = \lim x_n$. We may assume that $x_n \in U$ for all $n \geq N$ so $\sigma(f_n(x_n), f(x)) < \varepsilon$ for $n \geq N$ and $\langle f_n \rangle$ converges continuously to f at x. Conversely, suppose there exists $\varepsilon > 0$ such that for any N and any neighbourhood U of x, there exists $n \geq N$ and $x' \in U$ with $\sigma(f_n(x'), f(x)) \geq \varepsilon$. For each n, let $U_n = B_{x,1/n}$. There exists $n_1 \geq 1$ and $x_{n_1} \in U_1$ with $\sigma(f_{n_1}(x_{n_1}), f(x)) \geq \varepsilon$. Suppose $x_{n_1}, \ldots, x_{n_k}$ have been chosen. There exists $n_{k+1} \geq n_k$ and $x_{n_k} \in U_k$ with $\sigma(f_{n_{k+1}}(x_{n_{k+1}}), f(x)) \geq \varepsilon$. Then $x = \lim x_{n_k}$ by construction but $f(x) \neq \lim f_{n_k}(x_{n_k})$ so $\langle f_n \rangle$ does not converge continuously to f.

40b. Let $Z = \{1/n\} \cup \{0\}$. Define $g : X \times Z \to Y$ by $g(x, 1/n) = f_n(x)$ and $g(x, 0) = f(x)$. Suppose g is continuous at $\langle x_0, 0 \rangle$ in the product metric. Let $\langle x_n \rangle$ be a sequence with $x_0 = \lim x_n$. Then $\langle x_0, 0 \rangle = \lim \langle x_n, 1/n \rangle$ and $f(x) = g(x_0, 0) = \lim g(x_n, 1/n) = \lim f_n(x_n)$ so $\langle f_n \rangle$ converges continuously to f at x_0. Conversely, suppose $\langle f_n \rangle$ converges continuously to f at x_0. Let $\langle (x_n, z_n) \rangle$ be a sequence in $X \times Z$ converging to $\langle x_0, 0 \rangle$. Then $x_0 = \lim x_n$, $0 = \lim z_n$ and $f(x_0) = \lim f_n(x_n)$. i.e. $g(x_0, 0) = \lim g(x_n, 1/n)$. Since $0 = \lim z_n$, it follows that $g(x_0, 0) = \lim g(x_n, z_n)$ so g is continuous at $\langle x_0, 0 \rangle$.

40c. Let $\langle f_n \rangle$ converge continuously to f at x. By part (b), the function g is continuous at $\langle x, 0 \rangle$. If $\langle x_n \rangle$ is a sequence converging to x, then $f(x) = g(x, 0) = \lim g(x_n, 0) = \lim f(x_n)$. Hence f is continuous at x.

***40d.** Let $\langle f_n \rangle$ be a sequence of continuous maps. Suppose $\langle f_n \rangle$ converges continuously to f at x. By part (a), given $\varepsilon > 0$, there exists N and a neighbourhood U of x such that $\sigma(f_n(x'), f(x)) < \varepsilon/2$ for $n \geq N$ and $x' \in U$. By part (c), f is continuous at x so we may assume that $\sigma(f(x'), f(x)) < \varepsilon/2$ for $x' \in U$. Thus $\sigma(f_n(x'), f(x')) < \varepsilon$ for $n \geq N$ and $x' \in U$. Conversely, suppose that given $\varepsilon > 0$, there exist N and a neighbourhood U of x such that $\sigma(f_n(x'), f(x')) < \varepsilon/4$ for all $n \geq N$ and all $x' \in U$. In particular, $\sigma(f_N(x), f(x)) < \varepsilon/4$. Let $\langle x_k \rangle$ be a sequence with $x = \lim x_k$. We may assume that $x_k \in U$ for all k and $\sigma(f_N(x_k), f_N(x)) < \varepsilon/4$ for $k \geq N$. Then $\sigma(f(x_k), f(x)) \leq \sigma(f(x_k), f_N(x_k)) + \sigma(f_N(x_k), f_N(x)) + \sigma(f_N(x), f(x)) < 3\varepsilon/4$ for $k \geq N$. Thus $\sigma(f_k(x_k), f(x)) \leq \sigma(f_k(x_k), f(x_k)) + \sigma(f(x_k), f(x)) < \varepsilon$ for $k \geq N$ so $f(x) = \lim f_k(x_k)$ and $\langle f_n \rangle$ converges continuously to f at x.

40e. Let $\langle f_n \rangle$ be a sequence of continous maps. Suppose $\langle f_n \rangle$ converges continuously to f on X. For each $x \in X$ and each $\varepsilon > 0$, there exists N_x and a neighbourhood U_x of x such that $\sigma(f_n(x'), f(x')) < \varepsilon$ for each $n \geq N_x$ and $x' \in U_x$. Then $X = \bigcup U_x$ so for any compact subset $K \subset X$, $K \subset \bigcup_{i=1}^m U_{x_i}$ for some $x_1, \ldots, x_m$. Thus for $n \geq \max N_{x_i}$ and $x' \in K$, we have $\sigma(f_n(x'), f(x')) < \varepsilon$ so $\langle f_n \rangle$ converges uniformly to f on K. Conversely, suppose $\langle f_n \rangle$ converges uniformly to f on each compact subset of X. Then $\langle f_n \rangle$ converges uniformly to f on $\{x\}$ for each $x \in X$. Given $\varepsilon > 0$, there exists N such that $\sigma(f_n(x), f(x)) < \varepsilon/3$ for $n \geq N$. Since each f_n is continuous, f is also continuous. Then there exists a neighbourhood U of x such that $\sigma(f_n(x'), f_n(x)) < \varepsilon/3$ and $\sigma(f(x'), f(x)) < \varepsilon/3$ for $x' \in U$. Thus $\sigma(f_n(x'), f(x')) < \varepsilon$ for $n \geq N$ and $x' \in U$. Hence $\langle f_n \rangle$ converges continuously to f on X.

40f. Let X be a complete metric space and $\langle f_n \rangle$ a sequence of continuous maps of X into a metric space Y such that $f(x) = \lim f_n(x)$ for each $x \in X$. For $m, n \in \mathbb{N}$, define $F_{m,n} = \{x \in X : \sigma(f_k(x), f_l(x)) \leq 1/m \text{ for all } k, l \geq n\}$. Let x' be a point of closure of $F_{m,n}$. There is a sequence $\langle x_i \rangle$ in $F_{m,n}$ converging to x'. For any $\varepsilon > 0$ and $k, l \geq n$, there exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\sigma(f_k(x), f_k(x')) < \varepsilon$ and $\sigma(f_l(x), f_l(x')) < \varepsilon$. Then there exists N such that $\rho(x_i, x') < \delta$ for $i \geq N$. Thus $\sigma(f_k(x'), f_l(x')) \leq \sigma(f_k(x'), f_k(x_N)) + \sigma(f_k(x_N), f_l(x_N)) + \sigma(f_l(x_N), f_l(x')) < 2\varepsilon + 1/m$. Since ε is arbitrary, $\sigma(f_k(x'), f_l(x')) \leq 1/m$ and $x' \in F_{m,n}$ so $F_{m,n}$ is closed.

For any $x \in X$, there exists n such that $\sigma(f_k(x), f(x)) < 1/2m$ for $k \geq n$. Then for $k, l \geq n$, $\sigma(f_k(x), f_l(x)) \leq \sigma(f_k(x), f(x)) + \sigma(f(x), f_l(x)) < 1/m$ so $x \in F_{m,n}$. Hence $X = \bigcup_n F_{m,n}$. By Proposition 31, $O_m = \bigcup_n F_{m,n}^\circ$ is open and since X is complete, O_m is also dense.

40g. Let $x \in O_m$. Then $x \in F_{m,n}^\circ$ for some n. Thus there exists a neighbourhood U of x such that $U \subset F_{m,n}$. It follows that for any $k, l \geq n$ and any $x' \in U$, $\sigma(f_k(x'), f_l(x')) \leq 1/m$. Since $f(x') = \lim f_l(x')$, we have $\sigma(f_k(x'), f(x')) \leq 1/m$ for any $k \geq n$ and $x' \in U$.

40h. Let $E = \bigcap O_m$. Since each O_m is open and dense, E is a dense G_δ by Baire's theorem. If $x \in E$, then $x \in O_m$ for all m. Given $\varepsilon > 0$, choose $m > 1/\varepsilon$. There exists a neighbourhood U of x and an n such that $\sigma(f_k(x'), f(x')) \leq 1/m < \varepsilon$ for any $k \geq n$ and $x' \in U$.

40i. Let X be a complete metric space and $\langle f_n \rangle$ a sequence of continuous functions of X into a metric space Y. Suppose $\langle f_n \rangle$ converges pointwise to f. By parts (f), (g), (h) and (a), there exists a dense G_δ in X on which $\langle f_n \rangle$ converges continuously to f.

41a. Let (X, ρ) and (Y, σ) be complete metric spaces and $f : X \times Y \to Z$ be a mapping into a metric space (Z, τ) that is continuous in each variable. Fix $y_0 \in Y$. Set $F_{m,n} = \{x \in X : \tau[f(x,y), f(x,y_0)] \leq 1/m$ for all y with $\sigma(y, y_0) \leq 1/n\}$. Let x' be a point of closure of $F_{m,n}$. There is a sequence $\langle x_k \rangle$ in $F_{m,n}$ converging to x'. For each k and any y with $\sigma(y, y_0) \leq 1/n$, $\tau[f(x_k, y), f(x_k, y_0)] \leq 1/m$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, x') < \delta$ implies $\tau[f(x,y), f(x',y)] < \varepsilon$ and $\tau[f(x,y_0), f(x',y_0)] < \varepsilon$. Then there exists N such that $\rho(x_k, x') < \delta$ for $k \geq N$. Thus $\tau[f(x',y), f(x',y_0)] \leq \tau[f(x',y), f(x_N,y)] + \tau[f(x_N,y), f(x_N,y_0)] + \tau[f(x_N,y_0), f(x',y_0)] < 2\varepsilon + 1/m$. Since ε is arbitrary, $\tau[f(x',y), f(x',y_0)] \leq 1/m$ and $x' \in F_{m,n}$ so $F_{m,n}$ is closed.

For any $x \in X$ and m, there exists $\delta > 0$ such that $\sigma(y, y_0) < \delta$ implies $\tau[f(x,y), f(x,y_0)] < 1/m$. Choose $n > 1/\delta$. Then for any y with $\sigma(y, y_0) < 1/n$, $\tau[f(x,y), f(x,y_0)] < 1/m$ so $x \in F_{m,n}$. Hence $X = \bigcup_n F_{m,n}$.

41b. Let $O_m = \bigcup_n F^\circ_{m,n}$. By Proposition 31, O_m is open and since X is complete, O_m is also dense. Let $x \in O_m$. Then $x \in F^\circ_{m,n}$ for some n. Thus there exists a neighbourhood U' of x such that $U' \subset F_{m,n}$. For any y with $\sigma(y, y_0) \leq 1/n$ and any $x' \in U'$, $\tau[f(x',y), f(x',y_0)] \leq 1/m$. We may assume that $\tau[f(x',y_0), f(x,y_0)] \leq 1/m$ for $x' \in U'$ by continuity. Then $\tau[f(x',y), f(x,y_0)] \leq 2/m$ for $x' \in U'$. Hence there is a neighbourhood U of $\langle x, y_0 \rangle$ in $X \times Y$ such that $\tau[f(p), f(x,y_0)] \leq 2/m$ for all $p \in U$.

41c. Let $G = \bigcap O_m$. Since each O_m is open and dense, G is a dense G_δ by Baire's theorem. If $x \in G$, then $x \in O_m$ for each m. Given $\varepsilon > 0$, choose $m > 2/\varepsilon$. There exists a neighbourhood U of $\langle x, y_0 \rangle$ such that $\tau[f(p), f(x,y_0)] \leq 2/m < \varepsilon$ for $p \in U$. Hence f is continuous at $\langle x, y_0 \rangle$ for each $x \in G$.

41d. Let $E \subset X \times Y$ be the set of points at which f is continuous. If f is continuous at $z \in X \times Y$, then for each n, there exists $\delta_{n,z} > 0$ such that $f[B_{z,\delta_{n,z}}] \subset B_{f(z),1/n}$. Let $E_n = \bigcup_{z \in S} B_{z,\delta_{z,1/n}/2}$. Then $E = \bigcap_n E_n$ (c.f. Q2.7.53) and E is a G_δ set. Take $\langle x_0, y_0 \rangle \in X \times Y$ and let $\varepsilon > 0$ be given. By part (c), there exists a dense G_δ set $G \subset X$ such that f is continuous at $\langle x, y_0 \rangle$ for each $x \in G$. Since G is dense, there exists $x_1 \in G$ such that $\rho(x_0, x_1) < \varepsilon$. Then $\langle x_1, y_0 \rangle \in E$ and $\tau'(\langle x_1, y_0 \rangle, \langle x_0, y_0 \rangle) = \rho(x_1, x_0) < \varepsilon$. Thus E is dense in $X \times Y$.

41e. Given a finite product $X_1 \times \cdots \times X_p$ of complete metric spaces, we may regard the product as $(X_1 \times \cdots \times X_{p-1}) \times X_n$ where $X_1 \times \cdots \times X_{p-1}$ is a complete metric space. Let $f : X_1 \times \cdots \times X_p \to Z$ be a mapping into a metric space Z that is continuous in each variable. By a similar argument as in part (d), there exists a dense $G_\delta \subset X_1 \times \cdots \times X_p$ on which f is continuous.

42a. Let X and Y be complete metric spaces. Also let $G \subset X$ and $H \subset Y$ be dense G_δ's. Then $G = \bigcap G_n$ and $H = \bigcap H_n$ where each G_n and H_n is open. Thus $G \times H = \bigcap G_n \times \bigcap H_n = \bigcap (G_n \times H_n)$ where each $G_n \times H_n$ is open in $X \times Y$. Hence $G \times H$ is a G_δ in $X \times Y$. Given $\varepsilon > 0$ and $\langle x, y \rangle \in X \times Y$, there exist $x_0 \in G$ and $y_0 \in H$ such that $\rho(x, x_0) < \varepsilon/2$ and $\sigma(y, y_0) < \varepsilon/2$. Then $\tau(\langle x, y \rangle, \langle x_0, y_0 \rangle) < \varepsilon$. Hence $G \times H$ is dense in $X \times Y$.

***42b.**

42c. Let E be a dense G_δ in $X \times Y$. Then $E = \bigcap O_n$ where each O_n is a dense open set in $X \times Y$. By part (b), for each n, there exists a dense G_δ set $G_n \subset X$ such that $\{y : \langle x, y \rangle \in O_n\}$ is a dense open subset of Y for each $x \in G_n$. Let $G = \bigcap G_n$. Then G is still a countable intersection of dense open sets in X. Thus G is a dense G_δ set in X such that $\{y : \langle x, y \rangle \in E\} = \bigcap \{y : \langle x, y \rangle \in O_n\}$ is a dense G_δ for each $x \in G$.

***42d.**

***43.** Let (X, ρ) be a complete metric space and $f : X \to \mathbb{R}$ be an upper semicontinuous function. Let E be the set of points at which f is continuous. Then E is a G_δ. Suppose f is discontinuous at x. Then f is not lower semicontinuous at x so $f(x) > \varliminf_{y \to x} f(y)$. There exists $q \in \mathbb{Q}$ such that $f(x) \geq q > \varliminf_{y \to x} f(y)$. Let $F_q = \{x : f(x) \geq q\}$. Then $x \in F_q \setminus F_q^\circ$. Conversely, if $x \in F_q \setminus F_q^\circ$ for some $q \in \mathbb{Q}$, then $f(x) \geq q$ and for any $\delta > 0$, there exists y with $\rho(x,y) < \delta$ and $f(y) < q$ so $f(x) > \varliminf_{y \to x} f(y)$. Thus $E^c = \bigcup_{q \in \mathbb{Q}} F_q \setminus F_q^\circ$. Since each $F_q \setminus F_q^\circ$ is nowhere dense, E^c is of the first category and E is residual. Hence E is a dense G_δ in X

7.9 Absolute G_δ's

44. Let $\sigma = \sum 2^{-n} \sigma_n$. Then $\sigma(x,y) = \sum 2^{-n} \sigma_n(x,y) \leq \sum 2^{-n} = 1$ for all $x, y \in E$. Since $\sigma_n \geq 0$ for all n, $\sigma \geq 0$. Also, $\sigma(x,y) = 0$ if and only if $\sigma_n(x,y) = 0$ for all n if and only if $x = y$. Since

$\sigma_n(x,y) = \sigma_n(y,x)$ for all n, $\sigma(x,y) = \sigma(y,x)$. Also, $\sigma(x,y) = \sum 2^{-n}\sigma_n(x,y) \leq \sum 2^{-n}\sigma_n(x,z) + \sum 2^{-n}\sigma_n(z,y) = \sigma(x,z) + \sigma(z,y)$. Thus σ is a metric on E.

Let $\varepsilon > 0$ and fix $x \in E$. For each n, there exists $\delta_n > 0$ such that $\sigma_n(x,y) < \delta_n$ implies $\rho(x,y) < \varepsilon/2$ and $\rho(x,y) < \delta_n$ implies $\sigma_n(x,y) < \varepsilon/2$. There exists N such that $\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon/2$. Let $\delta = \min(\delta_1/2, \dots, \delta_N/2^N)$. If $\rho(x,y) < \delta$, then $\sigma_n(x,y) < \varepsilon/2$ for $n = 1, \dots, N$ so $\sum_{n=1}^{N} 2^{-n}\sigma_n(x,y) < \varepsilon/2$ and $\sigma(x,y) < \varepsilon$. If $\sigma(x,y) < \delta$, then $\sigma_1(x,y) < 2\delta \leq \delta_1$ so $\rho(x,y) < \varepsilon$. Hence σ is equivalent to ρ on E.

If each σ_n is uniformly equivalent to ρ, then σ is uniformly equivalent to ρ.

45. Let $A \subset B \subset \bar{A}$ be subsets of a metric space and let g and h be continuous maps of B into a metric space X. Suppose $g(u) = h(u)$ for all $u \in A$. Let $u \in B$. Then $u \in \bar{A}$ so there is a sequence $\langle u_n \rangle$ in A converging to u. Since g and h are continuous on B, $\langle g(u_n) \rangle$ and $\langle h(u_n) \rangle$ converge to $f(u)$ and $g(u)$ respectively. But $g(u_n) = h(u_n)$ for all n so $g(u) = h(u)$. Hence $g \equiv h$.

***46a.**

46b. Starting from E, let $I_1, I_2, \dots$ be disjoint open intervals satisfying the conditions of part (a) with $\varepsilon = 1/2$. For $E_{I_j} = E \cap I_j$, repeat the process with $\varepsilon = 1/4$, getting intervals $I_{j,1}, I_{j,2}, \dots$. Continuing, at the n-th stage getting intervals $I_{j,k,\dots,l}$ with $\varepsilon = 2^{-n}$. Given $x \in E$, there is a unique integer k_1 such that $x \in I_{k_1}$. Then $x \in E_{I_{k_1}}$ and there is a unique integer k_2 such that $x \in I_{k_1,k_2}$ and so on. Thus there is a unique sequence of integers $k_1, k_2, \dots$ such that for each n we have $x \in I_{k_1,\dots,k_n}$.

46c. Given a sequence of integers $k_1, k_2, \dots$, suppose there are $x, y \in E$ such that $x, y \in I_{k_1,\dots,k_n}$ for all n. By construction, the diameter of $I_{k_1,\dots,k_n}$ is less than 2^{-n} for each n. Thus $\sigma(x,y) < 2^{-n}$ for all n so $\sigma(x,y) = 0$ and $x = y$.

46d. Let $\mathbb{N}^\omega$ be the space of infinite sequences of integers and make $\mathbb{N}$ into a metric space by setting $\rho(i,j) = \delta_{ij}$. Let τ be the product metric on $\mathbb{N}^\omega$. Given $\varepsilon > 0$, choose N such that $2^{-N} < \varepsilon$. Let $\delta < 2^{-N}$. When $\sigma(x,y) < \delta$, $k_n(x) = k_n(y)$ for $n = 1, \dots, N$ so $\tau(\langle k_n(x) \rangle, \langle k_n(y) \rangle) \leq \sum_{i=N+1}^{\infty} 2^{-i}\rho^*(k_n(x), k_n(y)) \leq 2^{-N} < \varepsilon$. When $\tau(\langle k_n(x) \rangle, \langle k_n(y) \rangle) < \delta$, $k_n(x) = k_n(y)$ for $n = 1, \dots, N$. Then $x, y \in I_{k_1(x),\dots,k_N(x)}$ so $\sigma(x,y) < 2^{-N} < \varepsilon$. Hence the correspondence between $\mathbb{N}^\omega$ and E given by parts (b) and (c) is a uniform homeomorphism between $(\mathbb{N}^\omega, \tau)$ and (E, σ).

***46e.** By parts (a)-(d), any dense G_δ E in $(0,1)$ whose complement is dense is uniformly homeomorphic to $\mathbb{N}^\omega$, which is in turn homeomorphic to the set of irrationals in $(0,1)$ by the continued fraction expansion. Hence E is homeomorphic to the set of irrationals in $(0,1)$.

7.10 The Ascoli-Arzelá Theorem

47. Let X be a metric space. Let $\langle f_n \rangle$ be a sequence of continuous functions from X to a metric space Y which converge to a function f uniformly on each compact subset K of X. Let $x \in X$ and let $\langle x_k \rangle$ be a sequence in X converging to x. Then $K = \{x_k\}_{k=1}^{\infty} \cup \{x\}$ is a compact subset of X. Given $\varepsilon > 0$, there exists N such that $\sigma(f_N(x'), f(x')) < \varepsilon/3$ for any $x' \in K$. Also, there exists N' such that $\sigma(f_N(x_k), f_N(x)) < \varepsilon/3$ for $k \geq N'$. Thus $\sigma(f(x_k), f(x)) \leq \sigma(f(x_k), f_N(x_k)) + \sigma(f_N(x_k), f_N(x)) + \sigma(f_N(x), f(x)) < \varepsilon$ for $k \geq N'$. Hence f is continuous on X.

48a. Let X be a separable, locally compact metric space, and (Y, σ) any metric space. Let $\{x_n\}$ be a countable dense subset of X. For each n, there is an open set U_n such that $x_n \in U_n$ and $\overline{U_n}$ is compact. Then $X = \overline{\{x_n\}} \subset \bigcup \overline{U_n}$ so $X = \bigcup \overline{U_n}$. For each n and any $x \in \overline{U_n}$, there is an open set V_x containing x with $\overline{V_x}$ compact. Then $\overline{U_n} \subset \bigcup_{x \in \overline{U_n}} V_x$ and since $\overline{U_n}$ is compact, there is a finite number of V_x's covering $\overline{U_n}$. Let O_n be the union of the finite number of V_x's. Then $X = \bigcup O_n$.

48b. Let $\sigma^*(f,g) = \sum 2^{-n}\sigma_n^*(f,g)$ where $\sigma_n^*(f,g) = \sup_{\overline{O_n}} \frac{\sigma(f(x),g(x))}{1+\sigma(f(x),g(x))}$. Since $\sigma_n^*(f,g) \leq 1$ for all n, $\sigma^*(f,g) \leq 1 < \infty$ and since $\sigma_n^*(f,g) \geq 0$ for all n, $\sigma^*(f,g) \geq 0$. Since $\sigma_n^*(f,g) = \sigma_n^*(g,f)$ for all n, $\sigma^*(f,g) = \sigma^*(g,f)$. For each n, $\sigma_n^*(f,g) = \sup_{\overline{O_n}} \frac{\sigma(f(x),g(x))}{1+\sigma(f(x),g(x))} = \sup_{\overline{O_n}}[1 - \frac{1}{1+\sigma(f(x),g(x))}] \leq \sup_{\overline{O_n}}[1 - \frac{1}{1+\sigma(f(x),h(x))+\sigma(h(x),g(x))}] = \sup_{\overline{O_n}} \frac{\sigma(f(x),h(x))+\sigma(h(x),g(x))}{1+\sigma(f(x),h(x))+\sigma(h(x),g(x))} \leq \sigma_n^*(f,h) + \sigma_n^*(h,g)$. Thus $\sigma^*(f,g) \leq \sigma^*(f,h) + \sigma^*(h,g)$. Hence the set of functions from X into Y becomes a metric space under σ^*.

49. Let $\mathfrak{F}$ be an equicontinuous family of functions from X to Y, and let $\mathfrak{F}^+$ be the family of all pointwise limits of functions in $\mathfrak{F}$, that is of f for which there is a sequence $\langle f_n \rangle$ from $\mathfrak{F}$ such that

$f(x) = \lim f_n(x)$ for each $x \in X$. Given $x \in X$ and $\varepsilon > 0$, there is an open set O containing x such that $\sigma(f(x), f(y)) < \varepsilon/3$ for all $y \in O$ and $f \in \mathfrak{F}$. Now if $f \in \mathfrak{F}^+$ and $y \in O$, then there is a sequence $\langle f_n \rangle$ from $\mathfrak{F}$ such that $f(x) = \lim f_n(x)$ so there is an N such that $\sigma(f_N(x), f(x)) < \varepsilon/3$ and $\sigma(f_N(y), f(y)) < \varepsilon/3$. Then $\sigma(f(x), f(y)) \leq \sigma(f(x), f_N(x)) + \sigma(f_N(x), f_N(y)) + \sigma(f_N(y), f(y)) < \varepsilon$. Hence $\mathfrak{F}^+$ is also an equicontinuous family of functions.

50. Let $0 < \alpha \leq 1$ and let $\mathfrak{F} = \{f : ||f||_\alpha \leq 1\}$. If $f \in \mathfrak{F}$, then $\max |f(x)| + \sup |f(x) - f(y)|/|x-y|^\alpha \leq 1$. In particular, $|f(x) - f(y)| \leq |x-y|^\alpha$ for all x, y. Thus $f \in C[0,1]$. Also, $\mathfrak{F}$ is an equicontinuous family of functions on the separable space $[0,1]$ and each $f \in \mathfrak{F}$ is bounded. By the Ascoli-Arzelá Theorem, each sequence $\langle f_n \rangle$ in $\mathfrak{F}$ has a subsequence that converges pointwise to a continuous function. Hence $\mathfrak{F}$ is a sequentially compact, and thus compact, subset of $C[0,1]$.

***51a.** Let $\mathfrak{F}$ be the family of functions that are holomorphic on the unit disk $\Delta = \{z : |z| < 1\}$ with $|f(z)| \leq 1$. Let $f \in \mathfrak{F}$ and fix $z' \in \Delta$. Let U be an open ball of radius $\min(|z'|, 1 - |z'|)$ centred at z'. Then $U \subset \Delta$. Let C be the circle of radius $r/2$ centred at z'. For every z within $r/4$ of z', $f(z) - f(z') = \frac{1}{2\pi i} \int_C [\frac{f(w)\,dw}{w-z} - \frac{f(w)\,dw}{w-z'}] = \frac{z-z'}{2\pi i} \int_C \frac{f(w)\,dw}{(w-z)(w-z')}$ by Cauchy's integral formula. Since $|f(z)| \leq 1$ for all $f \in \mathfrak{F}$ and all $z \in \Delta$, $|f(z) - f(z')| \leq 4r^{-1}|z - z'|$ for all $f \in \mathfrak{F}$. It follows that $\mathfrak{F}$ is equicontinuous.

***51b.** By the Ascoli-Arzelá Theorem, any sequence $\langle f_n \rangle$ in $\mathfrak{F}$ has a subsequence that converges uniformly to a function f on each compact subset of Δ. Furthermore f is holomorphic on Δ.

***51c.** Let $\langle f_n \rangle$ be a sequence of holomorphic functions on Δ such that $f_n(z) \to f(z)$ for all $z \in \Delta$. For each $z \in \Delta$, there is an M_z such that $|f_n(z)| \leq M_z$ for all n. Let $E_m = \bigcap_n E_{m,n}$ where $E_{m,n} = \{z : |f_n(z)| \leq m\}$. Then E_m is closed and $\Delta = \bigcup_m E_m$. There is an E_m that is not nowhere dense so it has nonempty interior. Then $O = \bigcup_m E_m^\circ$ is a dense open subset of Δ and $\langle f_n \rangle$ is locally bounded on O. By an argument similar to that in part (a), $\langle f_n \rangle$ is equicontinuous on O so there is a subsequence that converges uniformly to a function f on each compact subset of O. Furthermore, f is holomorphic on O.

8 Topological Spaces

8.1 Fundamental notions

1a. Given a set X, define ρ on $X \times X$ by $\rho(x,y) = 0$ if $x = y$ and $\rho(x,y) = 1$ otherwise. Then for any set $A \subset X$, $A = \bigcup_{x \in A} B_{x,1/2}$ so A is open. Thus the associated topological space is discrete. If X has more than one point, then there is no metric on X such that the associated topological space is trivial because in a metric space, any two distinct points can be enclosed in disjoint open sets.

1b. Let X be a space with a trivial topology. If f is a continuous mapping of X into $\mathbb{R}$, then $f^{-1}[I]$ is either $\emptyset$ or X for any open interval I. Take $c \in \mathbb{R}$. Then $f^{-1}[c] = f^{-1}[(c-\varepsilon, c+\varepsilon)]$ is either $\emptyset$ or X. Thus f must be a constant function. Conversely, if $f(x) = c$ for all $x \in X$, then for any open interval I, $f^{-1}[I] = \emptyset$ if $c \notin I$ and $f^{-1}[I] = X$ if $c \in I$. Since any open set of real numbers is a countable union of disjoint open intervals, it follows that f is continuous. Hence the only continuous mappings from X into $\mathbb{R}$ are the constant functions.

1c. Let X be a space with a discrete topology. Let f be a mapping of X into $\mathbb{R}$. Since $f^{-1}[O] \subset X$ for any open set $O \subset \mathbb{R}$, f is continuous. Hence all mappings of X into $\mathbb{R}$ are continuous.

2. $\bar{E}$ is the intersection of all closed sets containing E so $E \subset \bar{E}$. Also, $\bar{E} \subset \bar{\bar{E}}$. On the other hand, $\bar{\bar{E}}$ is the intersection of all closed sets containing $\bar{E}$, one of which is $\bar{E}$ itself, so $\bar{\bar{E}} \subset \bar{E}$. Thus $\bar{\bar{E}} = \bar{E}$.

$\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$ so $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. On the other hand, $A \subset A \cup B$ and $B \subset A \cup B$ so $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$. Thus $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Hence $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

If F is closed, then F is a closed set containing F so $\bar{F} \subset F$. Since we always have $F \subset \bar{F}$, $F = \bar{F}$. Conversely, if $F = \bar{F}$, then since $\bar{F}$ is closed, F is closed.

If $x \in \bar{E}$, then x is in every closed set containing E. Let O be an open set containing x. If $O \cap E = \emptyset$, then $E \subset O^c$. Since O^c is closed, $x \in O^c$. Contradiction. Hence $O \cap E \neq \emptyset$ and x is a point of closure of E. Conversely, suppose x is a point of closure of E. If $x \notin \bar{E}$, then x is not in some closed set F containing E so $x \in F^c$. Since F^c is an open set containing x, $F^c \cap E \neq \emptyset$. Contradiction. Hence $x \in \bar{E}$.

E° is the union of all open sets contained in E so $E^\circ \subset E$. Also, $E^{\circ\circ} \subset E^\circ$. On the other hand, $E^{\circ\circ}$ is the union of all open sets contained in E°, one of which is E° itself, so $E^\circ \subset E^{\circ\circ}$. Thus $E^{\circ\circ} = E^\circ$.

$A^\circ \cap B^\circ$ is an open set contained in $A \cap B$ so $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. On the other hand, $A \cap B \subset A$ and $A \cap B \subset B$ so $(A \cap B)^\circ \subset A^\circ$ and $(A \cap B)^\circ \subset B^\circ$. Thus $(A \cap B)^\circ \subset A^\circ \cap B^\circ$. Hence $(A \cap B)^\circ = A^\circ \cap B^\circ$.

If $x \in E^\circ$, then x is in some open set O contained in E so x is an interior point of E. Conversely, if x is an interior point of E, then $x \in O \subset E$ for some open set O so $x \in E^\circ$.

If $x \in (E^c)^\circ$, then x is in some open set $O \subset E^c$ so $x \notin O^c \supset E$ and thus $x \in \bar{E}^c$. Conversely, if $x \in \bar{E}^c$, then x is not in some closed set $F \supset E$ so $x \in F^c \subset E^c$ and thus $x \in (E^c)^\circ$. Hence $(E^c)^\circ = \bar{E}^c$.

3. Suppose $A \subset X$ is open. Then given $x \in A$, $x \in O \subset A$ with $O = A$. Conversely, suppose that given $x \in A$, there is an open set O_x such that $x \in O_x \subset A$. Then $A = \bigcup_{x \in A} O_x$, which is open.

4. Let f be a mapping of X into Y. Suppose f is continuous. For any closed set F, $f^{-1}[F^c]$ is open. But $f^{-1}[F^c] = (f^{-1}[F])^c$ so $f^{-1}[F]$ is closed. Conversely, suppose the inverse image of every closed set is closed. For any open set O, $f^{-1}[O^c]$ is closed. But $f^{-1}[O^c] = (f^{-1}[O])^c$ so $f^{-1}[O]$ is open. Thus f is continuous.

5. Suppose f is a continuous mapping of X into Y and g is a continuous mapping of Y into Z. For any open set $O \subset Z$, $g^{-1}[O]$ is open in Y so $f^{-1}[g^{-1}[O]]$ is open in X. But $f^{-1}[g^{-1}[O]] = (g \circ f)^{-1}[O]$. Hence $g \circ f$ is a continuous mapping of X into Z.

6. Let f and g be two real-valued continuous functions on X and let $x \in X$. Given $\varepsilon > 0$, there is an open set O containing x such that $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$ whenever $y \in O$. Then $|(f+g)(x) - (f+g)(y)| = |f(x) - f(y) + g(x) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon$ whenever $y \in O$. Thus $f + g$ is continuous at x. There is an open set O' such that $|g(x) - g(y)| < \varepsilon/2|f(x)|$ and $|f(x) - f(y)| < \varepsilon/2 \max(|g(x) - \varepsilon/2|f(x)||, |g(x) + \varepsilon/2|f(x)||)$ whenever $y \in O'$. Then $|(fg)(x) - (fg)(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)| < \varepsilon$ whenever $y \in O'$. Thus fg is continuous at x.

7a. Let F be a closed subset of a topological space and $\langle x_n \rangle$ a sequence of points from F. If x is a cluster point of $\langle x_n \rangle$, then for any open set O containing x and any N, there exists $n \geq N$ such that $x_n \in O$. Suppose $x \notin F$. Then x is in the open set F^c so there exists $x_n \in F^c$. Contradiction. Hence $x \in F$.

7b. Suppose f is continuous and $x = \lim x_n$. For any open set U containing $f(x)$, there is an open set O containing x such that $f[O] \subset U$. There also exists N such that $x_n \in O$ for $n \geq N$ so $f(x_n) \in U$ for $n \geq N$. Hence $f(x) = \lim f(x_n)$.

7c. Suppose f is continuous and x is a cluster point of $\langle x_n \rangle$. For any open set U containing $f(x)$, there is an open set O containing x such that $f[O] \subset U$. For any N, there exists $n \geq N$ such that $x_n \in O$ so $f(x_n) \in U$. Hence $f(x)$ is a cluster point of $\langle f(x_n) \rangle$.

***8a.** Let E be an arbitrary set in a topological space X. $\overline{(\overline{E^\circ})^\circ}$ is the intersection of all closed sets containing $(\overline{E^\circ})^\circ$, one of which is $\overline{E^\circ}$, so $\overline{(\overline{E^\circ})^\circ} \subset \overline{E^\circ}$. On the other hand, $\overline{E^\circ}$ is the intersection of all closed sets containing E°, one of which is $\overline{(\overline{E^\circ})^\circ}$ since $E^\circ = E^{\circ\circ} \subset (\overline{E^\circ})^\circ$, so $\overline{E^\circ} \subset \overline{(\overline{E^\circ})^\circ}$. Hence $\overline{(\overline{E^\circ})^\circ} = \overline{E^\circ}$.

Now since $\bar{\bar{E}} = \bar{E}$ and $(E^c)^c = E$, new sets can only be obtained if the operations are performed alternately. Also, $(\bar{E})^c = (E^c)^\circ$ so $\left(\overline{\left(\overline{(\bar{E})^c}\right)^c}\right)^c = \overline{\left(\overline{(\bar{E})^c}\right)^\circ} = \overline{\left(\overline{(E^c)^\circ}\right)^\circ} = \overline{(E^c)^\circ} = \overline{(\bar{E})^c}$. Hence the distinct sets that can be obtained are at most E, $\bar{E}$, $(\bar{E})^c$, $\overline{(\bar{E})^c}$, $\left(\overline{(\bar{E})^c}\right)^c$, $\overline{\left(\overline{(\bar{E})^c}\right)^c}$, $\left(\overline{\left(\overline{(\bar{E})^c}\right)^c}\right)^c$, E^c, $\overline{E^c}$, $(\overline{E^c})^c$, $\overline{(\overline{E^c})^c}$, $\left(\overline{(\overline{E^c})^c}\right)^c$, $\overline{\left(\overline{(\overline{E^c})^c}\right)^c}$, $\left(\overline{\left(\overline{(\overline{E^c})^c}\right)^c}\right)^c$.

***8b.** Let $E = \{(0,0)\} \cup \{z : 1 < |z| < 2\} \cup \{z : 2 < |z| < 3\} \cup \{z : 4 < |z| < 5 \text{ and } z = (p,q) \text{ where } p, q \in \mathbb{Q}\} \subset \mathbb{R}^2$. Then E gives 14 different sets by repeated use of complementation and closure.

9. Let f be a function from a topological space X to a topological space Y. Suppose f is continuous and let $x \in X$. For any open set U containing $f(x)$, $f^{-1}[U]$ is open and $x \in f^{-1}[U]$. If $y \in f^{-1}[U]$, then $f(y) \in U$. Thus f is continuous at x. Conversely, suppose f is not continuous. There exists an open set O such that $f^{-1}[O]$ is not open. By Q3, there exists $x \in f^{-1}[O]$ such that U is not a subset of $f^{-1}[O]$ for any open set U containing x. Then O is an open set containing $f(x)$ and $f[U]$ is not a subset of O

for any open set U containing x. Hence f is not continuous at x.

10a. Let the subset A of a topological space X be the union of two sets A_1 and A_2 both of which are closed (resp. both of which are open). Let f be a map of A into a topological space Y such that the restrictions $f|_{A_1}$ and $f|_{A_2}$ are each continuous. For any closed (resp. open) set $E \subset Y$, $f^{-1}[E] = (f^{-1}[E] \cap A_1) \cup (f^{-1}[E] \cap A_2) = f|_{A_1}^{-1}[E] \cup f|_{A_2}^{-1}[E] = (A_1 \cap F) \cup (A_2 \cap F')$ for some closed (resp. open) sets $F, F' \subset A$. Hence $f^{-1}[E]$ is closed (resp. open) and f is continuous.

10b. Consider the subset $(0, 2]$ of $\mathbb{R}$. Then $(0, 2]$ is the union of $(0, 1)$, which is open, and $[1, 2]$, which is closed. Let f be the map defined by $f(x) = x$ on $(0, 1)$ and $f(x) = x + 1$ on $[1, 2]$. Then $f|_{(0,1)}$ and $f|_{[1,2]}$ are each continuous but f is not continuous (at 1).

***10c.** Suppose $A = A_1 \cup A_2$, $\overline{A_1 \setminus A_2} \cap (A_2 \setminus A_1) = \emptyset$ and $\overline{A_2 \setminus A_1} \cap (A_1 \setminus A_2) = \emptyset$. Let f be a map of A into a topological space Y such that the restrictions $f|_{A_1}$ and $f|_{A_2}$ are each continuous. Let F be a closed subset of Y and let $F' = f^{-1}[F] \subset A$. Then $\overline{F'} = \overline{F' \cap (A_1 \setminus A_2 \cup (A_1 \cap A_2) \cup A_2 \setminus A_1)} = \overline{F' \cap (A_1 \setminus A_2)} \cup \overline{F' \cap A_1 \cap A_2} \cup \overline{F' \cap (A_2 \setminus A_1)}$. Note that $\overline{F' \cap (A_1 \setminus A_2)} \subset A_1$ and $\overline{F' \cap (A_2 \setminus A_1)} \subset A_2$. Thus $f[\overline{F'}] = f[\overline{F' \cap (A_1 \setminus A_2)}] \cup f[\overline{F' \cap A_1 \cap A_2}] \cup f[\overline{F' \cap (A_2 \setminus A_1)}] = \overline{f[F']} \subset F$ and $\overline{F'} \subset f^{-1}[F] = F'$. Hence $F' = f^{-1}[F]$ is closed and f is continuous.

8.2 Bases and countability

11a. Let $\mathfrak{B}$ be a base for the topological space $(X, \mathcal{T})$. If $x \in \bar{E}$, then for every $B \in \mathfrak{B}$ with $x \in B$, since B is open, $B \cap E \neq \emptyset$. i.e. there is a $y \in B \cap E$. Conversely, suppose that for every $B \in \mathfrak{B}$ with $x \in B$ there is a $y \in B \cap E$. For each open set O in X with $x \in O$, there is a $B \in \mathfrak{B}$ with $x \in B \subset O$. Then there is a $y \in B \cap E \subset O \cap E$. Thus $x \in \bar{E}$.

11b. Let X satisfy the first axiom of countability. If there is a sequence $\langle x_n \rangle$ in E converging to x, then for any open set O containing x, there exists N such that $x_n \in O$ for $n \geq N$. Thus $O \cap E \neq \emptyset$ so $x \in \bar{E}$. Conversely, suppose $x \in \bar{E}$. There is a countable base $\langle B_n \rangle$ at x. By considering $B_1, B_1 \cap B_2, B_1 \cap B_2 \cap B_3, \ldots$, we may assume $B_1 \supset B_2 \supset B_3 \cdots$. Now each B_n contains an $x_n \in E$. For any open set O containing x, $O \supset B_N \supset B_{N+1} \supset \cdots$ for some N so $x_n \in O$ for $n \geq N$ and $\langle x_n \rangle$ is a sequence in E converging to x.

11c. Let X satisfy the first axiom of countability and let $\langle x_n \rangle$ be a sequence from X. If $\langle x_n \rangle$ has a subsequence converging to x, it follows from the definition that x is a cluster point of $\langle x_n \rangle$. Conversely, suppose x is a cluster point of $\langle x_n \rangle$. There is a countable base $\langle B_n \rangle$ at x. We may assume $B_1 \supset B_2 \supset \cdots$. Since B_1 is open and contains x, $x_{n_1} \in B_1$ for some $n_1 \geq 1$. Suppose $x_{n_1}, \ldots, x_{n_k}$ have been chosen. Then there exists $n_{k+1} \geq n_k$ such that $x_{n_{k+1}} \in B_{k+1}$. For any open set O containing x, $O \supset B_N \supset B_{N+1} \cdots$ for some N so $x_{n_k} \in O$ for $k \geq N$ and the subsequence $\langle x_{n_k} \rangle$ converges to x.

12a. See Q9.

12b. Let $\mathfrak{B}_x$ be a base at x and $\mathfrak{C}_y$ be a base at $y = f(x)$. Suppose f is continuous at x. For each $C \in \mathfrak{C}_y$, since C is open and contains $f(x)$, there is an open set U containing x such that $U \subset f^{-1}[C]$. Then there exists $B \in \mathfrak{B}_x$ such that $B \subset U \subset f^{-1}[C]$. Conversely, suppose that for each $C \in \mathfrak{C}_y$ there exists $B \in \mathfrak{B}_x$ such that $B \subset f^{-1}[C]$. For any open set O containing $f(x)$, there exists $C \in \mathfrak{C}_y$ such that $C \subset O$. Then there exists $B \in \mathfrak{B}_x$ such that $B \subset f^{-1}[C] \subset f^{-1}[O]$. Hence f is continuous at x.

13. Let $\mathfrak{C}$ be any collection of subsets of X. Let $\mathfrak{B}$ consist of X and all finite intersections of sets in $\mathfrak{C}$. By Proposition 5, $\mathfrak{B}$ is a base for some topology $\mathcal{S}$ on X. For any topology $\mathcal{T}$ on X containing $\mathfrak{C}$, we have $\mathfrak{B} \subset \mathcal{T}$. Since any set in $\mathcal{S}$ is a union of sets in $\mathfrak{B}$, $\mathcal{S} \subset \mathcal{T}$. Hence $\mathcal{S}$ is the weakest topology on X containing $\mathfrak{C}$.

14. Let X be an uncountable set of points and let $\mathcal{T}$ consist of the empty set and all subsets of X whose complement is finite. Then $X \in \mathfrak{T}$ since $X^c = \emptyset$, which is finite. If $O_1, O_2 \in \mathcal{T}$, then we may assume $O_1, O_2 \neq \emptyset$ so $(O_1 \cap O_2)^c = O_1^c \cup O_2^c$, which is finite. Thus $O_1 \cap O_2 \in \mathcal{T}$. If $O_\alpha \in \mathcal{T}$, then $(\bigcup O_\alpha)^c = \bigcap O_\alpha^c$, which is finite. Thus $\bigcup O_\alpha \in \mathcal{T}$. Hence $\mathcal{T}$ is a topology for X.

Suppose $(X, \mathcal{T})$ satisfies the first axiom of countability. Take $x \in X$. There is a countable base $\langle B_n \rangle$ at x. Note that B_n^c is finite for all n so $\bigcup B_n^c$ is countable. Thus there exists $y \in (\bigcup B_n^c)^c = \bigcap B_n$ with $y \neq x$ since $(\bigcup B_n^c)^c$ is uncountable. Now $X \setminus \{y\}$ is open and contains x so there exists $B_N \subset X \setminus \{y\}$. But $y \in B_N \subset X \setminus \{y\}$. Contradiction. Hence $(X, \mathcal{T})$ does not satisfy the first axiom of countability.

15. Let X be the set of real numbers and let $\mathfrak{B}$ be the set of all intervals of the form $[a, b)$. For any $x \in X$, we have $x \in [x, x+1)$. If $x \in [a, b) \cap [c, d)$, then $x \in [c, b)$ (or $[a, d)$). Hence $\mathfrak{B}$ is a base of a topology $\mathfrak{T}$ (the half-open interval topology) for X.

For any $x \in X$, let $B_{x,q} = [x, x+q)$ where q is a positive rational number. Then $\{B_{x,q}\}$ is a countable base at x. Thus $(X, \mathfrak{T})$ satisfies the first axiom of countability. Let $\mathfrak{B}$ be a base for $(X, \mathfrak{T})$. For any $x \in X$, there exist $B_x \in \mathfrak{B}$ such that $x \in B_x \subset [x, x+1)$. If $x \neq y$, then $B_x \neq B_y$. Thus $\{B_x\}$ is uncountable and $(X, \mathfrak{T})$ does not satisfy the second axiom of countability. For any $x \in X$ and any open set O containing x, there is an interval $[a, b) \subset O$ containing x. Furthermore, there is a rational $q \in [a, b)$. Thus the rationals are dense in $(X, \mathcal{T})$. Now $(X, \mathcal{T})$ is not metrizable because it is separable and a separable metric space satisfies the second axiom of countability.

16. Suppose X is second countable. Let $\mathfrak{B}$ be a countable base for the topology on X and let $\mathcal{U}$ be an open cover of X. For each $x \in X$, there exists $U_x \in \mathcal{U}$, such that $x \in U_x$. Then there exists $B_x \in \mathfrak{B}$ such that $x \in B_x \subset U_x$. Thus $X = \bigcup_{x \in X} B_x$. Since $\mathfrak{B}$ is countable, we may choose countably many U_x's to form a countable subcover of $\mathcal{U}$. Hence X is Lindelöf.

17a. Let $X_1 = \mathbb{N} \times \mathbb{N}$ and take $X = X_1 \cup \{\omega\}$. For each sequence $s = \langle m_k \rangle$ of natural numbers define $B_{s,n} = \{\omega\} \cup \{\langle j, k \rangle : j \geq m_k \text{ if } k \geq n\}$. Clearly, any $x \in X$ is in some $B_{s,n}$ or $\{\langle j, k \rangle\}$. If $x \in B_{s,n} \cap B_{s',n'}$, then either $x = \omega$ or $x = \langle j, k \rangle$. If $x = \langle j, k \rangle$, then $\{\langle j, k \rangle\} \subset B_{s,n} \cap B_{s'n'}$. If $x = \omega$, let $s'' = \langle m''_k \rangle$ be the sequence where $m''_k = \max(m_k, m'_k)$ for all k and let $n'' = \min(n, n')$. Then $\omega \in B_{s'',n''} \subset B_{s,n} \cap B_{s',n'}$. Other possible intersections are similarly dealt with. Hence the sets $B_{s,n}$ together with the sets $\{\langle j, k \rangle\}$ form a base for a topology on X.

***17b.** For any open set O containing ω, there is some $B_{s,n}$ such that $\omega \in B_{s,n} \subset O$. Since $B_{s,n}$ contains some $\langle j, k \rangle \in X_1$, so does O. Thus ω is a point of closure of X_1.

17c. X is countable so it is separable. If it satisfies the first axiom of countability, then since ω is a point of closure of X_1, by Q11b, there is a sequence from X_1 converging to ω and by Q11c, ω is a cluster point of that sequence, contradicting part (b). Hence X does not satisfy the first axiom of countability and thus it also does not satisfy the second axiom of countability.

17d. X is countable so it is Lindelöf.

8.3 The separation axioms and continuous real-valued functions

18a. Given two distinct points x, y in a metric space (X, ρ), let $r = \rho(x, y)/2$. Then the open balls $B_{x,r}$ and $B_{y,r}$ are disjoint open sets with $x \in B_{x,r}$ and $y \in B_{y,r}$. Hence every metric space is Hausdorff.

18b. Given two disjoint closed sets F_1, F_2 in a metric space (X, ρ), let $O_1 = \{x : \rho(x, F_1) < \rho(x, F_2)\}$ and let $O_2 = \{x : \rho(x, F_2) < \rho(x, F_1)\}$. The sets O_1 and O_2 are open since the functions $f(x) = \rho(x, F_1)$ and $g(x) = \rho(x, F_2)$ are continuous. Thus O_1 and O_2 are disjoint open sets with $F_1 \subset O_1$ and $F_2 \subset O_2$. Hence every metric space is normal.

19. Let X consist of $[0, 1]$ and an element $0'$. Consider the sets (α, β), $[0, \beta)$, $(\alpha, 1]$ and $\{0'\} \cup (0, \beta)$ where $\alpha, \beta \in [0, 1]$. Clearly, any $x \in [0, 1]$ belongs to one of the sets. Note that $(\alpha, \beta) \subset [0, \beta) \cap (\alpha, 1]$. If $x \in (\alpha, \beta) \cap [0, \beta')$, then $x \in (\alpha, \min(\beta, \beta')) \subset (\alpha, \beta) \cap [0, \beta')$. If $x \in (\alpha, \beta) \cap (\alpha', 1]$, then $x \in (\max(\alpha, \alpha'), \beta) \subset (\alpha, \beta) \cap (\alpha', 1]$. If $x \in (\alpha, \beta) \cap (\{0'\} \cup (0, \beta'))$, then $x \in (\alpha, \min(\beta, \beta')) \subset (\alpha, \beta) \cap (\{0'\} \cup (0, \beta'))$. If $x \in [0, \beta) \cap (\{0'\} \cup (0, \beta'))$, then $x \in (0, \min(\beta, \beta')) \subset [0, \beta) \cap (\{0'\} \cup (0, \beta'))$. If $x \in (\alpha, 1] \cap (\{0'\} \cup (0, \beta))$, then $x \in (\alpha, \beta) \subset (\alpha, 1] \cap (\{0'\} \cup (0, \beta))$. Hence the sets form a base for a topology on X.

Given two distinct points $x, y \in X$, if neither of them is 0', then $[0, (x+y)/2)$ is an open set containing one but not the other. If $y = 0'$, then $\{0'\} \cup (0, x/2)$ is an open set containing y but not x. Hence X is T_1. Consider the distinct points 0 and 0'. If O is an open set containing 0 and O' is an open set containing 0', then $0 \in [0, \beta) \subset O$ and $0' \in \{0'\} \cup (0, \beta') \subset O'$ for some β, β'. Since $[0, \beta) \cap (0, \beta') \neq \emptyset$, $O \cap O' \neq \emptyset$. Hence X is not Hausdorff.

20. Let f be a real-valued function on a topological space X. Suppose f is continuous. Then for any real number a, $\{x : f(x) < a\} = f^{-1}[(-\infty, a)]$ and $\{x : f(x) > a\} = f^{-1}[(a, \infty)]$. Since $(-\infty, a)$ and (a, ∞) are open, the sets $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ are open. Conversely, suppose the sets $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ are open for any real number a. Since any open set $O \subset \mathbb{R}$ is a union of open intervals and any open interval is an intersection of at most two of the sets, $f^{-1}[O]$ is open so f is continuous.

Since $\{x : f(x) \geq a\}^c = \{x : f(x) < a\}$, we also have f is continuous if and only if for any real number a, the set $\{x : f(x) > a\}$ is open and the set $\{x : f(x) \geq a\}$ is closed.

21. Suppose f and g are continuous real-valued functions on a topological space X. For any real number a, $\{x : f(x) + g(x) < a\} = \bigcup_{q \in \mathbb{Q}}(\{x : g(x) < q\} \cap \{x : f(x) < a - q\})$, which is open, and $\{x : f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}}(\{x : g(x) > q\} \cap \{x : f(x) > a - q\})$, which is open, so $f + g$ is continuous. $\{x : f(x)g(x) < a\} = \bigcup_{q \in \mathbb{Q}}(\{x : g(x) < q\}) \cap \{x : f(x) < a/q\}$, which is open, and $\{x : f(x)g(x) > a\} = \bigcup_{q \in \mathbb{Q}}(\{x : g(x) > q\}) \cap \{x : f(x) > a/q\}$, which is open, so fg is continuous. $\{x : (f \vee g)(x) < a\} = \{x : f(x) < a\} \cap \{x : g(x) < a\}$, which is open, and $\{x : (f \vee g)(x) > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\}$, which is open, so $f \vee g$ is continuous. $\{x : (f \wedge g)(x) < a\} = \{x : f(x) < a\} \cup \{x : g(x) < a\}$, which is open, and $\{x : (f \wedge g)(x) > a\} = \{x : f(x) > a\} \cap \{x : g(x) > a\}$, which is open, so $f \wedge g$ is continuous.

22. Let $\langle f_n \rangle$ be a sequence of continuous functions from a topological space X to a metric space Y. Suppose $\langle f_n \rangle$ converges uniformly to a function f. Let $\varepsilon > 0$ and let $x \in X$. There exists N such that $\rho(f_n(x), f(x)) < \varepsilon/3$ for all $n \geq N$ and all $x \in X$. Since f_N is continuous, there is an open set O containing x such that $f_N[O] \subset B_{f_N(x),\varepsilon/3}$. If $y \in O$, then $\rho(f(y), f(x)) \leq \rho(f(y), f_N(y)) + \rho(f_N(y), f_N(x)) + \rho(f_N(x), f(x)) < \varepsilon$. Thus $f[O] \subset B_{f(x),\varepsilon}$. Since any open set in Y is a union of open balls, f is continuous.

23a. Let X be a Hausdorff space. Suppose X is normal. Given a closed set F and an open set O containing F, there exist disjoint open sets U and V such that $F \subset U$ and $O^c \subset V$. Now $\bar{U}$ is disjoint from O^c since if $y \in O^c$, then V is an open set containing y that is disjoint from U. Thus $\bar{U} \subset O$. Conversely, suppose that given a closed set F and an open set O containing F, there is an open set U such that $F \subset U$ and $\bar{U} \subset O$. Let F and G be disjoint closed sets. Then $F \subset G^c$ and there is an open set U such that $F \subset U$ and $\bar{U} \subset G^c$. Equivalently, $F \subset U$ and $G \subset \bar{U}^c$. Since U and $\bar{U}^c$ are disjoint open sets, X is normal.

***23b.** Let F be a closed subset of a normal space contained in an open set O. Arrange the rationals in $(0, 1)$ of the form $r = p2^{-n}$ in a sequence $\langle r_n \rangle$. Let $U_1 = O$. By part (a), there exists an open set U_0 such that $F \subset U_0$ and $\overline{U_0} \subset O = U_1$. Let P_n be the set containing the first n terms of the sequence. Since $\overline{U_0} \subset U_1$, there exists an open set U_{r_1} such that $\overline{U_0} \subset U_{r_1}$ and $\overline{U_{r_1}} \subset U_1$. Suppose open sets U_r have been defined for rationals r in P_n such that $\overline{U_p} \subset U_q$ whenever $p < q$. Now r_{n+1} has an immediate predecessor r_i and an immediate successor r_j (under the usual order relation) in $P_{n+1} \cup \{0, 1\}$. Note that $\overline{U_{r_i}} \subset U_{r_j}$. Thus there is an open set $U_{r_{n+1}}$ such that $\overline{U_{r_i}} \subset U_{r_{n+1}}$ and $\overline{U_{r_{n+1}}} \subset U_{r_j}$. Now if $r \in P_n$, then either $r \leq r_i$, in which case $\overline{U_r} \subset U_{r_i} \subset U_{r_{n+1}}$, or $r \geq r_j$, in which case $\overline{U_{r_{n+1}}} \subset U_{r_j} \subset U_r$. By induction, we have a sequence $\{U_r\}$ of open sets, one corresponding to each rational in $(0, 1)$ of the form $p = r2^{-n}$, such that $F \subset U_r \subset O$ and $\overline{U_r} \subset U_s$ for $r < s$.

23c. Let $\{U_r\}$ be the family constructed in part (b) with $U_1 = X$. Let f be the real-valued function on X defined by $f(x) = \inf\{r : x \in U_r\}$. Clearly, $0 \leq f \leq 1$. If $x \in F$, then $x \in U_r$ for all r so $f(x) = 0$. If $x \in O^c$, then $x \notin U_r$ for any $r < 1$ so $f(x) = 1$. Given $x \in X$ and an open interval (c, d) containing $f(x)$, choose rationals r_1 and r_2 of the form $p2^{-n}$ such that $c < r_1 < f(x) < r_2 < d$. Consider the open set $U = U_{r_2} \setminus \overline{U_{r_1}}$. Since $f(x) < r_2$, $x \in U_r \subset U_{r_2}$ for some $r < r_2$. If $x \in \overline{U_{r_1}}$, then $x \in U_r$ for all $r > r_1$ so $f(x) \leq r_1$. Since $f(x) > r_1$, $x \notin \overline{U_{r_1}}$. Thus $x \in U$. If $y \in U$, then $y \in U_{r_2}$ so $f(y) \leq r_2 < d$. Also, $y \notin \overline{U_{r_1}}$ so $f(y) \geq r_1 > c$. Thus $f[U] \subset (c, d)$. Hence f is continuous.

23d. Let X be a Hausdorff space. Suppose X is normal. For any pair of disjoint closed sets A and B on X, B^c is an open set containing A. By the constructions in parts (b) and (c), there is a continuous real-valued function f on X such that $0 \leq f \leq 1$, $f \equiv 0$ on A and $f \equiv 1$ on $(B^c)^c = B$.

(*) Proof of Urysohn's Lemma.

24a. Let A be a closed subset of a normal topological space X and let f be a continuous real-valued function on A. Let $h = f/(1 + |f|)$. Then $|h| = |f|/(1 + |f|) < 1$.

24b. Let $B = \{x : h(x) \leq -1/3\}$ and let $C = \{x : h(x) \geq 1/3\}$. By Urysohn's Lemma, there is a continuous function h_1 which is $-1/3$ on B and $1/3$ on C while $|h_1(x)| \leq 1/3$ for all $x \in X$. Then $|h(x) - h_1(x)| \leq 2/3$ for all $x \in A$.

24c. Suppose we have continuous functions h_n on X such that $|h_n(x)| < 2^{n-1}/3^n$ for all $x \in X$ and $|h(x) - \sum_{i=1}^n h_i(x)| < 2^n/3^n$ for all $x \in A$. Let $B' = \{x : h(x) - \sum_{i=1}^n h_i(x) \leq -2^n/3^{n+1}\}$ and let $C' = \{x : h(x) - \sum_{i=1}^n h_i(x) \geq 2^n/3^{n+1}\}$. By Urysohn's Lemma, there is a continuous function h_{n+1} which is $-2^n/3^{n+1}$ on B' and $2^n/3^{n+1}$ on C' while $|h_{n+1}(x)| \leq 2^n/3^{n+1}$ for all $x \in X$. Then

$|h(x) - \sum_{i=1}^{n+1} h_i(x)| < 2^{n+1}/3^{n+1}$ for all $x \in A$.

24d. Let $k_n = \sum_{i=1}^{n} h_i$. Then each k_n is continuous and $|h(x) - k_n(x)| < 2^n/3^n$ for all $x \in A$. Also, $|k_n(x) - k_{n-1}(x)| < 2^{n-1}/3^n$ for all n and all $x \in X$. If $m > n$, then $|k_m(x) - k_n(x)| = |\sum_{i=n+1}^{m} h_i(x)| \le \sum_{i=n+1}^{m} 2^{i-1}/3^i < (2/3)^n$. Let $k(x) = \sum_{i=1}^{\infty} h_i(x)$. Then $|k(x) - k_n(x)| \le (2/3)^n$ for all $x \in X$. Thus $\langle k_n \rangle$ converges uniformly to k. i.e. $\langle h_n \rangle$ is uniformly summable to k and since each k_n is continuous, k is continuous. Also, since $|k_n| \le 1$ for each n, $|k| \le 1$. Now $|h(x) - k_n(x)| < 2^n/3^n$ for all n and all $x \in A$ so letting $n \to \infty$, $|h(x) - k(x)| = 0$ for all $x \in A$. i.e. $k = h$ on A.

24e. Since $|k| = |h| < 1$ on A, A and $\{x : k(x) = 1\}$ are disjoint closed sets. By Urysohn's Lemma, there is a continuous function φ on X which is 1 on A and 0 on $\{x : k(x) = 1\}$.

24f. Set $g = \varphi k/(1 - |\varphi k|)$. Then g is continuous and $g = k/(1 - |k|) = h/(1 - |h|) = f$ on A.

(*) Proof of Tietze's Extension Theorem.

25. Let $\mathfrak{F}$ be a family of real-valued functions on a set X. Consider the sets of the form $\{x : |f_i(x) - f_i(y)| < \varepsilon$ for some $\varepsilon > 0$, some $y \in X$, and some finite set $f_1, \dots, f_n$ of functions in $\mathfrak{F}\}$. The weak topology on X generated by $\mathfrak{F}$ has $\{f^{-1}[O] : f \in \mathfrak{F}, O$ open in $\mathbb{R}\}$ as a base. Now if $x' \in f^{-1}[O]$ for some $f \in \mathfrak{F}$ and O open in $\mathbb{R}$, we may assume O is an open interval (c, d). If $x \in \{x : |f(x) - f(x')| < \min(f(x') - c, d - f(x'))\}$, then $f(x) \in (c, d)$. i.e. $x \in f^{-1}[O]$. Thus $x' \in \{x : |f(x) - f(x')| < \min(f(x') - c, d - f(x'))\} \subset f^{-1}[O]$. Hence the sets of the form $\{x : |f_i(x) - f_i(y)| < \varepsilon$ for some $\varepsilon > 0$, some $y \in X$, and some finite set $f_1, \dots, f_n$ of functions in $\mathfrak{F}\}$ is a base for the weak topology on X generated by $\mathfrak{F}$.

Suppose this topology is Hausdorff. For any pair $\{x, y\}$ of distinct points in X, there are disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. Then there are sets B_x and B_y of the form $\{x : |f_i(x) - f_i(z)| < \varepsilon$ for some $\varepsilon > 0$, some $z \in X$, and some finite set $f_1, \dots, f_n$ of functions in $\mathfrak{F}\}$ such that $x \in B_x \subset O_x$ and $y \in B_y \subset O_y$. Suppose $f(x) = f(y)$ for all $f \in \mathfrak{F}$. Then $x, y \in B_x \cap B_y$. Contradiction. Hence there is a function $f \in \mathfrak{F}$ such that $f(x) \neq f(y)$. Conversely, suppose that for each pair $\{x, y\}$ of distinct points in X there exists $f \in \mathfrak{F}$ such that $f(x) \neq f(y)$. Then $x \in \{x' : |f(x') - f(x)| < |f(y) - f(x)|/2\}$ and $y \in \{x' : |f(x') - f(y)| < |f(y) - f(x)|/2\}$ with the two sets being disjoint open sets. Hence the topology is Hausdorff.

26. Let $\mathfrak{F}$ be a family of real-valued continuous functions on a topological space $(X, \mathcal{T})$. Since f is continuous for each $f \in \mathfrak{F}$, the weak topology generated by $\mathfrak{F}$ is contained in $\mathcal{T}$. Suppose that for each closed set F and each $x \notin F$ there is an $f \in \mathfrak{F}$ such that $f(x) = 1$ and $f \equiv 0$ on F. Then for each $O \in \mathcal{T}$ and each $x \in O$, there is an $f \in \mathfrak{F}$ such that $f(x) = 1$ and $f \equiv 0$ on O^c. i.e. $x \in f^{-1}[(1/2, 3/2)] \subset O$. Thus O is in the weak topology generated by $\mathfrak{F}$.

27. Let X be a completely regular space. Given a closed set F and $x \notin F$, there is a function $f \in C(X)$ such that $f(x) = 1$ and $f \equiv 0$ on F. Then $F \subset f^{-1}[(-\infty, 1/2)]$, $x \in f^{-1}[(1/2, \infty)]$, and the sets $f^{-1}[(-\infty, 1/2)]$ and $f^{-1}[(1/2, \infty)]$ are disjoint open sets. Hence X is regular.

28. Let X be a Hausdorff space and let Y be a subset of X. Given two distinct points x and y in Y, there are disjoint open sets O_1 and O_2 in X such that $x \in O_1$ and $y \in O_2$. Then $x \in O_1 \cap Y$ and $y \in O_2 \cap Y$, with $O_1 \cap Y$ and $O_2 \cap Y$ being disjoint open sets in Y. Hence Y is Hausdorff.

***29.** In $\mathbb{R}^n$ let $\mathfrak{B}$ be the family of sets $\{x : p(x) \neq 0\}$ where p is a polynomial in n variables. Let $\mathcal{T}$ be the family of all finite intersections $O = B_1 \cap \cdots \cap B_k$ from $\mathfrak{B}$. By considering the constant polynomials, we see that $\emptyset$ and X are in $\mathcal{T}$. It is also clear that if $O_1, O_2 \in \mathcal{T}$, then $O_1 \cap O_2 \in \mathcal{T}$. Now if $O_\alpha \in \mathcal{T}$,

Given two distinct points $x, y \in \mathbb{R}^n$, say $x = \langle x_1, \dots, x_n \rangle$ and $y = \langle y_1, \dots, y_n \rangle$, let p be the polynomial $\sum_{i=1}^{n} (X_i - x_i)^2$. Then $p(x) = 0$ and $p(y) \neq 0$. Thus there is an open set containing y but not x. Hence $\mathcal{T}$ is T_1.

Any two open sets are not disjoint since for any finite collection of polynomials there is an x that is not a root of any of the polynomials. Hence $\mathcal{T}$ is not T_2.

30. Let $A \subset B \subset \bar{A}$ be subsets of a topological space, and let f and g be two continuous maps of B into a Hausdorff space X with $f(u) = g(u)$ for all $u \in A$. For any $v \in B$, we have $v \in \bar{A}$. Suppose $f(v) \neq g(v)$. Then there are disjoint open sets O_1 and O_2 such that $f(v) \in O_1$ and $g(v) \in O_2$. Since f and g are continuous, there are open sets U_1 and U_2 such that $v \in U_1$, $f[U_1] \subset O_1$, $v \in U_2$, $g[U_2] \subset O_2$. Now there exists $u \in A$ with $u \in U_1 \cap U_2$. Then $f(u) \in O_1$ and $g(u) \in O_2$ but $f(u) = g(u)$ so $O_1 \cap O_2 \neq \emptyset$. Contradiction. Hence $f(v) = g(v)$. i.e. $f \equiv g$ on B.

31. Omitted.

8.4 Connectedness

32. Let $\{C_\alpha\}$ be a collection of connected sets and suppose that any two of them have a point in common. Let $G = \bigcup C_\alpha$. Suppose O_1 and O_2 is a separation of G. For any pair C_α and $C_{\alpha'}$, there is a $p \in C_\alpha \cap C_{\alpha'}$. Then $p \in O_1$ or $p \in O_2$. Suppose $p \in O_1$. Since C_α is connected and $p \in C_\alpha$, we have $C_\alpha \subset O_1$. Now any other $C_{\alpha''}$ has a point in common with C_α so by the same argument, $C_{\alpha''} \subset O_1$. Thus $G \subset O_1$, contradicting $O_2 \neq \emptyset$. Hence G is connected.

33. Let A be a connected subset of a topological space and suppose that $A \subset B \subset \bar{A}$. Suppose O_1 and O_2 is a separation of B. Since $A \subset B = O_1 \cup O_2$ and A is connected, we may assume that $A \subset O_1$. Then $B = B \cap \bar{A} \subset B \cap \overline{O_1} = O_1$, contradicting $O_2 \neq \emptyset$. Hence B is connected.

34a. Let E be a connected subset of $\mathbb{R}$ having more than one point. Suppose $x, y \in E$ with $x < y$. Suppose there exists $z \in (x, y)$ such that $z \notin E$. Then $E \cap (-\infty, z)$ and $E \cap (z, \infty)$ are a separation of E, contradicting the connectedness of E. Thus $[x, y] \subset E$. Let $a = \inf E$ and $b = \sup E$. Clearly, $E \subset [a, b]$. If $z \in (a, b)$, then there exists $z' \in E$ such that $a \leq z' < z$ and there exists $z'' \in E$ such that $z < z'' \leq b$ so that $z \in E$. Thus $(a, b) \subset E \subset [a, b]$. Hence E is an interval.

34b. Let $I = (a, b)$ and let O be a subset of I that is both open and closed in I. Clearly $\sup\{y : (a, y) \subset O\} \leq b$. Suppose $d = \sup\{y : (a, y) \subset O\} < b$. If $d \in O$, then $(d - \varepsilon, d + \varepsilon) \subset O \subset I$ for some $\varepsilon > 0$ since O is open. There exists $z > d - \varepsilon$ such that $(a, z) \subset O$. Then $(a, d + \varepsilon) = (a, z) \cup (d - \varepsilon, d + \varepsilon) \subset O$. Contradiction. If $d \notin O$, then $(d - \varepsilon, d + \varepsilon) \subset O^c$ for some $\varepsilon > 0$ since O^c is open. But there exists $z > d - \varepsilon$ such that $(d - \varepsilon, z) \subset (a, z) \subset O$. Contradiction. Hence $d = b$. Thus for any $c < b$, there exists $c' > c$ such that $(a, c') \subset O$. i.e. $c \in O$. Hence $O = I$ and I is connected. It follows from Q33 that intervals of the form $(a, b]$, $[a, b)$, $[a, b]$ are also connected.

35a. Let X be an arcwise connected space. Suppose O_1 and O_2 is a separation of X. Take $x \in O_1$ and $y \in O_2$. There is a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Since $[0, 1]$ is connected, $f[0, 1]$ is connected so we may assume $f[0, 1] \subset O_1$. Then $y \in O_1 \cap O_2$. Contradiction. Hence X is connected.

***35b.** Consider $X = \{\langle x, y\rangle : x = 0, -1 \leq y \leq 1\} \cup \{\langle x, y\rangle : y = \sin(1/x), 0 < x \leq 1\}$. Let $I = \{\langle x, y\rangle : x = 0, -1 \leq y \leq 1\}$ and $S = \{\langle x, y\rangle : y = \sin(1/x), 0 < x \leq 1\}$. Since S is the image of $(0, 1]$ under a continuous map, S is connected. Let $\langle 0, y\rangle \in I$. Given $\varepsilon > 0$, choose n such that $1/2n\pi < \varepsilon$. Since $\sin \frac{1}{2}(4n + 1)\pi = 1$ and $\sin \frac{1}{2}(4n + 3)\pi = -1$, $\sin(1/x)$ takes on every value between -1 and 1 in the interval $[2/(4n + 3)\pi, 2/(4n + 1)\pi]$. In particular, $\sin(1/x_0) = y$ for some x_0 in the interval. Then $\rho[\langle 0, y\rangle, \langle x_0, y\rangle] = x_0 < \varepsilon$ so $\langle x_0, y\rangle \in B_{\langle 0,y\rangle,\varepsilon} \cap S$. Hence $I \subset \bar{S}$ and by Q33, $X = I \cup S$ is connected.

Suppose X is arcwise connected. Then there is a path f from $\langle 0, 0\rangle$ to some point of S. The set of t such that $f(t) \in I$ is closed so it has a largest element. We may assume that $t = 0$. Let $f(t) = \langle x(t), y(t)\rangle$. Then $x(0) = 0$ while $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. For each n, choose u with $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then there exists $t_n \in (0, 1/n)$ such that $x(t_n) = u$. Now $\langle t_n\rangle$ converges to 0 but $\langle y(t_n)\rangle$ does not converge, contradicting the continuity of f. Hence X is not arcwise connected.

(*) (Closed) topologist's sine curve

35c. Let G be a connected open set in $\mathbb{R}^n$. Let $x \in G$ and let H be the set of points in G that can be connected to x by a path. There exists $\delta > 0$ such that $B_{x,\delta} \subset G$. For $y \in B_{x,\delta}$, $f(t) = (1 - t)x + ty$ is a path connecting x and y. Thus $H \neq \emptyset$. For each $y \in H$, there exists $\delta' > 0$ such that $B_{y,\delta'} \subset G$. There is a path f connecting y to x. For each $z \in B_{y,\delta'}$, there is a path g connecting z to y. Then h given by $h(t) = g(2t)$ for $t \in [0, 1/2]$ and $h(t) = f(2t - 1)$ for $t \in [1/2, 1]$ is a path connecting z to x. Thus $B_{y,\delta'} \subset H$ and H is open. If $y \in \bar{H}$, then $y \in B_{y,\delta} \subset G$ for some $\delta > 0$ and $B_{y,\delta}$ contains a point z of H. Now there is a path connecting z to x and there is a path connecting y to z. Thus there is a path connecting y to x so $y \in H$ and H is closed. Hence $H = G$. Since x is arbitrary, G is arcwise connected.

36. Let X be a locally connected space and let C be a component of X. If $x \in C$, then there is a connected basic set B such that $x \in B$. Since B is connected, $B \subset C$. Thus C is open.

***37.** Let X be as in Q35b. Sufficiently small balls centred at $\langle 0, 0\rangle$ do not contain connected open sets so X is not locally connected.

8.5 Products and direct unions of topological spaces

38a. Let $Z = \overset{\circ}{\bigcup} X_\alpha$. Suppose $f : Z \to Y$ is continuous. For any open set $O \subset Y$, $f^{-1}[O]$ is open in Z so $f^{-1}[O] \cap X_\alpha$ is open in X_α for each α. i.e. $f|_{X_\alpha}^{-1}[O]$ is open in X_α for each α. Thus each restriction $f|_{X_\alpha}$ is continuous. Conversely, suppose each restriction $f|_{X_\alpha}$ is continuous. For any open set $O \subset Y$, $f^{-1}[O] = \bigcup f|_{X_\alpha}^{-1}[O]$. Each of the sets $f|_{X_\alpha}^{-1}[O]$ is open so $f^{-1}[O]$ is open and f is continuous.

38b. $F \subset Z$ is closed if and only if F^c is open if and only if $F^c \cap X_\alpha$ is open in X_α for each α if and only if $F \cap X_\alpha$ is closed in X_α for each α.

38c. Suppose Z is Hausdorff. Given $x, y \in X_\alpha$, $x, y \in Z$ so there are disjoint open sets $O_1, O_2 \subset Z$ such that $x \in O_1$ and $y \in O_2$. Then $O_1 \cap X_\alpha$ and $O_2 \cap X_\alpha$ are disjoint open sets in X_α containing x and y respectively. Thus X_α is Hausdorff. Conversely, suppose each X_α is Hausdorff. Given $x, y \in Z$, $x \in X_\alpha$ and $y \in X_\beta$ for some α and β. If $\alpha = \beta$, then since X_α is Hausdorff, there are disjoint open sets $O_1, O_2 \subset X_\alpha$ such that $x \in O_1$ and $y \in O_2$. The sets O_1 and O_2 are also open in Z so it follows that Z is Hausdorff. If $\alpha \neq \beta$, then since $X_\alpha \cap X_\beta = \emptyset$ and the sets X_α and X_β are open in Z, it follows that Z is Hausdorff.

38d. Suppose Z is normal. Given disjoint closed sets $F_1, F_2 \subset X_\alpha$, F_1 and F_2 are closed in Z so there are disjoint open sets $O_1, O_2 \subset Z$ such that $F_1 \subset O_1$ and $F_2 \subset O_2$. Then $F_1 \subset O_1 \cap X_\alpha$ and $F_2 \subset O_2 \cap X_\alpha$ with $O_1 \cap X_\alpha$ and $O_2 \cap X_\alpha$ being disjoint open sets in X_α. Thus X_α is normal. Conversely, suppose each X_α is normal. Given disjoint closed sets $F_1, F_2 \subset Z$, the sets $F_1 \cap X_\alpha$ and $F_2 \cap X_\alpha$ are closed in X_α for each α so there are disjoint open sets $O_{1,\alpha}, O_{2,\alpha} \subset X_\alpha$ such that $F_1 \cap X_\alpha \subset O_{1,\alpha}$ and $F_2 \cap X_\alpha \subset O_{2,\alpha}$. Then $F_1 \subset \bigcup O_{1,\alpha}$ and $F_2 \subset \bigcup O_{2,\alpha}$ with $\bigcup O_{1,\alpha}$ and $\bigcup O_{2,\alpha}$ being disjoint open sets in Z. Hence Z is normal.

39a. Let $X_1 \subset X$ be a direct summand. Then it follows from the definition that X_1 is open. Let X_2 be another direct summand. Then X_2 is open and $X_1 \subset X_2^c$. Now X_2^c is closed so $X_1 = X_2^c \cap X_1$ is closed.

39b. Let X_1 be a subset of X that is both open and closed. Let $X_2 = X \setminus X_1$. Then X_2 is open, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. If O is open in X, then $O \cap X_i$ is open in X_i for $i = 1, 2$. On the other hand, if O is open in $X_1 \overset{\circ}{\cup} X_2$, then $O \cap X_i$ is open in X_i for $i = 1, 2$ and thus open in X. Then $O = (O \cap X_1) \cup (O \cap X_2)$ is open in X. Hence $X = X_1 \overset{\circ}{\cup} X_2$.

***40a.** If X has a base, each element being Tychonoff, let x, y be distinct points in X. There is a basic element B containing x. Since B is Tychonoff, there exists O open in B such that $x \in O$ but $y \notin O$. Then O is also open in X so X is Tychonoff.

Consider $E = \mathbb{R} \times \{0, 1\}/\sim$ where $\sim$ is the smallest equivalence relation with $\langle x, 0\rangle\ \langle x, 1\rangle$ for $x \in \mathbb{R} \setminus \{0\}$. Let $q : \mathbb{R} \times \{0, 1\} \to E$ be the quotient map and consider the base $q[(a, b) \times \{e\}]$ for $a < b$ and $e \in \{0, 1\}$. Then E is not Hausdorff and thus not regular and not completely regular but the elements in the base are.

(*) Error in book.

***40b.**

***40c.**

41. Let (X, ρ) be a metric space with an extended real-valued metric and X_α its parts. i.e. equivalence classes under the equivalence relation $\rho(x, y) < \infty$. Then X is the disjoint union of its parts and by Q7.3b, each part is open (and closed). Thus X is the direct union $X = \overset{\circ}{\bigcup} X_\alpha$.

42. Let $\langle X_\alpha, \mathcal{T}_\alpha\rangle$ be a family of Hausdorff spaces. Given distinct points $x, y \in \prod_\alpha X_\alpha$, we have $x_\alpha \neq y_\alpha$ for some α. Since X_α is Hausdorff, there are disjoint open sets $O_1, O_2 \subset X_\alpha$ such that $x \in O_1$ and $y \in O_2$. Then $\pi_\alpha^{-1}[O_1]$ and $\pi_\alpha^{-1}[O_2]$ are disjoint open sets in $\prod_\alpha X_\alpha$ containing x and y respectively.

43. Note that $\prod_\alpha X_\alpha$ is a basic element. If $x \in B_1 \cap B_2$ where $B_i = \prod_\alpha O_{\alpha,i}$, then $x_\alpha \in O_{\alpha,1} \cap O_{\alpha,2}$ for each α. Now each $O'_\alpha = O_{\alpha,1} \cap O_{\alpha,2}$ is open in X_α and only finitely many of them are not X_α. Thus $x \in \prod_\alpha O'_\alpha \subset B_1 \cap B_2$.

Let (X, ρ) and (Y, σ) be two metric spaces. Since ρ_∞ is equivalent to the usual product metric, we consider $(X \times Y, \rho_\infty)$. The open ball $B_{\langle x,y\rangle,\varepsilon}$ is the same as $B_{x,\varepsilon} \times B_{y,\varepsilon}$ so $B_{\langle x,y\rangle,\varepsilon}$ is open in the product topology. For any $U \times V$ with U open in X and V open in Y, given $\langle x, y\rangle \in U \times V$, there are open balls $B_{x,\varepsilon}$ and $B_{y,\delta}$ such that $x \in B_{x,\varepsilon} \subset U$ and $y \in B_{y,\delta} \subset V$. Let $\eta = \min(\varepsilon, \delta)$. Then

$B_{\langle x,y\rangle,\eta} \subset B_{x,\varepsilon} \times B_{y,\delta} \subset U \times V$. Thus $U \times V$ is open in the metric topology. Hence the product topology on $X \times Y$ is the same as the topology induced by the product metric.

44. By definition, $X^A = \prod_{\alpha\in A} X_\alpha$. We may identify each $x \in \prod_{\alpha\in A} X_\alpha$ with the function $f : A \to X$ given by $f(\alpha) = x_\alpha$. We may then also identify each basic element $\prod_{\alpha\in A} O_\alpha$ with $\{f : f(\alpha) \in O_\alpha$ for each $\alpha\}$. But since all but finitely many of the sets O_α are X_α, we only need to consider the sets $\{f : f(\alpha_1) \in O_1, \ldots, f(\alpha_n) \in O_n\}$ where $\{\alpha_1, \ldots, \alpha_n\}$ is some finite subset of A and $\{O_1, \ldots, O_n\}$ is a finite collection of open subsets of X.

Suppose a sequence $\langle f_n\rangle$ converges to f in X^A. Then we may regard this as a sequence $\langle x_n\rangle$ converging to x under the identification described above. Now since π_α is continuous, $\langle \pi_\alpha(x_n)\rangle$ converges to $\pi_\alpha(x)$. i.e. $\langle x_{n,\alpha}\rangle$ converges to x_α for each α. Under the identification again, $\langle f_n(\alpha)\rangle$ converges to $f(\alpha)$ for each α. Conversely, suppose $\langle f_n(\alpha)\rangle$ converges to $f(\alpha)$ for each α. Let B be a basic element containing f. Then $f(\alpha_i) \in O_i$ for some finite subset $\{\alpha_1, \ldots, \alpha_m\} \subset A$ and some finite collection $\{O_1, \ldots, O_m\}$ of open subsets of X. Then there exists N such that $f_n(\alpha_i) \in O_i$ for $n \geq N$ and $i = 1, \ldots, m$. Thus $f_n \in B$ for $n \geq N$ and $\langle f_n\rangle$ converges to f in X^A.

45. Suppose X is metrizable and A is countable. We may enumerate A as $\{\alpha_1, \alpha_2, \ldots\}$. Then by Q7.11a, X can be given an equivalent metric ρ that is bounded by 1. Define σ on X^A by $\sigma(x,y) = \sum_n 2^{-n}\rho(x_{\alpha_n}, y_{\alpha_n})$. Then σ is a metric on X^A so X^A is metrizable.

46. Given an open set $O_{\alpha_0} \subset X_{\alpha_0}$, $\pi_\alpha^{-1}[O_{\alpha_0}] = \{x \in \prod_\alpha X_\alpha : x_{\alpha_0} \in O_{\alpha_0}\}$, which is open in $\prod_\alpha X_\alpha$. Hence each π_α is continuous. If $\mathcal{T}$ is a topology on X^A such that each π_α is continuous, then each basic element in the product topology is a finite intersection of preimages under some π_α of open sets in X. Thus each basic element in the product topology is in $\mathcal{T}$ and thus the product topology is contained in $\mathcal{T}$. Hence the product topology is the weakest topology such that each π_α is continuous.

47. The ternary expansion of numbers gives a homeomorphism between 2^ω and the Cantor ternary set.

48a. Each $x \in X$ can be identified with the element in $I^{\mathfrak{F}}$ with f-th coordinate $f(x)$. Let F be the mapping of X onto its image in $I^{\mathfrak{F}}$. Suppose each $f \in \mathfrak{F}$ is continuous. Given a basic element $\prod_f O_f \subset I^{\mathfrak{F}}$, $F^{-1}[\prod_f O_f] = \{x : F(x) \in \prod_f O_f\} = \{x : f(x) \in O_f$ for each $f\} = \bigcap f^{-1}[O_f]$. The intersection is in fact a finite intersection since all but finitely many of the sets O_f are I. Thus $F^{-1}[\prod_f O_f]$ is open and F is continuous. Further suppose that given a closed set F and $x \notin F$ there is $f \in \mathfrak{F}$ such that $f[F] = 0$ and $f(x) = 1$. Let U be open in X and let $y \in F[U]$. Now $y = F(x)$ for some $x \in U$ and there is $f \in \mathfrak{F}$ such that $f[U^c] = 0$ and $f(x) = 1$. Let $V = \pi_f^{-1}[(0, \infty)]$ and let $W = V \cap F[X]$. Then W is open. Also, $\pi_f(y) = f(x) = 1$ so $y \in W$. If $z \in W$, then $z = F(x)$ for some $x \in X$ with $f(x) > 0$ so $x \in U$. Thus $W \subset F[U]$. Thus $F[U]$ is open in $I^{\mathfrak{F}}$. Hence F is a homeomorphism.

***48b.** Suppose X is a normal space satisfying the second axiom of countability. Let $\{B_n\}$ be a countable base for X. For each pair of indices n, m such that $\overline{B_n} \subset B_m$, by Urysohn's Lemma, there exists a continuous function $g_{n,m}$ on X such that $g_{n,m} \equiv 1$ on $\overline{B_n}$ and $g_{n,m} \equiv 0$ on B_m^c. Given a closed set F and $x \notin F$, choose a basic element B_m such that $x \in B_m \subset F^c$. By Q23a, there exists B_n such that $x \in B_n$ and $\overline{B_n} \subset B_m$. Then $g_{n,m}$ is defined with $g_{n,m}(x) = 1$ and $g_{n,m}[F] = 0$. Furthermore the family $\{g_{n,m}\}$ is countable.

48c. Suppose X is a normal space satisfying the second axiom of countability. By part (b), there is a countable family $\mathfrak{F}$ of continuous functions with the property in part (a). Then there is a homeomorphism between X and $I^{\mathfrak{F}}$. By Q45, $I^{\mathfrak{F}}$ is metrizable and thus X is metrizable.

***49.** First we consider finite products of connected spaces. Suppose X and Y are connected and choose $\langle a, b\rangle \in X \times Y$. The subspaces $X \times \{b\}$ and $\{x\} \times Y$ are connected, being homeomorphic to X and Y respectively. Thus $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected for each $x \in X$. Then $\bigcup_{x\in X} T_x$ is the union of a collection of connected spaces having the point $\langle a, b\rangle$ in common so it is connected. But this union is $X \times Y$ so $X \times Y$ is connected. The result for any finite product follows by induction.

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of connected spaces and let $X = \prod_\alpha X_\alpha$. Fix a point $a \in X$. For any finite subset $K \subset A$, let X_K be the subspace consisting of points x such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Then X_K is homeomorphic to the finite product $\prod_{\alpha\in K} X_\alpha$ so it is connected. Let Y be the union of the sets X_K. Since any two of them have a point in common, by Q32, Y is connected. Let $x \in X$ and let $U = \prod_\alpha O_\alpha$ be a basic element containing x. Now $O_\alpha = X_\alpha$ for all but finitely many α so let K be that finite set. Let y be the element with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for $\alpha \notin K$. Then $y \in X_K \subset Y$ and $y \in U$. Hence $X = \bar{Y}$ so X is connected.

8.6 Topological and uniform properties

50a. Let $\langle f_n \rangle$ be a sequence of continuous maps from a topological space X to a metric space (Y, σ) that converges uniformly to a map f. Given $\varepsilon > 0$, there exists N such that $\sigma(f_n(x), f(x)) < \varepsilon/3$ for $n \geq N$ and $x \in X$. Given $x \in X$, there is an open set O containing x such that $\sigma(f_N(x), f_N(y)) < \varepsilon/3$ for $y \in O$. Then $\sigma(f(x), f(y)) \leq \sigma(f(x), f_N(x)) + \sigma(f_N(x), f_N(y)) + \sigma(f_N(y), f(y)) < \varepsilon$ for $y \in O$. Hence f is continuous.

50b. Let $\langle f_n \rangle$ be a sequence of continuous maps from a topological space X to a metric space (Y, σ) that is uniformly Cauchy. Suppose Y is complete. For each $x \in X$, the sequence $\langle f_n(x) \rangle$ is Cauchy so it converges since Y is complete. Let $f(x)$ be the limit of the sequence. Then $\langle f_n \rangle$ converges to f. Given $\varepsilon > 0$, there exists N such that $\sigma(f_n(x), f_m(x)) < \varepsilon/2$ for $n, m \geq N$ and $x \in X$. For each $x \in X$, there exists $N_x \geq N$ such that $\sigma(f_{N_x}(x), f(x)) < \varepsilon/2$. Then for $n \geq N$ and $x \in X$, we have $\sigma(f_n(x), f(x)) \leq \sigma(f_n(x), f_{N_x}(x)) + \sigma(f_{N_x}(x), f(x)) < \varepsilon$. Thus $\langle f_n \rangle$ converges uniformly to f and by part (a), f is continuous.

51. The proofs of Lemmas 7.37-39 remain valid when X is a separable topological space. Thus the Ascoli-Arzelá Theorem and its corollary are still true.

8.7 Nets

52. Suppose X is Hausdorff and suppose a net $\langle x_\alpha \rangle$ in X has two limits x, y. Then there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Since x and y are limits, there exist α_0 and α_1 such that $x_\alpha \in U$ for $\alpha \succ \alpha_0$ and $x_\alpha \in V$ for $\alpha \succ \alpha_1$. Choose α' such that $\alpha' \succ \alpha_0$ and $\alpha' \succ \alpha_1$. Then $x_\alpha \in U \cap V$ for $\alpha \succ \alpha'$. Contradiction. Hence every net in X has at most one limit. Conversely, suppose X is not Hausdorff. Let x, y be two points that cannot be separated and let the directed system be the collection of all pairs $\langle A, B \rangle$ of open sets with $x \in A$, $y \in B$. Choose $x_{\langle A,B \rangle} \in A \cap B$. Let O be an open set containing x and let O' be an open set containing y. For $\langle A, B \rangle \succ \langle O, O' \rangle$, we have $A \subset O$ and $B \subset O'$ so $x_{\langle A,B \rangle} \in A \cap B \subset O \cap O'$. Thus both x and y are limits.

53. Suppose f is continuous. Let $\langle x_\alpha \rangle$ be a net that converges to x. For any open set O containing $f(x)$, we have $x \in f^{-1}[O]$, which is open. There exists α_0 such that $x_\alpha \in f^{-1}[O]$ for $\alpha \succ \alpha_0$. Then $f(x_\alpha) \in O$ for $\alpha \succ \alpha_0$. Hence $\langle f(x_\alpha) \rangle$ converges to $f(x)$. Conversely, suppose that for each net $\langle x_\alpha \rangle$ converging to x the net $\langle f(x_\alpha) \rangle$ converges to $f(x)$. Then in particular the statement holds for all sequences. It follows that f is continuous.

***54.** Let X be any set and f a real-valued function on X. Let A be the system consisting of all finite subsets of X, with $F \prec G$ meaning $F \subset G$. For each $F \in A$, let $y_F = \sum_{x \in F} f(x)$. Suppose that $f(x) = 0$ except for x in a countable subset $\{x_n\}$ and $\sum |f(x_n)| < \infty$. Given $\varepsilon > 0$, there exists N such that $\sum_{n=N+1}^{\infty} |f(x_n)| < \varepsilon$. Let $y = \sum f(x_n)$. For any open interval $(y - \varepsilon, y + \varepsilon)$, let $F_0 = \{x_1, \ldots, x_N\}$. For $F \succ F_0$, $|y - y_F| \leq \sum_{n=N+1}^{\infty} |f(x_n)| < \varepsilon$ so $y_F \in (y - \varepsilon, y + \varepsilon)$. Thus $\lim y_F = y$. Conversely, if $f(x) \neq 0$ on an uncountable set G, then for some n, $|f(x)| > 1/n$ for uncountably many x. Thus by considering arbitrarily large finite subsets of G, we see that $\langle y_F \rangle$ does not converge. Hence $f(x) = 0$ except on a countable set. Now if $\sum |f(x_n)| = \infty$, then we only have $\sum f(x_n) < \infty$ and it follows that the limit is not unique. Hence $\sum |f(x_n)| < \infty$.

55. Let $X = \prod_\alpha X_\alpha$. Suppose a net $\langle x_\beta \rangle$ in X converges to x. Since each projection is continuous, each coordinate of x_β converges to the corresponding coordinate of x. Conversely, suppose each coordinate of x_β converges to the corresponding coordinate of x. Let $\prod_\alpha O_\alpha$ be a basic element containing x. Then $\pi_\alpha(x) \in O_\alpha$ for each α. For each α, there exists β_α such that $\pi_\alpha(x_\beta) \in O_\alpha$ for $\beta \succ \beta_\alpha$. Since all but finitely many of the O_α are X_α, we only need to consider a finite set $\beta_{\alpha_1}, \ldots, \beta_{\alpha_n}$. In particular, we may choose β_0 such that $\beta_0 \succ \beta_{\alpha_i}$ for all i. For $\beta \succ \beta_0$, we have $\pi_{\alpha_i}(x_\beta) \in O_{\alpha_i}$. For $\alpha \neq \alpha_i$, we also have $\pi_\alpha(x_\beta) \in O_\alpha = X_\alpha$. Thus $x_\beta \in \prod_\alpha O_\alpha$ for $\beta \succ \beta_0$. Hence the net $\langle x_\beta \rangle$ converges to x.

9 Compact and Locally Compact Spaces

9.1 Compact spaces

1. Suppose X is compact. Then every open cover has a finite subcovering. In particular, it has a finite refinement. Conversely, suppose every open cover has a finite refinement. Since every element in the refinement is a subset of an element in the open cover, the open cover has a finite subcovering so X is compact.

***2.** Let $\langle K_n \rangle$ be a decreasing sequence of compact sets with K_0 Hausdorff. Let O be an open set with $\bigcap K_n \subset O$. Suppose K_n is not a subset of O for any n. Then $K_n \setminus O$ is nonempty and closed in K_n. Since K_n is a compact subset of the Hausdorff space K_0, K_n is closed in K_0 so $K_n \setminus O$ is closed in K_0. Also, $\langle K_n \setminus O \rangle$ is a decreasing sequence so it has the finite intersection property. Now $\emptyset \neq (\bigcap K_n) \setminus O = \bigcap (K_n \setminus O) \subset \bigcap K_n \subset O$. Contradiction. Hence $K_n \subset O$ for some n.

(*) Assume K_0 is Hausdorff.

3. Suppose X is a compact Hausdorff space. Let F be a closed subset and let $x \notin F$. For each $y \in F$, there are disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. Now F is compact and $\{V_y : y \in F\}$ is an open cover for F. Thus there is a finite subcovering $\{V_{y_1}, \ldots, V_{y_n}\}$. Let $U = \bigcap_{i=1}^n U_{y_i}$ and $V = \bigcup_{i=1}^n V_{y_i}$. Then U and V are disjoint open sets with $x \in U$ and $F \subset V$. Hence X is regular.

4. Suppose X is a compact Hausdorff space. Let F and G be disjoint closed subsets. By Q3, for each $y \in G$, there are disjoint open sets U_y and V_y such that $F \subset U_y$ and $y \in V_y$. Now G is compact and $\{V_y : y \in G\}$ is an open cover for G. Thus there is a finite subcovering $\{V_{y_1}, \ldots, V_{y_n}\}$. Let $U' = \bigcap_{i=1}^n U_{y_i}$ and $V' = \bigcup_{i=1}^n V_{y_i}$. Then U' and V' are disjoint open sets with $F \subset U'$ and $G \subset V'$. Hence X is normal.

5a. If $(X, \mathcal{T})$ is a compact space, then for $\mathcal{T}_\infty$ weaker than $\mathcal{T}$, any open cover from $\mathcal{T}_1$ is an open cover from $\mathcal{T}$ so it has a finite subcovering. Thus $(X, \mathcal{T}_1)$ is compact.

5b. If $(X, \mathcal{T})$ is a Hausdorff space, then for $\mathcal{T}_\in$ stronger than $\mathcal{T}$, any two distinct points in X can be separated by disjoint sets in $\mathcal{T}$, which are also sets in $\mathcal{T}_\in$. Thus $(X, \mathcal{T}_2)$ is Hausdorff.

5c. Suppose $(X, \mathcal{T})$ is a compact Hausdorff space. If $\mathcal{T}_1$ is a weaker topology, then $id : (X, \mathcal{T}) \to (X, \mathcal{T}_1)$ is a continuous bijection. If $(X, \mathcal{T}_1)$ is Hausdorff, then id is a homeomorphism. Contradiction. Hence $(X, \mathcal{T}_1)$ is not Hausdorff. If $\mathcal{T}_2$ is a stronger topology, then $id : (X, \mathcal{T}_2) \to (X, \mathcal{T})$ is a continuous bijection. If $(X, \mathcal{T}_2)$ is compact, then id is a homeomorphism. Contradiction. Hence $(X, \mathcal{T}_2)$ is not compact.

6. Let X be a compact space and $\mathfrak{F}$ an equicontinuous family of maps from X to a metric space (Y, σ). Let $\langle f_n \rangle$ be a sequence from $\mathfrak{F}$ such that $f_n(x) \to f(x)$ for all $x \in X$. For each $x \in X$, given $\varepsilon > 0$, there exists N_x such that $\sigma(f_n(x), f(x)) < \varepsilon/3$ for $n \geq N_x$. Also, there exists an open set O_x containing x such that $\sigma(f_n(x), f_n(y)) < \varepsilon/3$ for $y \in O_x$ and all n. Then we also have $\sigma(f(x), f(y)) < \varepsilon/3$ for $y \in O_x$. Now $\{O_x : x \in X\}$ is an open cover for X so there is a finite subcovering $\{O_{x_1}, \ldots, O_{x_k}\}$. Let $N = \max_{1 \leq i \leq k} N_{x_i}$. For $n \geq N$, $\sigma(f_n(x_i), f(x_i)) < \varepsilon/3$ for $1 \leq i \leq k$. For each $x \in X$, $x \in O_{x_i}$ for some i so $\sigma(f_n(x), f(x)) \leq \sigma(f_n(x), f_n(x_i)) + \sigma(f_n(x_i), f(x_i)) + \sigma(f(x_i), f(x)) < \varepsilon$ for $n \geq N$. Hence $\langle f_n \rangle$ converges uniformly to f on X.

***7.** Let X be a Hausdorff space and $\langle C_n \rangle$ a decreasing sequence of compact and connected sets. Let $C = \bigcap C_n$. For any open cover $\mathcal{U}$ of C, $\mathcal{U}$ is an open cover of some C_n by Q2. Thus it has a finite subcovering, which also covers C. Hence C is compact.

Suppose C is disconnected with A and B being a separation for C. Then A and B are nonempty disjoint closed subsets of C. Since C is an intersection of closed sets, C is closed. Thus A and B are closed in C_0. Since C_0 is compact and Hausdorff, it is normal. Thus there are disjoint open sets $U, V \subset C_0$ such that $A \subset U$ and $B \subset V$. Then $C = A \cup B \subset U \cup V$. By Q2, $C_n \subset U \cup V$ for some n. Hence C_n is disconnected. Contradiction. Hence C is connected.

(*) Assume X is Hausdorff

8a. Let $\langle f_n \rangle$ be a sequence of maps from X to Y that converge in the compact-open topology to f. For $x \in X$, let O be an open set containing $f(x)$. Then $N_{\{x\},O}$ is open in the compact-open topology and contains f. There exists N' such that $f_n \in N_{\{x\},O}$ for $n \geq N'$. i.e. $f_n(x) \in O$ for $n \geq N'$. Hence $f(x) = \lim f_n(x)$.

***8b.** Let $\langle f_n \rangle$ be a sequence of continuous maps from a topological space X to a metric space (Y, σ). Suppose $\langle f_n \rangle$ converges to f uniformly on each compact subset C of X. Let $N_{K,O}$ be a subbasic element containing f. Then $f[K]$ is a compact set disjoint from O^c so $\sigma(f[K], O^c) > 0$. Let $\varepsilon = \sigma(f[K], O^c)$. There exists N' such that $\sigma(f_n(x), f(x)) < \varepsilon$ for $n \geq N'$ and $x \in K$. Then $f_n(x) \in O$ for $n \geq N'$ and $x \in K$. i.e. $f_n \in N_{K,O}$ for $n \geq N'$. Hence $\langle f_n \rangle$ converges to f in the compact-open topology.

Conversely, suppose $\langle f_n \rangle$ converges to f in the compact-open topology. Let C be a compact subset of X and let $\varepsilon > 0$ be given. Since C is compact and f is continuous, $f[C]$ is compact. Thus there exist $x_1, \ldots, x_n \in C$ such that the open balls $B_{f(x_1),\varepsilon/4}, \ldots, B_{f(x_n),\varepsilon/4}$ cover $f[C]$. For each i, let $C_i = C \cap f^{-1}[\overline{B_{f(x_i),\varepsilon/4}}]$ and $O_i = B_{f(x_i),\varepsilon/4}$. Then C_i is compact and $f[C_i] \subset O_i$. Thus $f \in \bigcap_{i=1}^n N_{C_i,O_i}$. There exists N' such that $f_n \in \bigcap_{i=1}^n N_{C_i,O_i}$ for $n \geq N'$. For any $x \in C$, $x \in C_i$ for some i so $\sigma(f(x_i), f(x)) < \varepsilon/4$. Also, $f_n(x) \in O_i$ for $n \geq N'$ so $\sigma(f_n(x), f(x_i)) < \varepsilon/4$ for $n \geq N'$. Thus $\sigma(f_n(x), f(x)) < \varepsilon$ for $n \geq N'$ and $x \in C$. Hence $\langle f_n \rangle$ converges uniformly to f on C.

(*) Assume f to be continuous.

9.2 Countable compactness and the Bolzano-Weierstrass property

9a. A real-valued function f on X is continuous if and only if $\{x : f(x) < \alpha\}$ and $\{x : f(x) > \alpha\}$ are open for any real number α if and only if f is both upper semicontinuous and lower semicontinuous.

9b. Suppose f and g are upper semicontinuous. Then $\{x : f(x) < \alpha\}$ and $\{x : g(x) < \alpha\}$ are open for any real number α. Now for any real number α, $\{x : f(x) + g(x) < \alpha\} = \bigcup_{q \in \mathbb{Q}} [\{x : g(x) < q\} \cap \{x : f(x) < \alpha - q\}]$, which is open. Hence $f + g$ is upper semicontinuous.

9c. Let $\langle f_n \rangle$ be a decreasing sequence of upper semicontinuous functions which converge pointwise to a real-valued function f. If $f(x) < \alpha$, then there exists N such that $f_n(x) - f(x) < \alpha - f(x)$ for $n \geq N$. i.e. $f_n(x) < \alpha$ for $n \geq N$. On the other hand, if $f_n(x) < \alpha$ for some n, then $f(x) \leq f_n(x) < \alpha$. Thus $\{x : f(x) < \alpha\} = \bigcup_n \{x : f_n(x) < \alpha\}$, which is open. Hence f is upper semicontinuous.

9d. Let $\langle f_n \rangle$ be a decreasing sequence of upper semicontinuous functions on a countably compact space, and suppose that $\lim f_n(x) = f(x)$ where f is a lower semicontinuous real-valued function. By part (c), f is also upper semicontinuous so f is continuous. Now $\langle f_n - f \rangle$ is a sequence of upper semicontinuous functions on a countably compact space such that for each x, $\langle f_n(x) - f(x) \rangle$ decreases to zero. Thus by Proposition 11 (Dini), $\langle f_n - f \rangle$ converges to zero uniformly. i.e. $\langle f_n \rangle$ converges to f uniformly.

9e. Suppose a sequence $\langle f_n \rangle$ of upper semicontinuous functions converges uniformly to a function f. Fix $y \in X$. Given $\varepsilon > 0$, there exists N such that $|f_N(x) - f(x)| < \varepsilon/3$ for all $x \in X$. Since $\{x : f_N(x) < f_N(y) + \varepsilon/3\}$ is open, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $f_N(x) - f_N(y) < \varepsilon/3$. Now if $|x - y| < \delta$, then $f(x) - f(y) = [f(x) - f_N(x)] + [f_N(x) - f_N(y)] + [f_N(y) - f(y)] < \varepsilon$. Given $\alpha \in \mathbb{R}$, pick $y \in \{x : f(x) < \alpha\}$. There exists $\delta > 0$ such that $|x - y| < \delta$ implies $f(x) - f(y) < \alpha - f(y)$. i.e. $f(x) < \alpha$. Hence $\{x : f(x) < \alpha\}$ is open and f is upper semicontinuous.

***10** (i) $\Rightarrow$ (iii) by Proposition 9. Suppose that every bounded continuous real-valued function on X assumes its maximum. Let f be a continuous function and suppose it is unbounded. We may assume, by taking $\max(1, f)$, that $f \geq 1$. Then $-1/f$ is a bounded continuous function with no maximum. Thus (iii) $\Rightarrow$ (ii). Suppose X is no countably compact. Then it does not have the Bolzano-Weierstrass property. There is a sequence $\langle x_n \rangle$ in X with no cluster point. Thus the sequence has no limit points. Let $A = \{x_n\}$. Then A is closed and since all subsets of A are also closed, A is discrete. Define $f(x_n) = n$ for $x_n \in A$. Then f is continuous on the closed set A and since X is normal, by Tietze's Extension Theorem, there is a continuous function g on X such that $g|_A = f$. Then g is an unbounded continuous function on X. Thus (ii) $\Rightarrow$ (i).

11a. Let X be the set of ordinals less than the first uncountable ordinal and let $\mathcal{B}$ be the collection of sets of the form $\{x : x < a\}, \{x : a < x < b\}, \{x : a < x\}$. For any $x_0 \in X$, $x_0 \in \{x : x < x_0 + 1\}$. Now $\{x : x < b\} \cap \{x : a < x\} = \{x : a < x < b\}$, $\{x : x < c\} \cap \{x : a < x < b\} = \{x : a < x < \min(b, c)\}$, $\{x : \max(a, c) < x < b\} = \{x : a < x < b\} \cap \{x : c < x\}$. Hence $\mathcal{B}$ is a base for a topology on X.

***11b.** For any sequence $\langle x_n \rangle$ in X, let $x_0 = \sup x_n$. Then $x_n < x_0 + 1$ for all n. i.e. $x_n \in \{x : x < x_0 + 1\}$ for all n so the sequence converges to x_0. Hence X is sequentially compact. The sets $\{x : x < a\}$ form an open cover for X. If it has a finite subcovering, then there exists $a \in X$ such that $x < a$ for all $x \in X$. Contradiction. Hence X is not compact.

***11c.** Let f be a continuous real-valued function on X. We first show the existence of a sequence $\langle x_n \rangle$ in X such that $|f(y) - f(x_n)| < 1/n$ for $y > x_n$. If no such sequence exists, we may construct for some

N an increasing sequence $\langle z_n \rangle$ such that $|f(z_n) - f(z_{n-1})| \geq 1/N$ for each n. Then $\langle z_n \rangle$ converges to its supremum but $\langle f(z_n) \rangle$ does not converge, contradicting the continuity of f. Now let $x_0 = \sup x_n$. Then $f(x) = f(x_0)$ for $x \geq x_0$.

12a. Similar argument as Q11a.

***12b.** Let $\mathcal{U}$ be an open cover for Y consisting of basic sets. Let $L_a = \{x : x < a\}$ and $U_b = \{x : b < x\}$. Then $\omega_1 \in U_b$ for some $U_b \in \mathcal{U}$. Also, $0 \in L_a$ for some $L_a \in \mathcal{U}$. Let $a_0 = \sup\{a : L_a \in \mathcal{U}\}$. Then $a_0 \in U_{b_0}$ for some $U_{b_0} \in \mathcal{U}$ so $b_0 < a_0$. Then there exists $a_1 > b_0$ such that $L_{a_1} \in \mathcal{U}$. Since $y > b_0$ or $y < a_1$ for any $y \in Y$, $\{L_{a_1}, U_{b_0}\}$ is a finite subcovering of $\mathcal{U}$ that covers Y. Hence Y is compact.

Let D be a countable subset of Y. Then $y = \sup D$ exists and $y \in Y$. Now $\{x : y < x\}$ is an open set in Y that does not intersect D. Thus D is not dense in Y and Y is not separable. Suppose there is a countable base $\{U_n\}$ at ω_1. Then each U_n is of the form $\{x : x > a_n\}$. There exists $a < \omega_1$ such that $a > a_n$ for all n. Then the open set $\{x : x > a\}$ does not contain any U_n. Contradiction. Thus Y is not first countable.

9.3 Products of compact spaces

13. Each closed and bounded subset X of $\mathbb{R}^n$ is contained in a cube I^n where $I = [a, b]$. Each I is compact in $\mathbb{R}$ so I^n is compact in $\mathbb{R}^n$ by Tychonoff's Theorem. Now X is closed in I^n so X is compact.

***14.** Suppose X is compact and I is a closed interval. Let $\mathcal{U}$ be an open cover of $X \times I$ and let $t \in I$. Since $X \times \{t\}$ is homeomorphic to X, $X \times \{t\}$ is compact so there is a finite subcovering $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ such that $U = \bigcup_{i=1}^n U_i \supset X \times \{t\}$. Now $X \times \{t\}$ can be covered by finitely many basic sets $A_1 \times B_1, \ldots, A_k \times B_k \subset U$. Then $B = B_1 \cap \cdots \cap B_k$ is an open set containing t. If $\langle x, y \rangle \in X \times B$, then $\langle x, t \rangle \in A_j \times B_j$ for some j so $x \in A_j$ and $y \in \bigcap_{i=1}^n B_i \subset B_j$. Thus $\langle x, y \rangle \in A_j \times B_j$ and $X \times B \subset \bigcup_{i=1}^n (A_i \times B_i) \subset U$. Thus for each $t \in I$, there is an open set B_t containing t such that $X \times B_t$ can be covered by finitely many elements of $\mathcal{U}$. The collection of the sets B_t forms an open cover of X so there is a finite subcollection $\{B_{t_1}, \ldots, B_{t_m}\}$ covering X. Now $X \times I = \bigcup_{i=1}^m (B_{t_i} \times I)$, which can then be covered by finitely many elements of $\mathcal{U}$.

15. Let $\langle X_n \rangle$ be a countable collection of sequentially compact spaces and let $X = \prod X_n$. Given a sequence $\langle x_n \rangle$ in X, there is a subsequence $\langle x_n^{(1)} \rangle$ whose first coordinate converges. Then there is a subsequence $\langle x_n^{(2)} \rangle$ of $\langle x_n^{(1)} \rangle$ whose second coordinate converges. Consider the diagonal sequence $\langle x_n^{(n)} \rangle$. Each coordinate of this sequence converges so the sequence converges in X. Hence X is sequentially compact.

16. Let X be a compact Hausdorff space. Let $\mathfrak{F}$ be the family of continuous real-valued functions on X with values in $[0, 1]$. Let $Q = \prod_{f \in \mathfrak{F}} I_f$. Consider the mapping g of X into Q mapping x to the point whose f-th coordinate is $f(x)$. If $x \neq y$, then since X is compact Hausdorff, and thus normal, by Urysohn's Lemma, there exists $f \in \mathfrak{F}$ such that $f(x) = 0$ and $f(y) = 1$. Thus g is one-to-one. Since f is continuous for each $f \in \mathfrak{F}$, g is continuous. Now $g[X]$ is a compact subset of the Hausdorff space Q so $g[X]$ is closed in Q. Furthermore, g is a continuous bijection from the compact space X onto the Hausdorff space $g[X]$ so X is homeomorphic to $g[X]$.

***17.** Let $Q = I^A$ be a cube and let f be a continuous real-valued function on Q. Given $\varepsilon > 0$, cover $f[Q]$ by finitely many open intervals $I_1, \ldots, I_n$ of length ε. Consider $f^{-1}[I_j]$ for $j = 1, \ldots, n$. These sets cover Q and we may assume each of them is a basic set, that is, $f^{-1}[I_j] = \prod_\alpha U_\alpha^{(j)}$ where $U_\alpha^{(j)}$ is open in I and all but finitely many of the $U_\alpha^{(j)}$ are I. For each j, let $F_j = \{\alpha : U_\alpha^{(j)} \neq I\}$ and let $F = \bigcup_{j=1}^n F_j$, which is a finite set. Define $h : Q \to Q$ by $\pi_\alpha h(x) = x_\alpha$ for $\alpha \in F$ and 0 otherwise. Define $g = f \circ h$. Then g depends only on F and $|f - g| < \varepsilon$.

9.4 Locally compact spaces

18. Let X be a locally compact space and K a compact subset of X. For each $x \in K$, there is an open set O_x containing x with $\overline{O_x}$ compact. Since K is compact, there is a finite subcollection $\{O_{x_1}, \ldots, O_{x_n}\}$ that covers K. Let $O = \bigcup_{i=1}^n O_{x_i}$. Then $O \supset K$ and $\bar{O} = \bigcup_{i=1}^n \overline{O_{x_i}}$ is compact.

***19a.** Let X be a locally compact Hausdorff space and K a compact subset. Then K is closed in

X^*. By Q8.23a, there exists a closed set D containing ω with $D \cap K = \emptyset$. By Urysohn's Lemma, there is a continuous function g on X^* with $0 \le g \le 1$ that is 1 on K and 0 on D. Define $f(x) = \min(2(g(x) - 1/2), 0)$. Then $\overline{\{x : f(x) > 0\}} = \overline{\{x : g(x) > 1/2\}}$, which is compact because g is continuous on X^*.

(*) Alternatively, use Q16 to regard K as a closed subset of the compact Hausdorff space Q.

***19b.** Let K be a compact subset of a locally compact Hausdorff space X. Then by Q18, there is an open set $O \supset K$ with $\bar{O}$ compact. Now $X^* \setminus O$ and K are disjoint closed subsets of X^* so by Urysohn's Lemma, there is a continuous function g on X^* with $0 \le g \le 1$ that is 1 on K and 0 on $X^* \setminus O$. Then $f = g|_X$ is the required function.

20a. Let X^* be the Alexandroff one-point compactification of a locally compact Hausdorff space X. Consider the collection of open sets of X and complements of compact subsets of X. Then $\emptyset$ and X^* are in the collection. If U_1 and U_2 are open in X, then so is $U_1 \cap U_2$. If K_1 and K_2 are compact subsets of X, then $(X^* \setminus K_1) \cap (X^* \setminus K_2) = X^* \setminus (K_1 \cup K_2)$ where $K_1 \cup K_2$ is compact. Also, $U_1 \cap (X^* \setminus K_1) = U_1 \cap (X \setminus K_1)$, which is open in X. Thus the collection of sets is closed under finite intersection. If $\{U_\alpha\}$ is a collection of open sets in X, then $\bigcup U_\alpha$ is open in X. If $\{K_\beta\}$ is a collection of compact subsets of X, then $\bigcup(X^* \setminus K_\beta) = X^* \setminus \bigcap K_\beta$. Since each K_β is compact in the Hausdorff space X, each K_β is closed in X and $\bigcap K_\beta$ is closed in X. Thus $\bigcap K_\beta$ is closed in each K_β so $\bigcap K_\beta$ is compact. Finally, $\bigcup U_\alpha \cup \bigcup(X^* \setminus K_\beta) = U \cup (X^* \setminus K) = X^* \setminus (K \setminus U)$, where $U = \bigcup U_\alpha$ and $K = \bigcap K_\beta$. Since $K \setminus U$ is closed in the compact set K, $K \setminus U$ is compact. Thus the collection of sets is closed under arbitrary union. Hence the collection of sets forms a topology for X^*.

20b. Let id be the identity mapping from X to $X^* \setminus \{\omega\}$. Clearly id is a bijection. If U is open in $X^* \setminus \{\omega\}$, then $U = (X^* \setminus \{\omega\}) \cap U'$ for some U' open in X^*. If U' is open in X, then $U = U'$ so U is open in X. If $U' = X^* \setminus K$ for some compact $K \subset X$, then $U = X \setminus K$, which is open in X. Thus id is continuous. Also, any open set in X is open in X^* so id is an open mapping. Hence id is a homeomorphism.

20c. Let $\mathcal{U}$ be an open cover of X^*. Then $\mathcal{U}$ contains a set of the form $X^* \setminus K$ for some compact $K \subset X$. Take the other elements of $\mathcal{U}$ and intersect each of them with X to get an open cover of K. Then there is a finite subcollection that covers K. The corresponding finite subcollection of $\mathcal{U}$ together with $X^* \setminus K$ then covers X^*. Hence X^* is compact.

Let x, y be distinct points in X^*. If $x, y \in X$, then there are disjoint open sets in X, and thus in X^*, that separate x and y. If $x \in X$ and $y = \omega$, then there is an open set O containing x with $\bar{O}$ compact. The sets O and $X^* \setminus \bar{O}$ are disjoint open sets in X^* separating x and y. Hence X^* is Hausdorff.

***21a.** Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Let $p = \langle 0, \ldots, 0, 1\rangle \in \mathbb{R}^{n+1}$. Define $f : S^n - p \to \mathbb{R}^n$ by $f(x) = \frac{1}{1-x_{n+1}}\langle x_1, \ldots, x_n\rangle$. The map $g : \mathbb{R}^n \to S^n - p$ defined by $g(y) = \langle t(y)y_1, \ldots, t(y)y_n, 1 - t(y)\rangle$ where $t(y) = 2/(1+||y||^2)$ is the inverse of f. Thus $\mathbb{R}^n$ is homeomorphic to $S^n - p$ and the Alexandroff one-point compactification of $\mathbb{R}^n$ is homeomorphic to the Alexandroff one-point compactification of $S^n - p$, which is S^n.

21b. Let X be the space in Q11 and Y be the space in Q12. Define $f : X^* \to Y$ by $f(x) = x$ for $x \in X$ and $f(\omega) = \omega_1$. Then f is clearly a bijection. Consider the basic sets in Y. Now $f^{-1}[\{x : x < a\}] = \{x : x < a\}$, which is open in X and thus open in X^*. Similarly for sets of the form $\{x : a < x < b\}$. Also, $f^{-1}[\{x : a < x\}] = \{x \in X : a < x\} \cup \{\omega\}$, whose complement $\{x : x \le a\}$ is compact. Thus $f^{-1}[\{x : a < x\}]$ is open in X^*. Since f is a continuous bijection from the compact space X^* to the Hausdorff space Y, f is a homeomorphism. Hence the one-point compactification of X is Y.

22a. Let O be an open subset of a compact Hausdorff space X. By Q8.23a, for any $x \in O$, there is an open set U such that $x \in U$ and $\bar{U} \subset O$. Then the closure of U in X is the same as the closure of U in O and $\bar{U}$ is compact, being closed in a compact space. Hence O is locally compact.

22b. Let O be an open subset of a compact Hausdorff space X. Consider the mapping f of X to the one-point compactification of O which is identity on O and takes each point in $X \setminus O$ to ω. If U is open in O, then $f^{-1}[U] = U$ is open in X. If $K \subset O$ is compact, then $f^{-1}[O^* \setminus K] = X \setminus K$, which is open in X. Hence f is continuous.

23. Let X and Y be locally compact Hausdorff spaces, and f a continuous mapping of X into Y. Let X^* and Y^* be the one-point compactifications of X and Y, and f^* the mapping of X^* into Y^* whose restriction to X is f and which takes the point at infinity in X^* to the point at infinity in Y^*. Suppose

f is proper. If U is open in Y, then $(f^*)^{-1}[U] = f^{-1}[U]$, which is open in X and thus open in X^*. If $K \subset Y$ is compact, then $(f^*)^{-1}[Y^* \setminus K] = f^{-1}[Y \setminus K] \cup \{\omega_X\} = X^* \setminus f^{-1}[K]$, which is open in X^* since $f^{-1}[K] \subset X$ is compact. Hence f^* is continuous. Conversely, suppose f^* is continuous. Then $f = f^*|_X$ is continuous. Also, for any compact set $K \subset Y$, $(f^*)^{-1}[Y^* \setminus K] = X^* \setminus f^{-1}[K]$ is open in X^*. Hence $f^{-1}[K] \subset X$ is compact.

***24a.** Let X be a locally compact Hausdorff space. Suppose F is a closed subset of X. For each closed compact set K, $F \cap K$ is closed. Conversely, suppose F is not closed. Take $x \in \bar{F} \setminus F$. There is an open set O containing x with $\bar{O}$ compact. For any open set U containing x, $(O \cap U) \cap F \neq \emptyset$. Thus $U \cap (F \cap \bar{O}) \neq \emptyset$. Then $x \in \overline{F \cap \bar{O}} \setminus (F \cap \bar{O})$ so $F \cap \bar{O}$ is not closed.

***24b.** Let X be a Hausdorff space satisfying the first axiom of countability. Suppose F is a closed subset of X. For each closed compact set K, $F \cap K$ is closed. Conversely, suppose F is not closed. Take $x \in \bar{F} \setminus F$. Since X is first countable, there is a sequence $\langle x_n \rangle$ in F converging to x. Let $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Then K is compact in the Hausdorff space X and thus closed. Now $F \cap K = \{x_n : n \in \mathbb{N}\}$ is not closed.

25. Let $\mathfrak{F}$ be a family of real-valued continuous functions on a locally compact Hausdorff space X, and suppose that $\mathfrak{F}$ has the following properties: (i) If $f, g \in \mathfrak{F}$, then $f + g \in \mathfrak{F}$. (ii) If $f, g \in \mathfrak{F}$, then $f/g \in \mathfrak{F}$ provided that *support* $f \subset \{x \in X : g(x) \neq 0\}$. (iii) Given an open set $O \subset X$ and $x_0 \in O$, there is an $f \in \mathfrak{F}$ with $f(x_0) = 1, 0 \leq f \leq 1$ and *support* $f \subset O$.

Let $\{O_\lambda\}$ be an open covering of a compact subset K of a locally compact Hausdorff space X. Let O be an open set with $K \subset O$ and $\bar{O}$ compact. For each $x_0 \in K$, there is an $f_{x_0} \in \mathfrak{F}$ with $f_{x_0}(x_0) = 1, 0 \leq f_{x_0} \leq 1$ and *support* $f_{x_0} \subset O \cap O_\lambda$ for some λ. For each $x_0 \in \bar{O} \setminus K$, there is a $g_{x_0} \in \mathfrak{F}$ with $g_{x_0}(x_0) = 1, 0 \leq g_{x_0} \leq 1$ and *support* $g_{x_0} \subset K^c$. By compactness of $\bar{O}$, we may choose a finite number $f_1, \ldots, f_n, g_1, \ldots, g_m$ of these functions such that the sets where they are positive cover $\bar{O}$. Set $f = \sum_{i=1}^n f_i$ and $g = \sum_{i=1}^m g_i$. Then $f, g \in \mathfrak{F}$, $f > 0$ on K, *support* $f \subset O$, $f + g > 0$ on $\bar{O}$ and $g \equiv 0$ on K. Thus $f/(f+g) \in \mathfrak{F}$ is continuous and $\equiv 1$ on K. The functions $\varphi_i = f_i/(f+g) \in \mathfrak{F}, i = 1, \ldots, n$ form a finite collection of functions subordinate to the collection $\{O_\lambda\}$ and such that $\varphi_1 + \cdots \varphi_n \equiv 1$ on K.

***26.** Lemma: Let X be a locally compact Hausdorff space and U be an open set containing $x \in X$. Then there is an open set V containing x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Proof: There is an open set $O \subset X$ containing x such that $\bar{O}$ is compact. Then $U \cap O$ is open in $\bar{O}$ and contains x. Thus there is an open set $O' \subset \bar{O}$ such that $x \in O'$ and $\bar{O'}_{\bar{O}} \subset U \cap O$. Note that $\bar{O'}_{\bar{O}} = \bar{O} \cap \bar{O'} = \bar{O'}$. Thus $\bar{O'} \subset U \cap O$. Let $V = O \cap O'$. Then $x \in V$. Furthermore, since O' is open in $\bar{O}$, it is open in O and thus open in X so V is open in X. Also, $\bar{V} \subset \bar{O'} \subset U \cap O \subset U$. Since $\bar{V}$ is closed in $\bar{O}$, $\bar{V}$ is compact. $\square$

Let X be a locally compact Hausdorff space and $\{O_n\}$ a countable collection of dense open sets. Given an open set U, let x_1 be a point in $O_1 \cap U$. Let V_1 be an open set containing x_1 such that $\overline{V_1}$ is compact and $\overline{V_1} \subset O_1 \cap U$. Suppose $x_1, \ldots, x_n$ and $V_1, \ldots, V_n$ have been chosen. Let $x_{n+1} \in O_{n+1} \cap V_n$ and let V_{n+1} be an open set containing x_{n+1} such that $\overline{V_{n+1}}$ is compact and $\overline{V_{n+1}} \subset O_{n+1} \cap V_n$. Then $\overline{V_1} \supset \overline{V_2} \supset \cdots$ is a decreasing sequence of closed sets in the compact set $\overline{V_1}$. This collection of closed sets has the finite intersection property so $\bigcap \overline{V_n} \neq \emptyset$. Let $y \in \bigcap \overline{V_n}$. Then $y \in \bigcap O_n \cap U$. Hence $\bigcap O_n$ is dense in X.

(*) Assume that X is Hausdorff.

***27.** Let X be a locally compact Hausdorff space and let O be an open subset contained in a countable union $\bigcup F_n$ of closed sets. Note that O is a locally compact Hausdorff space (see Q29b). Also, $O = \bigcup (O \cap F_n)$, which is a union of sets closed in O. If $\bigcup (O \cap F_n)^\circ = \emptyset$, then $(O \cap F_n)^\circ = \emptyset$ for all n (with the interior taken in O) so $O \setminus F_n$ is dense and open in O for all n. By Q26, $\bigcap (O \setminus F_n)$ is dense in O. But $\bigcap (O \setminus F_n) = O \setminus (\bigcup F_n)^c = \emptyset$. Contradiction. Hence $\bigcup (O \cap F_n)^\circ \neq \emptyset$. Now $O \cap \bigcup F_n^\circ \supset \bigcup (O \cap F_n)^\circ$ so $O \cap \bigcup F_n^\circ \neq \emptyset$ and $\bigcup F_n^\circ$ is an open set dense in O.

(*) Assume that X is Hausdorff.

***28.** Let Y be a dense subset of a Hausdorff space X, and suppose that Y with its subspace topology is locally compact. Given $y \in Y$, there is an open set $U \subset Y$ containing y with $\bar{U}_Y = Y \cap \bar{U}$ compact. Since X is Hausdorff, $Y \cap \bar{U}$ is closed. Then since $U \subset Y \cap \bar{U}$, we have $\bar{U} \subset Y \cap \bar{U} \subset Y$. Now $U = V \cap Y$ for some open set $V \subset X$. Note that since Y is dense, $\bar{V} = \overline{V \cap Y}$. Then $x \in V$ and $V \subset \bar{V} = \overline{V \cap Y} = \bar{U} \subset Y$. Hence Y is open in X.

29a. Suppose F is closed in a locally compact space X. Given $x \in F$, there is an open set $O \subset X$

containing x with $\bar{O}$ compact. Then $O \cap F$ is an open set in F containing y and $\overline{O \cap F}_F = F \cap \overline{O \cap F}$ is closed in $\bar{O}$ and thus compact. Hence F is locally compact.

***29b.** Suppose O is open in a locally compact Hausdorff space X. Take $x \in O$. By the lemma in Q26, there is an open set U containing x such that $\bar{U}$ is compact and $\bar{U} \subset O$. Now $U \cap O$ is an open set in O containing x with $\overline{U \cap O}_O = O \cap \overline{U \cap O} = \overline{U \cap O}$ being compact since it is closed in the compact set $\bar{U}$. Hence O is locally compact.

29c. Suppose a subset Y of a locally compact Hausdorff space X is locally compact in its subspace topology. Then Y is dense in the Hausdorff space $\bar{Y}$. By Q28, Y is open in $\bar{Y}$. Conversely, suppose Y is open in $\bar{Y}$. By part (a), $\bar{Y}$ is locally compact since $\bar{Y}$ is closed in X. By part (b), Y is locally compact since Y is open in $\bar{Y}$.

9.5 σ-compact spaces

30. Let X be a locally compact Hausdorff space. Suppose there is a sequence $\langle O_n \rangle$ of open sets with $\overline{O_n}$ compact, $\overline{O_n} \subset O_{n+1}$ and $X = \bigcup O_n$. For each n, let φ_n be a continuous real-valued function with $\varphi_n \equiv 1$ on $\overline{O_{n-1}}$ and $support \varphi_n \subset O_n$. Define $\varphi : X \to [0, \infty)$ by $\varphi = \sum (1 - \varphi_n)$.

Given $y \in X$ and $\varepsilon > 0$, $y \in O_N$ for some N and thus $y \in O_n$ for $n \geq N$. There exists N' such that $\varphi(y) - \sum_{n=1}^{N'} (1 - \varphi_n(y)) < \varepsilon/2$. Let $N'' = \max(N, N')$. Take $x \in O_{N''}$. Then $\varphi_{N''+1}(x) = 1$. In fact, $\varphi_n(x) = 1$ for $n \geq N'' + 1$ so $\varphi(x) = \sum_{n=1}^{N''} (1 - \varphi_n(x))$. For each $n = 1, \ldots, N''$, there is an open set U_n containing y such that $|\varphi_n(y) - \varphi_n(x)| < \varepsilon/2N''$ for $x \in U_n$. Let $U = \bigcap_{n=1}^{N''} U_n \cap O_{N''}$. Then U is an open set containing y and for $x \in U$, $|\varphi(y) - \varphi(x)| \leq [\varphi(y) - \sum_{n=1}^{N''} (1 - \varphi_n(y))] + |\sum_{n=1}^{N''} (1 - \varphi_n(y)) - \varphi(x)| \leq [\varphi(y) - \sum_{n=1}^{N''} (1 - \varphi_n(y))] + \sum_{n=1}^{N''} |\varphi_n(y) - \varphi_n(x)| < \varepsilon$. Hence φ is continuous.

To show that φ is proper, we consider closed bounded intervals in $[0, \infty)$. By considering $\overline{O_{N''-1}}$ and open sets V_n with $\overline{V_n}$ compact and $\overline{V_n} \subset U_n$, we then apply a similar argument as above.

31a. Let (X, ρ) be a proper locally compact metric space. If K is a compact subset, then K is closed and bounded by Proposition 7.22. Conversely, suppose a subset K is closed and bounded. Since X is proper, the closed balls $\{x : \rho(x, x_0) \leq a\}$ are compact for some x_0 and all $a \in (0, \infty)$. Since K is bounded, there exist x_1 and b such that $\rho(x, x_1) \leq b$ for all $x \in K$. Then $\rho(x, x_0) \leq b + \rho(x_1, x_0)$ for all $x \in K$. Thus K is a closed subset of the compact set $\{x : \rho(x, x_0) \leq b + \rho(x_1, x_0)\}$ so K is compact.

31b. Let (X, ρ) be a proper locally compact metric space. Then the closed balls $\{x : \rho(x, x_0) \leq a\}$ are compact for some x_0 and all $x \in (0, \infty)$. Note that a compact subset $K \subset [0, \infty)$ is closed and bounded. Also, the function $f(x) = \rho(x, x_0)$ is continuous from X to $[0, \infty)$. Now $f^{-1}[K]$ is bounded since K is bounded and closed since f is continuous and K is closed. Thus by part (a), $f^{-1}[K]$ is compact. Hence $f : X \to [0, \infty)$ is a proper continuous map and X is σ-compact.

31c. Let (X, ρ) be a σ-compact and locally compact metric space. Then there is a proper continuous map $\varphi : X \to [0, \infty)$. Define $\rho^*(x, y) = \rho(x, y) + |\varphi(x) - \varphi(y)|$. Then ρ^* is a metric on X. Given $x \in X$ and $\varepsilon > 0$, there exists $\delta' > 0$ such that $\rho(x, y) < \delta'$ implies $|\varphi(x) - \varphi(y)| < \varepsilon/2$. Choose $\delta < \min(\delta', \varepsilon/2)$. When $\rho(x, y) < \delta$, $\rho^*(x, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and when $\rho^*(x, y) < \delta$, $\rho(x, y) \leq \rho^*(x, y) < \varepsilon$. Thus ρ and ρ^* are equivalent metrics. For a fixed $x_0 \in X$ and any $a \in (0, \infty)$, $\{x : \rho^*(x, x_0) \leq a\} \subset \{x : \varphi(x) \leq a + \varphi(x_0)\}$, which is compact. Thus $\{x : \rho(x, x_0) \leq a\}$ is compact. Hence ρ^* is a proper metric.

9.6 Paracompact spaces

32. Let $\{E_\lambda\}$ be a locally finite collection of subsets of a topological space X, and set $E = \bigcup E_\lambda$. Since $E_\lambda \subset E$ for all λ, $\overline{E_\lambda} \subset \bar{E}$ for all λ. Thus $\bigcup \overline{E_\lambda} \subset \bar{E}$. If $x \in \bar{E}$, then there is an open set containing x that meets only a finite number of sets $E_{\lambda_1}, \ldots, E_{\lambda_n}$ so $x \in \bigcup_{i=1}^n \overline{E_{\lambda_i}} \subset \bigcup \overline{E_\lambda}$. Thus $\bar{E} \subset \bigcup \overline{E_\lambda}$. Hence $\bar{E} = \bigcup \overline{E_\lambda}$.

(*) Proof of Lemma 22.

33. Let $\{E_\lambda\}$ be a locally finite collection of subsets of X and K a compact subset of X. For each $x \in X$, there is an open set O_x containing x that meets only a finite number of sets in $\{E_\lambda\}$. Now $K \subset \bigcup_{x \in K} O_x$ so $K \subset \bigcup_{i=1}^n O_{x_i}$ for some $x_1, \ldots, x_n \in K$. Since each O_{x_i} meets only a finite number of sets in $\{E_\lambda\}$, so does K.

(*) Proof of Lemma 23.

34a. Let X be a paracompact Hausdorff space. Let F be a closed subset and let $x \notin F$. For each $y \in F$, there are disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. Now $X \setminus F \cup \{V_y : y \in F\}$ is an open cover for X so it has a locally finite open refinement $\{E_\lambda\}$. Let $E = \bigcup\{E_\lambda : E_\lambda \cap F \neq \emptyset\}$. Then E is an open set containing F. Also, there is an open set U containing x that meets only a finite number of sets $E_{\lambda_1}, \ldots, E_{\lambda_n}$ in $\{E_\lambda\}$. Each of these sets must lie in some V_{y_i} where $y_i \in F$. Consider $O = U \cap \bigcap_{i=1}^n U_{y_i}$. Then O is an open set containing x and $O \cap E = \emptyset$. Hence X is regular.

34b. Let X be a paracompact Hausdorff space. Let F and G be disjoint closed subsets. By part (a), for each $y \in G$, there are disjoint open sets U_y and V_y such that $F \subset U_y$ and $y \in V_y$. Now $X \setminus G \cup \{V_y : y \in G\}$ is an open cover for X so it has a locally finite open refinement $\{E_\lambda\}$. Let $E = \bigcup\{E_\lambda : E_\lambda \cap F \neq \emptyset\}$. Then E is an open set containing F. For each $y \in G$, there is an open set O_y that meets only a finite number of sets $E_{\lambda_1}, \ldots, E_{\lambda_n}$ in $\{E_\lambda\}$. Each of these sets must lie in some V_{y_i} where $y_i \in G$. Let $O'_y = O_y \cap \bigcap_{i=1}^n U_{y_i}$. Then O'_y is an open set containing y and $O'_y \cap E = \emptyset$. Let $O = \bigcup_{y \in G} O'_y$. Then O is an open set containing G and $O \cap E = \emptyset$.

35. Let (X, ρ) be a locally compact metrizable space. Suppose it can be metrized by a proper extended metric ρ^*. By Q8.41, X is the direct union of its parts X_α. Now $(X_\alpha, \rho^*|_{X_\alpha})$ is a proper locally compact metric space for each α so by Q31b, each X_α is σ-compact. Hence X is the direct union of σ-compact spaces so it is paracompact. Conversely, suppose X is paracompact. Then X is the direct union of σ-compact spaces X_β. Each $(X_\beta, \rho|_{X_\beta})$ is a σ-compact and locally compact metric space so by Q31c, each X_β can be metrized by a proper metric ρ_β. Now define $\rho^{**}(x,y) = \rho_\beta(x,y)$ if $x, y \in X_\beta$ and $\rho^{**}(x,y) = \infty$ if $x \in X_{\beta_1}$ and $y \in X_{\beta_2}$ with $\beta_1 \neq \beta_2$. Then ρ^{**} is a proper extended metric on X.

9.7 Manifolds

36. Let $X = (-1,1) \cup [2,3)$, and make X into a topological space by taking as a base all open intervals $(a,b) \subset X$ and all sets of the form $(-\varepsilon, 0) \cup [2, 2+\varepsilon)$ for $0 < \varepsilon < 1$. Clearly all open intervals $(a,b) \subset X$ are open balls in $\mathbb{R}$. Also, sets of the form $(-\varepsilon, 0) \cup [2, 2+\varepsilon)$ are homeomorphic to $(-\varepsilon, \varepsilon)$. Hence X is locally Euclidean. There are no disjoint basic sets that separate 0 and 2 so X is not Hausdorff.

37a. A not necessarily connected manifold X is the disjoint union of its components. By Q8.36, since X is locally connected, each component of X is open. Thus X is the direct union of its components. Also, its components are Hausdorff and locally Euclidean. Hence X is the direct union of (connected) manifolds.

37b. For a not necessarily connected manifold, statements (ii), (iii), (v), (vi), (vii) are equivalent. Also, statements (i) and (iv) are equivalent. The first set of statements imply the second set of statements.

***38a.**

9.8 The Stone-Čech compactification

39a. Let f be a bounded continuous real-valued function on X with $|f| \leq 1$. The restriction to X of the projection π_f on $\beta(X)$ is f and π_f is continuous on $\beta(X)$ since $\beta(X)$ is a subspace of the product space $I^{\mathfrak{F}}$.

***39b.** Suppose X is a dense open subset of a compact Hausdorff space Y. Then Y is a subset of $I^{\mathfrak{G}}$ where $\mathfrak{G}$ is the space of continuous g on Y with $|g| \leq 1$. The inclusion $i : X \to Y$ induces a continuous function $F : I^{\mathfrak{F}} \to I^{\mathfrak{G}}$ as follows. If $g \in \mathfrak{G}$, then $g \circ i \in \mathfrak{F}$. so define $F(\langle t_f \rangle) = \langle (t_{g \circ i})_g \rangle$. Since $\pi_g \circ F = \pi_{g \circ i}$ is continuous for each g, F is continuous. Now $F[\beta(X)] \subset \overline{F[E]} \subset Y$. Let $\varphi = F|_{\beta(X)}$. Then φ is a continuous mapping of $\beta(X)$ onto Y with $\varphi(x) = x$ for all $x \in X$. Also, φ is unique as a map to a Hausdorff space is determined by its values on a dense subset.

39c. Suppose Z is a space with the same properties. By part (b), there is a unique continuous mapping ψ of $\beta(X)$ onto Z with $\psi(x) = x$ for all $x \in X$. Also, there is a unique continuous mapping φ of Z onto $\beta(X)$ with $\varphi(x) = x$ for all $x \in X$. Thus $\varphi \circ \psi = id|_X$ so φ and ψ are homeomorphisms.

***40.** Let X be the set of ordinals less than the first uncountable ordinal and let Y be the set of ordinals less than or equal to the first uncountable ordinal. Then X is dense in the compact Hausdorff space Y. By Q11, every continuous real-valued function on X is eventually constant so it extends to a continuous function on Y.

***41.** If $A \subset \mathbb{N}$, define $f : \mathbb{N} \to [0,1]$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \notin A$. Now f is continuous so it extends to a continuous function $\hat{f}$ on $\beta(\mathbb{N})$. Then $\bar{A} \cup \overline{\mathbb{N} \setminus A} = \beta(\mathbb{N})$. It follows that $\hat{f}^{-1}[\{1\}] = \bar{A}$ and $\hat{f}^{-1}[\{0\}] = \overline{\mathbb{N} \setminus A}$. Thus $\bar{A} \cap \overline{\mathbb{N} \setminus A} = \emptyset$ and $\bar{A}$ is open.

If $B \subset \mathbb{N}$ and $A \cap B = \emptyset$, then $B \subset \mathbb{N} \setminus A$ and $\bar{A} \cap \bar{B} = \emptyset$.

If V is an open subset of $\beta(\mathbb{N})$, then $\overline{V \cap \mathbb{N}}$ is an open subset of $\beta(\mathbb{N})$. Now if $x \in \bar{V}$ and W is an open neighbourhood of x, then $W \cap V \cap \mathbb{N} \neq \emptyset$ so $x \in \overline{V \cap \mathbb{N}}$. Thus $\bar{V} = \overline{V \cap \mathbb{N}}$ and $\bar{V}$ is open ($\beta(\mathbb{N})$ is extremally disconnected).

Let Y be a subset of $\beta(\mathbb{N})$ with at least two distinct points x and y. Since $\beta(\mathbb{N})$ is Hausdorff, there is an open set $U \subset \beta(\mathbb{N})$ such that $x \in U$ and $y \notin \bar{U}$. Then $Y = (Y \cap \bar{U}) \cup (Y \setminus \bar{U})$ is a separation of Y so Y is not connected. Hence $\beta(\mathbb{N})$ is totally disconnected.

Clearly, if a sequence in $\mathbb{N}$ converges in $\mathbb{N}$, then it converges in $\beta(\mathbb{N})$. Conversely, if it converges in $\beta(\mathbb{N}) \setminus \mathbb{N}$, then consider a function $f : \mathbb{N} \to [0,1]$ with $f(x_{2n}) = 0$ and $f(x_{2n+1}) = 1$ for all n. Now f is continuous since $\mathbb{N}$ is discrete so it has a continuous extension to $\beta(\mathbb{N})$. This is a contradiction as the sequence $\langle g(x_n) \rangle$ does not converge.

Hence $\beta(\mathbb{N})$ is compact but not sequentially compact as the sequence $x_n = n$ does not have a convergent subsequence.

9.9 The Stone-Weierstrass Theorem

42. Let A be the set of finite Fourier series φ given by $\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx), N \in \mathbb{N}$. Then A is a linear space of functions in $C(X)$ where X is taken to be the unit circle in $\mathbb{C}$. From the trigonometric identities $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$, $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$ and $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$, we see that A is a subalgebra of $C(X)$. Furthermore A separates the points of X and contains the constant functions. By the Stone-Weierstrass Theorem, given any continuous periodic real-valued function f on $\mathbb{R}$ with period 2π and any $\varepsilon > 0$, there is a finite Fourier series φ such that $|\varphi(x) - f(x)| < \varepsilon$ for all x.

***43.** Let A be an algebra of continuous real-valued functions on a compact space X, and assume that A separates the points of X. If for each $x \in X$ there is an $f_x \in A$ with $f_x(x) \neq 0$, then by continuity, there is an open neighbourhood O_x of x such that $f_x(y) \neq 0$ for $y \in O_x$. The sets $\{O_x\}$ cover X so by compactness, finitely many of them cover X, say $\{O_{x_1}, \dots, O_{x_n}\}$. Let $g = f_{x_1}^2 + \cdots + f_{x_n}^2$. Then $g \in A$ and $g \neq 0$ everywhere. The closure of the range of g is a compact set K not containing 0. The function h given by $h(t) = 1/t$ for $t \in K$ and $h(0) = 0$ is continuous on $K \cup \{0\}$ so it can be uniformly approximated by polynomials h_n so that $h_n \circ g \in A$ and $h_n \circ g \to 1/g$. Note that if h_n is uniformly within $\varepsilon/2$ of h, then $|h_n(0)| < \varepsilon/2$ but $h_n(0)$ is the constant term of h_n so subtracting the constant term results in a polynomial still within ε of h. Thus we may assume that the polynomials h_n have no constant term. Thus $1/g \in \bar{A}$ so $1 \in \bar{A}$ and $\bar{A}$ contains the constant functions. Hence $\bar{A} = C(X)$.

44. Let $\mathfrak{F}$ be a family of continuous real-valued functions on a compact Hausdorff space X, and suppose that $\mathfrak{F}$ separates the points of X. Let A be the set of polynomials in a finite number of functions of $\mathfrak{F}$. Then A is a subalgebra of $C(X)$ that separates the points of X and contains the constant functions. By the Stone-Weierstrass Theorem, A is dense in $C(X)$. Hence every continuous real-valued function on X can be uniformly approximated by a polynomial in a finite number of functions of $\mathfrak{F}$.

45a. Let X be a topological space and A a set of real-valued functions on X. Define $x \equiv y$ if $f(x) = f(y)$ for all $f \in A$. Clearly, $x \equiv x$ for all $x \in X$ and $y \equiv x$ if $x \equiv y$. If $x \equiv y$ and $y \equiv z$, then $f(x) = f(y) = f(z)$ for all $f \in A$ so $x \equiv z$. Hence $\equiv$ is an equivalence relation.

45b. Let $\widetilde{X}$ be the set of equivalence classes of $\equiv$ and φ the natural map of X into $\widetilde{X}$. Given $f \in A$, define $\tilde{f}$ on $\widetilde{X}$ by $\tilde{f}(\tilde{x}) = f(x)$. If $\tilde{x} = \tilde{y}$, then $x \equiv y$ so $f(x) = f(y)$ and $\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{y})$. Thus $\tilde{f}$ is well-defined and it is the unique function such that $f = \tilde{f} \circ \varphi$.

45c. Let $\widetilde{X}$ have the weak topology generated by the functions $\tilde{f}$ in part (b). Consider a basic set $\tilde{f}^{-1}[O]$. Now $\varphi^{-1}[\tilde{f}^{-1}[O]] = \{x : \tilde{f}(\varphi(x)) \in O\} = \{x : f(x) \in O\} = f^{-1}[O]$, which is open since f is continuous. Hence φ is continuous.

45d. Since φ is continuous and maps X onto $\widetilde{X}$, if X is compact, then so is $\widetilde{X}$. By definition of the weak topology on $\widetilde{X}$, the functions $\tilde{f}$ are continuous.

***45e.** Let X be a compact space and A a closed subalgebra of $C(X)$ containing the constant functions. Define $\widetilde{X}$ and φ as above. If $\tilde{x}$ and $\tilde{y}$ are distinct points in $\widetilde{X}$, then $f(x) \neq f(y)$ for some $f \in A$ so there are disjoint open sets O_x and O_y in $\mathbb{R}$ with $f(x) \in O_x$ and $f(y) \in O_y$. Then $\tilde{x} \in \tilde{f}^{-1}[O_x]$ and $\tilde{y} \in \tilde{f}^{-1}[O_y]$. Thus $\widetilde{X}$ is a compact Hausdorff space. Furthermore, φ induces a (continuous) mapping of $\varphi^* : A \to C(\widetilde{X}), f \mapsto \tilde{f}$, where $f = \tilde{f} \circ \varphi$. The image of A in $C(\widetilde{X})$ is a subalgebra containing the constant functions and separating the points of $\widetilde{X}$. Suppose $\langle \tilde{g_n} \rangle$ is a sequence in $\varphi^*[A]$ that converges to $\tilde{g}$ in $C(\widetilde{X})$. Then the sequence $\langle g_n \rangle$, where $g_n = \tilde{g_n} \circ \varphi$, converges to $\tilde{g} \circ \varphi$. Since each $g_n \in A$ and A is closed, $\tilde{g} \circ \varphi \in A$ so $\tilde{g} \in \varphi^*[A]$ by uniqueness. Thus $\varphi^*[A]$ is closed. By the Stone-Weierstrass Theorem, the image of A is $C(\widetilde{X})$. Hence A is the set of all functions of the form $\tilde{f} \circ \varphi$ with $\tilde{f} \in C(\widetilde{X})$.

46. Let X and Y be compact spaces. The set of finite sums of functions of the form $f(x)g(y)$ where $f \in C(X)$ and $g \in C(Y)$ is an algebra of continuous real-valued functions on $X \times Y$ that contains the constant functions and separates points in $X \times Y$. By the Stone-Weierstrass Theorem, this set is dense in $C(X \times Y)$. Thus for each continuous real-valued function f on $X \times Y$ and each $\varepsilon > 0$, there exist continuous functions $g_1, \dots, g_n$ on X and $h_1, \dots, h_n$ on Y such that $|f(x,y) - \sum_{i=1}^n g_i(x)h_i(y)| < \varepsilon$ for all $\langle x, y \rangle \in X \times Y$.

47. The functions of norm 1 in the algebra A give a mapping of X into the infinite-dimensional cube $\prod\{I_f : f \in A, ||f|| = 1\}$. By the Tietze Extension Theorem, each continuous function f on the image of X can be extended to a continuous function g on the cube and by Q17, g can be approximated by a continuous function h of only a finite number of coordinates. Then h can be regarded as a continuous function on a cube in $\mathbb{R}^n$, which can be uniformly approximated by a polynomial in (a finite number of) the coordinate functions.

48a. Let φ be the polynomial defined by $\varphi(x) = x + x(1-2x)(1-x)$. Then $\varphi'(x) = 6x^2 - 6x + 2 > 0$ for all x. Thus φ is monotone increasing and its fixed points are $0, \frac{1}{2}, 1$.

***48b.** Choose $\varepsilon > 0$. Note that $\varphi(x) > x$ on $(0, \frac{1}{2})$ and $\varphi(x) < x$ on $(\frac{1}{2}, 1)$. Let $[a_n, b_n] = \varphi_n[\varepsilon, 1-\varepsilon]$ for each n where φ_n is an iterate of φ. Then $a_n = \varphi_n(\varepsilon)$ increases to some a and $b_n = \varphi_n(1-\varepsilon)$ decreases to some b. Furthermore, $\varepsilon \leq a \leq b \leq 1-\varepsilon$ and a, b are fixed points of φ. Thus $a = b = \frac{1}{2}$. Hence some iterate φ_n is a polynomial with integral coefficients that is monotone increasing on $[0,1]$ and such that $|\varphi_n(x) - \frac{1}{2}| < \varepsilon$ for $x \in [\varepsilon, 1-\varepsilon]$.

***48c.** Given α with $0 < \alpha < 1$ and any $\varepsilon > 0$, it suffices to consider the case where α is a rational number $\frac{a}{b}$. Define $\varphi(x) = x + x(a - bx)(1-x)$. Then α is a fixed point of φ. By parts (a) and (b), some iterate $\psi = \varphi_n$ is a polynomial with integral coefficients (and no constant term) such that $0 \leq \psi(x) \leq 1$ in $[0,1]$ and $|\psi(x) - \alpha| < \varepsilon$ for all $x \in [\varepsilon, 1-\varepsilon]$.

***48d.** Let P be a polynomial with integral coefficients, and suppose that $P(-1) = P(0) = P(1) = 0$. Let β be any real number. We may assume that $0 < \beta < 1$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|P(x)| < \varepsilon/2$ for $x \in (-\delta, \delta)$, $x \in (1-\delta, 1]$ and $x \in [-1, -1+\delta)$. We may assume that $\delta < \varepsilon/||P||$. By part (c), there is a polynomial ψ with integral coefficients and no constant term such that $|\psi(x^2) - \beta| < \delta$ for all $x \in [\delta, 1-\delta]$. Then $|P(x)\psi(x^2) - \beta P(x)| < \delta||P|| < \varepsilon$ for all $x \in [-1, 1]$.

***48e.** Let $I = [-1, 1]$ and f a continuous real-valued function on I such that $f(-1), f(0), f(1)$ are integers and $f(1) \equiv f(-1) \mod 2$. Let $f(-1) = a, f(0) = b, f(1) = a + 2c$, where a, b, c are integers. Let $Q(x) = (a - b + c)x^2 + cx + b$. Replacing f by $f - Q$, we may assume that $a = b = c = 0$. Then use the Stone-Weierstrass Theorem to approximate f by a polynomial R with rational coefficients such that $R(-1) = R(0) = R(1) = 0$. Let N be the least common multiple of the denominators of the coefficients of R so that NR has integral coefficients and vanishes at -1,0,1. Let $\beta = 1/N$ and apply part (d) to the polynomial NR so that R, and thus f, can be approximated by a polynomial P with integral coefficients.

***49a.**

***49b.**

***50a.**

***50b.**

10 Banach Spaces

10.1 Introduction

1. Suppose $x_n \to x$. Then $|\,||x_n|| - ||x||\,| \le ||x_n - x|| \to 0$. Hence $||x_n|| \to ||x||$.

2. The metric $\rho(x,y) = [\sum_{i=1}^n (x_i - y_i)^2]^{1/2}$ is derived from the norm $||x|| = (\sum_{i=1}^n x_i^2)^{1/2}$. The metric $\rho^*(x,y) = \sum_{i=1}^n |x_i - y_i|$ is derived from the norm $||x||^* = \sum_{i=1}^n |x_i|$. The metric $\rho^+(x,y) = \max|x_i - y_i|$ is derived from the norm $||x||^+ = \max|x_i|$. Now $n^{-1}||x||^+ \le ||x||^+ \le ||x||^* \le n||x||^+$ so $||x||^*$ and $||x||^+$ are equivalent. Also, $(\sqrt{n})^{-1}||x||^+ \le ||x||^+ \le ||x|| \le (n(||x||^+)^2)^{1/2} = \sqrt{n}||x||^+$ so $||x||$ and $||x||^+$ are equivalent. Thus $||x||$ and $||x||^*$ are also equivalent.

3. Consider $+$ as a function from $X \times X$ into X. Since $||(x_1+y_1)-(x_2+y_2)|| \le ||x_1 - x_2|| + ||y_1 - y_2||$, $+$ is continuous. Consider $\cdot$ as a function from $\mathbb{R} \times X$ into X. Since $||cx - cy|| = |c|\,||x-y||$, $\cdot$ is continuous.

4. Let M be a nonempty set. Then $M \subset M + M$ since $m = m + \theta$ for all m, where θ is the zero vector. Also, $M \subset \lambda M$ since $m = 1 \cdot m$ for all m. If M is a linear manifold, then $M + M \subset M$ and $\lambda M \subset M$ for each λ so $M + M = M$ and $\lambda M = M$. Conversely, suppose $M + M = M$ and $\lambda M = M$. Then $\lambda x \in M$ for each $\lambda \in \mathbb{R}$ and $x \in M$. Thus also $\lambda_1 x_1 + \lambda_2 x_2 \in M$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in M$. Hence M is a linear manifold.

5a. Let $\{M_i : i \in I\}$ be a family of linear manifolds and let $M = \bigcap M_i$. For any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in M$, we have $x_1, x_2 \in M_i$ for all i. Since each M_i is a linear manifold, $\lambda_1 x_1 + \lambda_2 x_2 \in M_i$ for each i. i.e. $\lambda_1 x_1 + \lambda_2 x_2 \in M$. Hence M is a linear manifold.

5b. Given a set A in a vector space X, X is a linear manifold containing A. Consider the family of linear manifolds containing A. The intersection $\{A\}$ of this family is a linear manifold containing A and it is the smallest such linear manifold.

5c. Consider the set M of all finite linear combinations of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$ with $x_i \in A$. Then M is a linear manifold containing A. Also, any linear manifold containing A will contain M. Hence M is the smallest linear manifold containing A. i.e. $\{A\} = M$.

6a. Let M and N be linear manifolds. For $\lambda_1, \lambda_2 \in \mathbb{R}$, $m_1, m_2 \in M$ and $n_1, n_2 \in N$, $\lambda_1(m_1 + n_1) + \lambda_2(m_2 + n_2) = (\lambda_1 m_1 + \lambda_2 m_2) + (\lambda_1 n_1 + \lambda_2 n_2) \in M + N$. Hence $M + N$ is a linear manifold. Note that $M \subset M + N$ and $N \subset M + N$ so $M + N$ contains $M \cup N$. Also, any manifold containing $M \cup N$ will also contain $M + N$. Hence $M + N = \{M \cup N\}$.

6b. Let M be a linear manifold. For $\lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2 \in \bar{M}$ and $\delta > 0$, there exist $y_1, y_2 \in M$ such that $||x_1 - y_1|| < \delta/2\lambda_1$ and $||x_2 - y_2|| < \delta/2\lambda_2$ (We may assume $\lambda_1, \lambda_2 \ne 0$). Then $||(\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 y_1 + \lambda_2 y_2)|| \le |\lambda_1|\,||x_1 - y_1|| + |\lambda_2|\,||x_2 - y_2|| < \delta$. Thus $\lambda_1 x_1 + \lambda_2 x_2 \in \bar{M}$ so $\bar{M}$ is a linear manifold.

7. Let P be the set of all polynomials on $[0,1]$. Then $P \subset C[0,1]$. For $\lambda_1, \lambda_2 \in \mathbb{R}$ and $p_1, p_2 \in P$, $\lambda_1 p_1 + \lambda_2 p_2$ is still a polynomial on $[0,1]$ so $\lambda_1 p_1 + \lambda_2 p_2 \in P$. Thus P is a linear manifold in $C[0,1]$. The set P is not closed in $C[0,1]$ because by the Weierstrass Approximation Theorem, every continuous function on $[0,1]$ can be uniformly approximated by polynomials on $[0,1]$. i.e. $\bar{P}$ contains a continuous function that is not a polynomial.

The set of continuous functions f with $f(0) = 0$ is a closed linear manifold in $C[0,1]$.

8. Let M be a finite-dimensional linear manifold in a normed vector space X with $M = \{x_1, \ldots, x_n\}$. Each $x \in X$ can be written as a unique linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$. We may define a norm $||x||_1 = \sum_{i=1}^n |\lambda_i|$ and see that $||\cdot||_1$ is equivalent to the original norm on X. Thus convergence under the original norm on X is equivalent to convergence of each sequence of coefficients in $\mathbb{R}$. Let $\langle \sum_{i=1}^n \lambda_i^{(k)} x_i \rangle_k$ be a sequence in M converging to $x \in X$. Let $\lambda_i = \lim_k \lambda_i^{(k)}$ for each i. By continuity of addition and scalar multiplication, $x = \lim_k \sum_{i=1}^n \lambda_i^{(k)} x_i = \sum_{i=1}^n \lambda_i x_i \in M$. Hence M is closed.

9. Let $S = \{x : ||x|| < 1\}$. Given $x \in S$, let $\delta = (1 - ||x||)/2$. When $||y - x|| < \delta$, we have $||y|| \le ||y - x|| + ||x|| < (1 - ||x||)/2 + ||x|| < 1$ so $y \in S$. Hence S is open. For any sequence $\langle x_n \rangle$ in S that converges to some x, we have $||x_n|| \to ||x||$ so $||x|| \le 1$. Thus $\bar{S} \subset \{x : ||x|| \le 1\}$. On the other hand, if $||x|| = 1$ and $\delta > 0$, let $\lambda = \max(1 - \delta/2, 0)$. Then $||\lambda x|| \in S$ and $||x - \lambda x|| = |1 - \lambda| < \delta$. Thus $x \in \bar{S}$ and $\{x : ||x|| \le 1\} \subset \bar{S}$. Hence $\bar{S} = \{x : ||x|| \le 1\}$.

10. Define $x \equiv y$ if $||x - y|| = 0$. Then $x \equiv x$ since $||0x|| = 0||x|| = 0$. Also, $x \equiv y$ implies $y \equiv x$ since $||y - x|| = ||-(x - y)|| = |-1|\,||x - y|| = ||x - y||$. Finally, if $x \equiv y$ and $y \equiv z$, then

$||x-z|| \leq ||x-y|| + ||y-z|| = 0$ so $x \equiv z$. Thus $\equiv$ is an equivalence relation. If $x_1 \equiv y_1$ and $x_2 \equiv y_2$, then $||(x_1+x_2)-(y_1+y_2)|| \leq ||x_1-y_1|| + ||x_2-y_2|| = 0$ so $x_1+x_2 \equiv y_1+y_2$. If $x \equiv y$, then $||cx-cy|| = |c|\,||x-y|| = 0$ so $cx \equiv cy$ for $c \in \mathbb{R}$. Hence $\equiv$ is compatible with addition and scalar multiplication. If $x \equiv y$, then $|\,||x||-||y||\,| \leq ||x-y|| = 0$ so $||x|| = ||y||$.

Let X' be the set of equivalence classes under $\equiv$. Define $\alpha x' + \beta y'$ as the (unique) equivalence class which contains $\alpha x + \beta y$ for $x \in x'$ and $y \in y'$ and define $||x'|| = ||x||$ for $x \in x'$. Then X' becomes a normed vector space. The mapping φ of X onto X' that takes each element of X into the equivalence class to which it belongs is a homomorphism of X onto X' since $\varphi(\alpha x + \beta y) = \alpha x' + \beta y' = \alpha\varphi(x) + \beta\varphi(y)$. The kernel of φ consists of the elements of X that belong to the equivalence class containing the zero vector θ. These are the elements x with $||x|| = 0$.

On the L^p spaces on $[0,1]$ we have the pseudonorm $||f||_p = \left(\int_0^1 |f|^p\right)^{1/2}$. Then $f \equiv g$ if $\int_0^1 |f-g|^p = 0$. i.e. $f = g$ a.e. The kernel of the mapping φ consists of the functions that are 0 a.e.

11. Let X be a normed linear space (with norm $||\cdot||$) and M a linear manifold in X. Let $||x||_1 = \inf_{m\in M} ||x-m||$. Since $||x-m|| \geq 0$ for all $x \in X$ and $m \in M$, $||x||_1 \geq 0$ for all $x \in X$. For $x, y \in X$ and $\varepsilon > 0$, there exist $m, n \in M$ such that $||x-m|| < ||x||_1 + \varepsilon/2$ and $||y-m|| < ||y||_1 + \varepsilon/2$. Then $||x+y||_1 \leq ||(x+y)-(m+n)|| < ||x||_1 + ||y||_1 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $||x+y||_1 \leq ||x||_1 + ||y||_1$. Also, for $x \in X$ and $\alpha \in \mathbb{R}$, $||\alpha x||_1 = \inf_{m\in M} ||\alpha x - m|| = |\alpha| \inf_{m\in M} ||x-m|| = |\alpha|\,||x||_1$. Hence $||\cdot||_1$ is a pseudonorm on X.

Let X' be the normed linear space derived from X and the pseudonorm $||\cdot||_1$ using the process in Q10. The natural mapping φ of X onto X' has kernel $\bar{M}$ since it consists of the elements x with $||x||_1 = \inf_{m\in M} ||x-m|| = 0$. Let O be an open set in X. Take $x \in O$. Then there exists $\delta > 0$ such that $y \in O$ if $||y-x||_1 < \delta$. Now if $||z - \varphi(x)|| < \delta$, where $z = \varphi(y)$ for some $y \in X$, then $||y-x||_1 < \delta$ so $y \in O$ and $z \in \varphi[O]$. Hence $\varphi[O]$ is open. i.e. φ maps open sets into open sets.

12. Suppose X is complete and M is a closed linear manifold in X. Let $\langle\varphi(x_n)\rangle$ be an absolutely summable sequence in X/M. Then $\sum ||\varphi(x_n)|| < \infty$ so $\sum ||x_n||_1 < \infty$. Given $\varepsilon > 0$, for each n, there exists $m_n \in M$ such that $||x_n - m_n|| < ||x_n||_1 + 2^{-n}$. Then $\sum ||x_n - m_n|| \leq \sum ||x_n||_1 + 1 < \infty$. Since X is complete, the sequence $\langle x_n - m_n\rangle$ is summable in X, say $\sum(x_n - m_n) = x$. Now φ is continuous since $||\varphi(x)|| = ||x||_1 \leq ||x||$. Also, M is the kernel of φ. Thus $\sum \varphi(x_n) = \sum \varphi(x_n - m_n) = \varphi(\sum(x_n - m_n)) = \varphi(x) \in X/M$. Since any absolutely summable sequence in X/M is summable, X/M is complete.

10.2 Linear operators

13. Suppose $A_n \to A$ and $x_n \to x$. Then $||A_n - A|| \to 0$ and $||x_n - x|| \to 0$. Since $||A_n x_n - Ax|| \leq ||A_n x_n - Ax_n|| + ||Ax_n - Ax|| \leq ||A_n - A||\,||x_n|| + ||A||\,||x_n - x||$ and $||x_n||$ is bounded, $||A_n x_n - Ax|| \to 0$. i.e. $A_n x_n \to Ax$.

14. Let A be a linear operator and $\ker A = \{x : Ax = \theta\}$. If $x, y \in \ker A$ and $\alpha, \beta \in \mathbb{R}$, then $A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \theta$ so $\alpha x + \beta y \in \ker A$. Thus $\ker A$ is a linear manifold.

Suppose A is continuous. Let $\langle x_n\rangle$ be a sequence in $\ker A$ converging to some x. Since A is continuous, $\langle Ax_n\rangle$ converges to Ax. Now $Ax_n = \theta$ for all n so $Ax = \theta$ and $x \in \ker A$. Thus $\ker A$ is closed.

15a. Let X be a normed linear space and M a closed linear manifold. Let φ be the natural homomorphism of X onto X/M. Now $||\varphi(x)|| = ||x||_1 \leq ||x||$, where $||\cdot||_1$ is the pseudonorm in Q11. Thus $||\varphi|| \leq 1$. Given $\varepsilon > 0$ and $x \in X$, there exists $m \in M$ such that $||x-m|| < ||x||_1 + \varepsilon = ||\varphi(x)|| + \varepsilon = ||\varphi(x-m)|| + \varepsilon$. Let $y = (x-m)/||x-m||$. Then $1 = ||y|| < ||\varphi(y)|| + \varepsilon$. i.e. $||\varphi(y)|| > 1 - \varepsilon$. Since $||\varphi|| = \sup_{||x||=1} ||\varphi(x)||$, we have $||\varphi|| > 1 - \varepsilon$ for all $\varepsilon > 0$. Thus $||\varphi|| \geq 1$.

15b. Let X and Y be normed linear spaces and A a bounded linear operator from X into Y whose kernel is M. Define a mapping B from X/M into Y by $Bx' = Ax$ where x' is the equivalence class containing x. If $x' = y'$, then $||x-y||_1 = 0$. Thus for any $\varepsilon > 0$, there exists $m \in M$ such that $||x-y-m|| < \varepsilon$. Then $||Ax - Ay|| = ||A(x-y-m)|| \leq ||A||\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $||Ax - Ay|| = 0$. i.e. $Ax = Ay$. Thus B is well-defined and $A = B \circ \varphi$. Furthermore, it is the unique such mapping. If $x', y' \in X/M$ and $\alpha, \beta \in \mathbb{R}$, then $B(\alpha x' + \beta y') = A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha Bx' + \beta By'$ so B is a linear operator. Also, $||Bx'|| = ||Ax|| \leq ||A||\,||x|| = ||A||\,||x-m||$ for all $m \in M$. Thus $||Bx'|| \leq ||A||\,||x||_1 = ||A||\,||x'||$ so $||B|| \leq ||A||$ and B is bounded. For any $\varepsilon > 0$, there exists $x \in X$ with $||x|| = 1$ and $||Ax|| > ||A|| - \varepsilon$.

Then $||x'|| \leq 1$ and $||Bx'|| > ||A|| - \varepsilon$. Since $||B|| = \sup_{||x'|| \leq 1} ||Bx'||$, we have $||B|| \geq ||A||$. Hence $||A|| = ||B||$.

16. Let X be a metric space and Y the space of real-valued functions f on X vanishing at a fixed point $x_0 \in X$ and satisfying $|f(x) - f(y)| \leq M\rho(x,y)$ for some M (depending on f). Define $||f|| = \sup \frac{|f(x)-f(y)|}{\rho(x,y)}$. Clearly $||f|| \geq 0$. Also, $||f|| = 0$ if and only if $f(x) = f(y)$ for all $x, y \in X$ if and only if f is the zero function. Furthermore, $||f+g|| \leq ||f|| + ||g||$ since $|(f+g)x-(f+g)y| \leq |f(x)-f(y)|+|g(x)-g(y)|$ and $\sup A + B \leq \sup A + \sup B$. Similarly, $||\alpha f|| = |\alpha|\,||f||$. Thus $||\cdot||$ defines a norm on Y.

For each $x \in X$, define the functional F_x by $F_x(f) = f(x)$. Then $F_x(\alpha f + \beta g) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$ so F_x is a linear functional on Y. Also, $||F_x(f)|| = |f(x)| = |f(x) - f(x_0)| \leq \rho(x, x_0)||f||$ so F_x is bounded. Furthermore, $||F_x|| \leq \rho(x, x_0)$ so $||F_x - F_y|| \leq \rho(x, x_0) + \rho(y, x_0) \leq \rho(x, y)$. If $||f|| = 1$ and $\varepsilon > 0$, then there exist $x, y \in X$ such that $\frac{|f(x)-f(y)|}{\rho(x,y)} > 1 - \varepsilon$. i.e. $|f(x) - f(y)| > (1 - \varepsilon)\rho(x, y)$. Since $||F_x - F_y|| = \sup_{||f||=1} |(F_x - F_y)f| = \sup_{||f||=1} |f(x) - f(y)| > (1-\varepsilon)\rho(x,y)$ for all $\varepsilon > 0$, we have $||F_x - F_y|| \geq \rho(x, y)$. Hence $||F_x - F_y|| = \rho(x, y)$.

Thus X is isometric to a subset of the space Y^* of bounded linear operators from Y to $\mathbb{R}$. Since Y^* is complete, the closure of this subset gives a completion of X.

10.3 Linear functionals and the Hahn-Banach Theorem

17. Let f be a linear functional on a normed linear space. If f is bounded, then it is uniformly continuous and by Q14, its kernel is closed. Conversely, if f is unbounded, then there is a sequence $\langle x_n \rangle$ with $||x_n|| \leq 1$ for all n and $f(x_n) \to \infty$. Take $x \notin \ker f$ and consider $y_n = x - (f(x)/f(x_n))x_n$. Each y_n is in $\ker f$ and $y_n \to x$. Thus $\ker f$ is not closed.

18. Let T be a linear subspace of a normed linear space X and y a given element of X. If $y \in T$, then $\inf_{t \in T} ||y - t|| = 0 = \sup\{f(y) : ||f|| = 1, f(t) = 0 \,\text{for all}\, t \in T\}$. Thus we may assume $y \notin T$. Let $\delta = \inf_{t \in T} ||y - t||$. Then $||y - t|| \geq \delta$ for all $t \in T$. There is a bounded linear functional f on X such that $||f|| = 1$, $f(y) = \delta$ and $f(t) = 0$ for all $t \in T$. Thus $\delta \leq \sup\{f(y) : ||f|| = 1, f(t) = 0 \,\text{for all}\, t \in T\}$. If $\delta < \sup\{f(y) : ||f|| = 1, f(t) = 0 \,\text{for all}\, t \in T\}$, then there exists f with $||f|| = 1$ and $f(t) = 0$ for $t \in T$ such that $f(y) > \delta$ so there exists $t \in T$ such that $f(y) > ||y - t||$. But then $||y - t|| = ||f||\,||y - t|| \geq f(y - t) = f(y) > ||y - t||$. Contradiction. Thus $\delta \geq \sup\{f(y) : ||f|| = 1, f(t) = 0 \,\text{for all}\, t \in T\}$. Hence $\inf_{t \in T} ||y - t|| = \sup\{f(y) : ||f|| = 1, f(t) = 0 \,\text{for all}\, t \in T\}$.

19. Let T be a linear subspace of a normed linear space X and y an element of X whose distance to T is at least δ. Let S be the subspace consisting of multiples of y. Define $f(\lambda y) = \lambda\delta$. Then f is a linear functional on S. Let $p(x) = \inf_{t \in T} ||x - t||$. Then $f(\lambda y) = \lambda\delta \leq \lambda p(y) \leq p(\lambda y)$. By the Hahn-Banach Theorem, we may extend f to all of X so that $f(x) \leq p(x)$ for all $x \in X$. In particular, $f(y) = \delta$ and $f(t) = 0$ for all $t \in T$. Also, $f(x) \leq p(x) = \inf_{t \in T} ||x - t|| \leq ||x||$ so $||f|| \leq 1$.

20. Let ℓ^∞ be the space of all bounded sequences and let S be the subspace consisting of the constant sequences. Let G be the Abelian semigroup of operators generated by the shift operator A given by $A[\langle \xi_n \rangle] = \langle \xi_{n+1} \rangle$. If $\langle \xi_n \rangle \in S$, say $\xi_n = \xi$ for all n, define $f[\langle \xi_n \rangle] = \xi$. Then f is a linear functional on S. Define $p[\langle \xi_n \rangle] = \overline{\lim}\xi_n$. Then $f[\langle \xi_n \rangle] = p[\langle \xi_n \rangle]$ on S. Also, $p(A^n x) = p(x)$ for all $x \in X$. If $\langle \xi_n \rangle \in S$, then $A^n[\langle \xi_n \rangle] = \langle \xi_n \rangle \in S$ and $f(A^n[\langle \xi_n \rangle]) = f[\langle \xi_n \rangle]$. By Proposition 5, there is an extension F of f to a linear functional on X such that $F(x) \leq p(x)$ and $F(Ax) = F(x)$ for all $x \in X$.

In particular, $F[\langle \xi_n \rangle] \leq \overline{\lim}\xi_n$. Also, $-F[\langle \xi_n \rangle] = F[\langle -\xi_n \rangle] \leq \overline{\lim}(-\xi_n) = -\underline{\lim}\xi_n$ so $\underline{\lim}\xi_n \leq F[\langle \xi_n \rangle]$. By linearity, $F[\langle \xi_n + \eta_n \rangle] = F[\langle \xi_n \rangle + \langle \eta_n \rangle] = F[\langle \xi_n \rangle] + F[\langle \eta_n \rangle]$ and $F[\langle \alpha\xi_n \rangle] = F[\alpha\langle \xi_n \rangle] = \alpha F[\langle \xi_n \rangle]$. Finally, if $\eta_n = \xi_{n+1}$, then $F[\langle \eta_n \rangle] = F[A[\langle \xi_n \rangle]] = F[\langle \xi_n \rangle]$.

(*) The functional F is called a Banach limit and is often denoted by *Lim*.

***21.** Let X be the space of bounded real-valued functions on the unit circle and let S be the subspace of bounded Lebesgue measurable functions on the unit circle. For $s \in S$, define $f(s) = \int s$. Also define $p(x) = \inf_{x \leq s} f(s)$. Then f is a linear functional on S with $f \leq p$ on S. Let G consist of the rotations so that it is an Abelian semigroup of operators on X such that for every $A \in G$ we have $p(Ax) \leq p(x)$ for $x \in X$ while for $s \in S$ we have $As \in S$ and $f(As) = f(s)$. Then there is an extension of f to a linear functional F on X such that $F(x) \leq p(x)$ and $F(Ax) = F(x)$ for $x \in X$. For a subset P of the unit circle, let $\mu(P) = F(\chi_P)$. This will be a rotationally invariant measure on $[0, 2\pi]$. Then extend it to the bounded subsets of $\mathbb{R}$ to get the required set function.

22. Let X be a Banach space. Suppose X^* is reflexive. If X is not reflexive, then there is a nonzero function $y \in X^{***}$ such that $y(x') = 0$ for all $x' \in \varphi[X]$. But there exists $x^* \in X^*$ such that $y = \varphi^*(x^*)$. If $x \in X$, then $0 = y(\varphi(x)) = (\varphi^*(x^*))(\varphi(x)) = (\varphi(x))(x^*) = x^*(x)$. Thus $x^* = 0$ and so $y = 0$. Contradiction. Thus X is reflexive. Conversely, suppose X is reflexive. Let $x^{***} \in X^{***}$. Define $x^* \in X^*$ by $x^*(x) = x^{***}(\varphi(x))$. Then $\varphi^*(x^*)(x^{**}) = x^{**}(x^*) = (\varphi(x))(x^*) = x^*(x) = x^{***}(\varphi(x)) = x^{***}(x^{**})$. Thus X^* is reflexive.

23a. If $x, y \in S^\circ$ and $\alpha, \beta \in \mathbb{R}$, then $(\alpha x + \beta y)(s) = \alpha x(s) + \beta y(s) = 0$ for all $s \in S$ so S° is a linear subspace of X^*. Let $\langle y_n \rangle$ be a sequence in S° that converges to some $y \in X^*$. By Q13, for each $s \in S$, $y_n(s) \to y(s)$. Since $y_n(s) = 0$ for all n, we have $y(s) = 0$. Thus $y \in S^\circ$ and S° is closed.

***23b.** If $x \in \bar{S}$, then there is a sequence $\langle s_n \rangle$ in S converging to x. Let $y \in S^\circ$. Then $y(s_n) = 0$ for all n so $y(x) = 0$. Thus $\bar{S} \subset S^{\circ\circ}$. Suppose there exists $x \in S^{\circ\circ} \setminus \bar{S}$. Then there is a linear functional f with $||f|| \leq 1$, $f(x) = \inf_{t \in \bar{S}} ||x - t|| > 0$ and $f(t) = 0$ for $t \in \bar{S}$. Thus $f \in (\bar{S})^\circ = S^\circ$ so $f(x) = 0$. Contradiction. Hence $S^{\circ\circ} = \bar{S}$.

23c. Let S be a closed subspace of X and let $\varphi : X^* \to X^*/S^\circ$ be the natural homomorphism. Define $A : X^* \to S^*$ by $Ay = y|_S$. Then A is a bounded linear operator with kernel S°. By Q15b, there is a unique bounded linear operator $B : X^*/S^\circ \to S^*$ such that $A = B \circ \varphi$. By the Hahn-Banach Theorem, A is onto. Thus so is B. If $y|_S = z|_S$, then $y - z \in S^\circ$ so $\varphi(y) = \varphi(z)$ and B is one-to-one. Hence B is an isomorphism between S^* and X^*/S°.

23d. Let S be a closed subspace of a reflexive Banach space X. Let $\varphi : X \to X^{**}$ be the natural isomorphism and define $A : X^* \to S^*$ by $Ay = y|_S$. Let $s^{**} \in S^{**}$. Then $s^{**} \circ A \in X^{**}$ so $s^{**} \circ A = \varphi(x)$ for some $x \in X$. If $x \notin S$, then there exists $x^* \in X^*$ such that $x^*(x) > 0$ and $x^*(s) = 0$ for $s \in S$. Then $A(x^*) = 0$ so $x^*(x) = (\varphi(x))(x^*) = (s^{**} \circ A)(x^*) = 0$. Contradiction. Thus $x \in S$. Now for any $s^* \in S^*$, there exists $x^* \in X^*$ such that $A(x^*) = s^*$. Then $s^{**}(s^*) = (s^{**} \circ A)(x^*) = (\varphi(x))(x^*) = x^*(x) = s^*(x) = (\varphi_S(x))(s^*)$. i.e. $s^{**} = \varphi_S(x)$. Hence S is reflexive.

24. Let X be a vector space and P a subset of X such that $x, y \in P$ implies $x + y \in P$ and $\alpha x \in P$ for $\alpha > 0$. Define a partial order in X by defining $x \leq y$ to mean $y - x \in P$. A linear functional f on X is said to be positive (with respect to P) if $f(x) \geq 0$ for all $x \in P$. Let S be any subspace of X with the property that for each $x \in X$ there is an $s \in S$ with $x \leq s$. Let f be a positive linear functional on S. The family of positive linear functionals on S is partially ordered by setting $f \prec g$ if g is an extension of f. By the Hausdorff Maximal Principle, there is a maximal linearly ordered subfamily $\{g_\alpha\}$ containing f. Define a functional F on the union of the domains of the g_α by setting $F(x) = g_\alpha(x)$ if x is in the domain of g_α. Since the subfamily is linearly ordered, F is well-defined. Also, F is a positive linear functional extending f. Furthermore, F is a maximal extension since if G is any extension of F, then $g_\alpha \prec F \prec G$ implies that G must belong to $\{g_\alpha\}$ by maximality of $\{g_\alpha\}$. Thus $G \prec F$ so $G = F$.

Let T be a proper subspace of X with the property that for each $x \in X$ there is a $t \in T$ with $x \leq t$. We show that each positive linear functional g on T has a proper extension h. Let $y \in X \setminus T$ and let U be the subspace spanned by T and y. If h is an extension of g, then $h(\lambda y + t) = \lambda h(y) + h(t) = \lambda h(y) + g(t)$. There exists $t' \in T$ with $y \leq t'$. i.e. $t' - y \in T$. Then $\lambda(t' - y) + t \in T$ and $g(\lambda(t' - y) + t) \geq 0$. Define $h(y) = g(t' - y)$. Then $h(\lambda y + t) = \lambda h(y) + g(t) = g(\lambda(t' - y)) + g(t) = g(\lambda(t' - y) + t) \geq 0$. Thus h is a proper extension of g. Since F is a maximal extension, it follows that F is defined on X.

***25.** Let f be a mapping of the unit ball $S = \{x : ||x|| \leq 1\}$ into $\mathbb{R}$ such that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ whenever x, y and $\alpha x + \beta y$ are in S. Define $g(x) = ||x|| f(\frac{x}{||x||})$. If $||x|| \leq 1$, then $g(x) = ||x|| \frac{1}{||x||} f(x) = f(x)$. If $x, y \in X$, then $g(x + y) = ||x + y|| f(\frac{x+y}{||x+y||}) = ||x + y|| f(\frac{||x||}{||x+y||} \frac{x}{||x||} + \frac{||y||}{||x+y||} \frac{y}{||y||}) = ||x + y|| [\frac{||x||}{||x+y||} f(\frac{x}{||x||}) + \frac{||y||}{||x+y||} f(\frac{y}{||y||})] = ||x|| f(\frac{x}{||x||}) + ||y|| f(\frac{y}{||y||}) = g(x) + g(y)$. If $\alpha \in \mathbb{R}$ and $x \in X$, then $g(\alpha x) = ||\alpha x|| f(\frac{\alpha x}{||\alpha x||}) = |\alpha| \, ||x|| f(\frac{\alpha x}{|\alpha| \, ||x||}) = |\alpha| \, ||x|| \frac{\alpha}{|\alpha|} f(\frac{x}{||x||}) = \alpha g(x)$. Thus g is a linear functional on X extending f.

10.4 The Closed Graph Theorem

26. Let $\langle T_n \rangle$ be a sequence of continuous linear operators from a Banach space X to a normed vector space Y. Suppose that for each $x \in X$ the sequence $\langle T_n x \rangle$ converges to a value Tx. Now for each

$x \in X$ there exists M_x such that $||T_n x|| \le M_x$ for all n. Thus there exists M such that $||T_n|| \le M$ for all n. Given $\varepsilon > 0$, for each $x \in X$, there exists N such that $||T_N x - Tx|| < \varepsilon$. Then $||Tx|| \le ||T_N x - Tx|| + ||T_N x|| < \varepsilon + M||x||$. Thus $||Tx|| \le M||x||$ for all $x \in X$. i.e. T is a bounded linear operator.

27. Let A be a bounded linear transformation from a Banach space X to a Banach space Y, and let M be the kernel and S the range of A. Suppose S is isomorphic to X/M. Since X is complete and M is closed, X/M is complete by Q12. Thus X/M is closed and since S is isomorphic to X/M, S is also closed. Conversely, suppose S is closed. Then since Y is complete, so is S. Let φ be the natural homomorphism from X to X/M. There is a unique bounded linear operator $B : X/M \to S$ such that $A = B \circ \varphi$. Since A and φ are onto, so is B. Thus B is an open mapping. It remains to show that B is one-to-one. Suppose $Bx' = By'$. Then $x' = \varphi(x)$ and $y' = \varphi(y)$ for some $x, y \in X$ so $B(\varphi(x)) = B(\varphi(y))$. Then $Ax = Ay$ so $x - y \in M$ and $x' = y'$. Hence B is one-to-one and is thus an isomorphism.

28a. Let S be a linear subspace of $C[0,1]$ that is closed as a subspace of $L^2[0,1]$. Let $\langle f_n \rangle$ be a sequence in S converging to f in $C[0,1]$. i.e. $||f_n - f||_\infty \to 0$. Then since $||f_n - f||_2 \le ||f_n - f||_\infty$, we have $||f_n - f||_2 \to 0$. Thus $f \in S$. Hence S is closed as a subspace of $C[0,1]$.

28b. For any $f \in S$, we have $||f||_2 = (\int f^2)^{1/2} \le (\int ||f||_\infty^2)^{1/2} = ||f||_\infty$. Since S is closed in both $C[0,1]$ and $L^2[0,1]$, it is complete in both norms. Thus there exists M such that $||f||_\infty \le M||f||_2$.

***28c.** Let $y \in [0,1]$ and define $F(f) = f(y)$. Then F is a linear functional on $L^2[0,1]$. Also, $|F(f)| = |f(y)| \le ||f||_\infty \le M||f||_2$ so F is bounded. By the Riesz Representation Theorem, there exists $k_y \in L^2$ such that $f(y) = F(f) = \int k_y(x) f(x)\, dx$.

***29a.** Let $Y = C[0,1]$ and let X be the subspace of functions which have a continuous derivative. Let A be the differential operator. Let $x_n(t) = t^n$. Then $||x_n|| = 1$ and $Ax_n(t) = nt^{n-1}$ so $||Ax_n|| = n$. Thus A is unbounded and thus discontinuous. Let $x_n \in X$ such that $x_n \to x$ and $x_n' = Ax_n \to y$. Since we have uniform convergence, $\int y = \int \lim x_n' = \lim \int x_n' = x(t) - x(0)$ so $x(t) = x(0) + \int y$. Thus $x \in X$ and $Ax = x' = y$. Hence A has a closed graph.

***29b.** Consider $A : \mathbb{R} \to \mathbb{R}$ given by $A(x) = 1/x$ if $x \neq 0$ and $A(0) = 0$. Then A is a discontinuous operator from a Banach space to a normed linear space with a closed graph.

10.5 Topological vector spaces

30a. Let $\mathcal{B}$ be a collection of subsets containing θ. Suppose $\mathcal{B}$ is a base at θ for a translation invariant topology. By definition of a base, if $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that $W \subset U \cap V$ so (i) holds. If $U \in \mathcal{B}$ and $x \in U$, then $U - x$ is open so there exists $V \in \mathcal{B}$ such that $V \subset U - x$. Then $x + V \subset U$ so (ii) holds.

Conversely, suppose (i) and (ii) hold. Let $\mathcal{T} = \{O : x \in O \Rightarrow \exists y \in X \text{ and } U \in \mathcal{B} \text{ such that } x \in y + U \subset O\}$. It follows that $\mathcal{T}$ contains $\emptyset$ and X, and is closed under union. If $x \in O_1 \cap O_2$, then there exist $y_1, y_2 \in X$ and $U_1, U_2 \in \mathcal{B}$ such that $x \in y_i + U_i \in O_i, i = 1, 2$. Now $x - y_i \in U_i$ so by (ii), there exists $V_i \in \mathcal{B}$ such that $x - y_i + V_i \subset U_i$. i.e. $x + V_i \subset y_i + U_i \subset O_i$. Now by (i), there exists $W \in \mathcal{B}$ such that $W \subset V_1 \cap V_2$ so $x \in x + W \subset O_1 \cap O_2$. Thus $\mathcal{T}$ is closed under finite intersection. If $O \in \mathcal{T}$ and $y \in x + O$, then $y - x \in O$ so there exists $z \in X$ and $U \in \mathcal{B}$ such that $y - x \in z + U \subset O$. Thus $y \in x + z + U \subset x + O$ so $x + O \in \mathcal{T}$. Hence $\mathcal{T}$ is a translation invariant topology. Furthermore, if $\theta \in O$, then there exist $x \in X$ and $U \in \mathcal{B}$ such that $\theta \in x + U \subset O$. Thus $-x \in U$ so there exists $V \in \mathcal{B}$ such that $-x + V \subset U$. i.e. $V \subset x + U$. Then $\theta \in V \subset O$. Thus $\mathcal{B}$ is a base at θ.

30b. Let $\mathcal{B}$ be a base at θ for a translation invariant topology. Suppose addition is continuous from $X \times X$ to X. In particular, addition is continuous at $\langle \theta, \theta \rangle$. Thus for each $U \in \mathcal{B}$, there exists $V_1, V_2 \in \mathcal{B}$ such that $V_1 + V_2 \subset U$. Take $V \in \mathcal{B}$ with $V \subset V_1 \cap V_2$. Then $V + V \subset U$. Conversely, suppose (iii) holds. For $x_0, y_0 \in X$, $\{x_0 + y_0 + U : U \in \mathcal{B}\}$ is a base at $x_0 + y_0$. Now for each $U \in \mathcal{B}$, pick $V \in \mathcal{B}$ such that $V + V \subset U$. If $x \in x_0 + V$ and $y \in y_0 + V$, then $x + y \in x_0 + y_0 + U$. Thus addition is continuous from $X \times X$ to X.

30c. Suppose scalar multiplication is continuous (at $\langle 0, \theta \rangle$) from $\mathbb{R} \times X$ to X. Given $U \in \mathcal{B}$ and $x \in X$, there exist $\varepsilon > 0$ and $V \in \mathcal{B}$ such that $\beta(x + V) \subset U$ for $|\beta| < \varepsilon$. Let $\alpha = 2/\varepsilon$. Then $\frac{1}{\alpha}(x + V) \subset U$ so $x + V \subset \alpha U$. In particular, $x \in \alpha U$.

30d. Let X be a topological vector space and let $\mathcal{B}$ be the family of all open sets U that contain θ and such that $\alpha U \subset U$ for all α with $|\alpha| < 1$. If O is an open set containing θ, then continuity of scalar multiplication implies that there is an open set V containing θ and an $\varepsilon > 0$ such that $\lambda V \subset O$ for all $|\lambda| < \varepsilon$. Let $U = \bigcup_{|\lambda|<\varepsilon} \lambda V$. Then V is open, $\theta \in U \subset O$, and $\alpha U \subset U$ for α with $|\alpha| < 1$. Thus $\mathcal{B}$ is a local base for the topology and it satisfies (v) by its definition.

***30e.** Suppose $\mathcal{B}$ satisfies the conditions of the proposition. By part (a), $\mathcal{B}$ is a base at θ for a translation invariant topology and by part (b), addition is continuous. Given $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subset U$. Now given $x \in X$, there exists $\alpha \in \mathbb{R}$ such that $x \in \alpha V$. Let $\varepsilon = 1/|\alpha|$. If $|\lambda| < \min(\varepsilon, 1)$, then $\lambda(x + V) \subset U$ since $\lambda x \in \lambda\alpha V \subset V$ and $\lambda V \subset V$. Thus scalar multiplication is continuous at $\langle 0, x\rangle$. Given $U \in \mathcal{B}$ and $\alpha \in \mathbb{R}$, let β be such that $0 < |\beta| \leq 1$ and choose n such that $|\beta|^{-n} > |\alpha|$. Let $\varepsilon = |\beta|^n - |\alpha|$. Now when $|\lambda - \alpha| < \varepsilon$, we have $|\lambda| < |\beta|^{-n}$ since $|\alpha| = \max(\alpha, -\alpha)$. Then $\beta^n U \in \mathcal{B}$ and $\lambda\beta^n U \subset U$. Thus scalar multiplication is continuous at $\langle \alpha, \theta\rangle$. Now given $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subset U$.

Since scalar multiplication is continuous at $\langle 0, x\rangle$ and $\langle \alpha, \theta\rangle$, given $\alpha_0 \in \mathbb{R}$ and $x_0 \in X$, there exist $\varepsilon > 0$ and $W, W' \in \mathcal{B}$ such that $\alpha(x_0 + W) \subset V$ when $|\alpha| < \varepsilon$ and $\alpha W' \subset V$ when $|\alpha - \alpha_0| < \varepsilon$. Then $\alpha x - \alpha_0 x_0 = \alpha(x - x_0) + (\alpha - \alpha_0)x_0 \in V + V \subset U$ when $|\alpha - \alpha_0| < \varepsilon$ and $x \in x_0 + W'$. Hence scalar multiplication is continuous from $\mathbb{R} \times X$ to X.

30f. Suppose X is T_1. If $x \neq \theta$ and $x \in \bigcap\{U \in \mathcal{B}\}$, then any open set containing θ will also contain x. Contradiction. Hence (vi) holds. Conversely, suppose (vi) holds. Given two distinct points x and y, there exists $U \in \mathcal{B}$ such that $x - y \notin U$. Also, there exists $V \in \mathcal{B}$ such that $V + V \subset U$. If $(x+V) \cap (y+V) \neq \emptyset$, then $x - y \in V - V$. By (v), $-V \subset V$ so $x - y \in V + V \subset U$. Contradiction. Thus $x + V$ and $y + V$ are disjoint open sets separating x and y so X is Hausdorff.

(*) Proof of Proposition 14

31a. Suppose a linear transformation f from one topological vector space X to a topological vector space Y is continuous at one point. We may assume f is continuous at the origin. Let O be an open set containing the origin in Y. There exists an open set U containing the origin in X such that $f[U] \subset O$. Since f is linear, for any $x \in X$, $f[x + U] = f(x) + f[U] \subset f(x) + O$. Hence f is uniformly continuous.

***31b.** Let f be a linear functional on a topological vector space X. Suppose f is continuous. Let I be a bounded open interval containing 0. There exists an open set O containing the origin in X such that $f[O] \subset I$. Thus $f[O] \neq \mathbb{R}$. Conversely, suppose there is a nonempty open set O such that $f[O] \neq \mathbb{R}$. Take $x \in O$. Then $O - x$ is an open neighbourhood of θ so there is an open neighbourhood U of θ such that $U \subset O - x$ and $\alpha U \subset U$ for α with $|\alpha| < 1$. Now $f[U] \neq \mathbb{R}$ and $\alpha f[U] = f[\alpha U] \subset f[U]$ if $|\alpha| < 1$ so $f[U]$ is a bounded interval. Thus f is continuous at θ and thus continuous everywhere.

32. Let X be a topological vector space and M a closed linear subspace. Let φ be the natural homomorphism of X onto X/M, and define a topology on X/M by taking O to be open if and only if $\varphi^{-1}[O]$ is open in X. Clearly, φ is continuous. If U is open in X, then $\varphi^{-1}[\varphi[U]] = \bigcup_{m\in M}(m + U)$, which is open so $\varphi[U]$ is open. Now let O be an open set containing $x' + y' \in X/M$. Then $\varphi^{-1}[O]$ is an open set containing $x + y \in X$. There exist open sets U and V containing x and y respectively such that $U + V \subset \varphi^{-1}[O]$. Since φ is open, $\varphi[U]$ and $\varphi[V]$ are open sets containing x' and y' respectively. Then $\varphi[U] + \varphi[V] \subset \varphi[\varphi^{-1}[O]] = O$. Thus addition is continuous from $X/M \times X/M$ to X/M. Now let O be an open set containing $cx' \in X/M$. Then $\varphi^{-1}[O]$ is an open set containing $cx \in X$. There exist open sets U and V containing c and x respectively such that $UV \subset \varphi^{-1}[O]$. Since φ is open, $\varphi[V]$ is an open set containing x'. Then $U\varphi[V] \subset \varphi[\varphi^{-1}[O]] = O$. Thus scalar multiplication is continuous from $\mathbb{R} \times X/M$ to X/M. Hence X/M is a topological vector space.

33a. Suppose X is finite dimensional topological vector space. Let $x_1, \ldots, x_n$ be a vector space basis of X and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Define a linear map φ of $\mathbb{R}^n$ to X so that $\varphi(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i x_i$. Then φ is one-to-one and thus onto. If a sequence $\langle \sum a_i^{(j)} e_i\rangle$ in $\mathbb{R}^n$ converges to $\sum a_i e_i$, then $\langle a_i^{(j)}\rangle$ converges to a_i for each i. Since addition and scalar multiplication are continuous on X, the sequence $\langle \sum a_i^{(j)} x_i\rangle$ converges to $\sum a_i x_i$. Thus φ is continuous.

33b. Suppose X is Hausdorff. Let S and B be the subsets of $\mathbb{R}^n$ defined by $S = \{y : ||y|| = 1\}$ and $B = \{y : ||y|| < 1\}$. Since S is compact and φ is continuous, $\varphi[S]$ is compact in X and thus closed. Then $X \setminus \varphi[S]$ is open.

33c. Let $\mathcal{B}$ be a base at θ satisfying the conditions of Proposition 14. Since $X \setminus \varphi[S]$ is an open set containing θ, there exists $U \in \mathcal{B}$ such that $U \subset X \setminus \varphi[S]$. Furthermore, $\alpha U \subset U$ for each $|\alpha| < 1$ by condition (v) of Proposition 14.

33d. Suppose $u = \varphi(y) \in U$ for some y with $||y|| > 1$. Then $\frac{1}{||y||} < 1$ so $\frac{1}{||y||}u \in U$ but $\frac{1}{||y||}u = \frac{1}{||y||}\varphi(y) = \varphi(\frac{y}{||y||})$ where $||\frac{y}{||y||}|| = 1$ so $\frac{1}{||y||}u \in \varphi[S]$. Contradiction. Hence $U \subset \varphi[B]$. Thus $\varphi[B]$ is open and φ^{-1} is continuous.

(*) Proof of Proposition 15.

34. Let M be a finite dimensional subspace of a Hausdorff topological vector space X. If $x \notin M$, let N be the finite dimensional subspace spanned by x and M. Then N has the usual topology so x is not a point of closure of M. Hence M is closed.

35. Let A be a linear mapping of a finite dimensional Hausdorff vector space X into a topological vector space Y. The range of A is a finite dimensional subspace of Y so it has the usual topology. If $x_n \to x$ in X where $x_n = \sum a_i^{(n)} e_i$ and $x = \sum a_i e_i$, then $a_i^{(n)} \to a_i$ for each i. Since the range of A has the usual topology, $Ax_n = \sum a_i^{(n)} Ae_i \to \sum a_i Ae_i = Ax$. Hence A is continuous.

***36.** Let A be a linear mapping from a topological vector space X to a finite dimensional topological space Y. If A is continuous, then its kernel M is closed by Q14. Conversely, suppose M is closed. There is a unique linear mapping $B : X/M \to Y$ such that $A = B \circ \varphi$ where $\varphi : X \to X/M$ is the natural homomorphism, which is a continuous open map by Q32. Then $\ker \varphi$ is closed and φ is an open map so X/M is a Hausdorff topological vector space. Furthermore, B is one-to-one and Y is finite dimensional so X/M is finite dimensional. By Q35, B is continuous. Hence A is continuous.

***37.** Let X be a locally compact Hausdorff vector space. Let V be a neighbourhood of θ with $\bar{V}$ compact and $\alpha V \subset V$ for each α with $|\alpha| < 1$. The set $\{x + \frac{1}{3}V : x \in \bar{V}\}$ is an open cover of $\bar{V}$ so $\bar{V}$ can be covered by a finite number of translates $x_1 + \frac{1}{3}V, \ldots, x_n + \frac{1}{3}V$. To show that $x_1, \ldots, x_n$ span X, it suffices to show that they span V. Let $x \in V$. Then $x = x_{k_1} + \frac{1}{3}u_1$ for some $k_1 \in \{1, \ldots, n\}$ and $u_1 \in U$. Now $\frac{1}{3}U$ is covered by $\frac{1}{3}(x_1 + \frac{1}{3}V), \ldots, \frac{1}{3}(x_n + \frac{1}{3}V)$ so $x = x_{k_1} + \frac{1}{3}x_{k_2} + \frac{1}{9}u_2$. Continuing in this way, we have $x \in x_{k_1} + \frac{1}{3}x_{k_2} + \cdots + \frac{1}{3^{r-2}}x_{k_r} + \frac{1}{3^r}U$ for each r. Let $y_r = x_{k_1} + \frac{1}{3}x_{k_2} + \cdots + \frac{1}{3^{r-2}}x_{k_r}$. Then y_r is in the span of S, which is finite dimensional and thus closed by Q34.

Now for each $y \in \bar{V}$, there exists $\varepsilon_y > 0$ and an open neighbourhood U_y of y such that $\delta U_y \subset V$ whenever $|\delta| < \varepsilon_y$ since scalar multiplication is continuous at $\langle 0, y\rangle$. The open sets U_y cover $\bar{V}$ so $\bar{V} \subset U_{y_1} \cup \cdots \cup U_{y_n}$. Then $\delta\bar{V} \subset V$ whenever $|\delta| < \min_{1 \le k \le n} \varepsilon_{y_k}$.

Choose N such that $\delta_0 \bar{V} \subset V$ whenever $|\delta_0| < 3^{-N}$. Let $\delta > 0$ be given. Choose M such that $3^{-M}\delta^{-1} < 3^{-N}$. When $m \ge M$, we have $3^{-m}\delta^{-1} < 3^{-N}$ so $3^{-m}\delta^{-1}\bar{V} \subset V$. i.e. $3^{-m}\bar{V} \subset \delta V$. Thus $x - y_m \in \delta V$ for $m \ge M$. Thus $x - y_m \to \theta$ and $y_m \to x$. Hence x is in the span of S. It follows that $x_1, \ldots, x_n$ span X so X is finite dimensional.

10.6 Weak topologies

38a. Suppose $x_n \to x$ weakly. Then $f(x_n) \to f(x)$ for each $f \in X^*$. Thus $|f(x_n)| \le C_f$ for each $f \in X^*$ and each n. Now let $\varphi : X \to X^{**}$ be the natural homomorphism so that $\varphi(x_n)(f) = f(x_n)$. Then $|\varphi(x_n)(f)| \le C_f$ for each $f \in X^*$ and each n. Since X^* is a Banach space, $\langle ||\varphi(x_n)||\rangle$ is bounded but $||\varphi(x_n)|| = ||x_n||$ so $\langle ||x_n||\rangle$ is bounded.

***38b.** Let $\langle x_n\rangle$ be a sequence in $\ell^p, 1 < p < \infty$, and let $x_n = \langle \xi_{m,n}\rangle_{m=1}^{\infty}$. Suppose $\langle x_n\rangle$ converges weakly to $x = \langle \xi_m\rangle$. By part (a), $\langle ||x_n||\rangle$ is bounded. Each bounded linear functional F on ℓ^p is given by $F(x_n) = \sum_m \xi_{m,n}\eta_m$ for some $\langle \eta_m\rangle \in \ell^q$. Conversely, taking the m-th term to be 1 and the remaining terms to be 0 gives a sequence $\langle \eta_m\rangle \in \ell^q$ so that $F(x_n) = \sum \xi_{m,n}\eta_m = \xi_{m,n}$ is a bounded linear functional on ℓ^p. Thus $\langle F(x_n)\rangle$ converges to $F(x)$. i.e. $\langle \xi_{m,n}\rangle_{m=1}^{\infty}$ converges to ξ_m.

Conversely, suppose $\langle ||x_n||\rangle$ is bounded and for each m we have $\xi_{m,n} \to \xi_m$. Then $||x_n|| \le C$ and $||x|| \le C$ for some C. Let $F \in (\ell^p)^* = \ell^q$. Let e_n be the sequence having 1 as the n-th term and 0 elsewhere. Then $span\{e_n\}$ is dense in ℓ^q so there exists $\langle F_n\rangle$ with $F_n \in span\{e_n\}$ and $F_n \to F$. Given $\varepsilon > 0$, there exists N such that $||F_N - F|| < \varepsilon/3C$. Since $F_N \in span\{e_n\}$ and $\xi_{m,n} \to \xi_m$ for each n, there is an M such that $|F_N(x_n) - F_N(x)| < \varepsilon/3$ for $n \ge M$. Thus for $n \ge M$, we have $|F(x_n) - F(x)| \le |F(x_n) - F_N(x_n)| + |F_N(x_n) - F_N(x)| + |F_N(x) - F(x)| < \varepsilon$. Hence $\langle x_n\rangle$ converges weakly to x.

38c. Let $\langle x_n \rangle$ be a sequence in $L^p[0,1], 1 < p < \infty$. Suppose $\langle ||x_n|| \rangle$ is bounded and $\langle x_n \rangle$ converges to x in measure. By Corollary 4.19, every subsequence of $\langle x_n \rangle$ has in turn a subsequence that converges a.e. to x. By Q6.17, every subsequence of $\langle x_n \rangle$ has in turn a subsequence $\langle x_{n_{k_j}} \rangle$ such that for each $y \in L^q[0,1]$ we have $\int x_{n_{k_j}} y \to \int xy$. Now for each bounded linear functional F on $L^p[0,1]$, we have $F(x) = \int xy$ for some $y \in L^q[0,1]$. Thus every subsequence of $\langle x_n \rangle$ has in turn a subsequence $\langle x_{n_{k_j}} \rangle$ such that $F(x_{n_{k_j}}) \to F(x)$. By Q2.12, $F(x_n) \to F(x)$. Hence $\langle x_n \rangle$ converges weakly to x.

38d. Let $x_n = n\chi_{[0,1/n]}$ for each n. Then $x_n \to 0$ and $||x_n||_1 = 1$ for all n. In particular, $\langle x_n \rangle$ is a sequence in $L^1[0,1]$ converging to 0 in measure. Let $y = \chi_{[0,1]} \in L^\infty[0,1]$. Then $F(x) = \int xy$ is a bounded linear functional on $L^1[0,1]$ and $F(0) = 0$ while $F(x_n) = ||x_n||_1 = 1$ for each n so $F(x_n)$ does not converge to $F(0)$. Hence $\langle x_n \rangle$ does not converge weakly to 0.

(*) See Q6.17

38e. In $\ell^p, 1 < p < \infty$, let x_n be the sequence whose n-th term is one and whose remaining terms are zero. For any bounded linear functional F on ℓ^p, there exists $\langle y_n \rangle \in \ell^q$ such that $F(x_n) = y_n$. Then $F(x_n) = y_n \to 0$ since $\langle y_n \rangle \in \ell^q$. Thus $x_n \to 0$ in the weak topology. If $\langle x_n \rangle$ converges in the strong topology, then it must converge to 0 but $||x_n||_p = 1$ for all n so it does not converge to 0 and thus does not converge in the strong topology.

38f. Let x_n be as in part (e), and define $y_{n,m} = x_n + nx_m$. Let $F = \{y_{n,m} : m > n\}$. Note that the distance between any two points in F is at least 1 so there are no nonconstant sequences in F that converge in the strong topology. Any sequence in F that converges must be a constant sequence so its limit is in F. Hence F is strongly closed.

38g. Let F be as in part (f). The sets $\{x : |f_i(x)| < \varepsilon, i = 1, \dots, n\}$ where $\varepsilon > 0$ and $f_1, \dots, f_n \in (\ell^p)^*$ form a base at θ for the weak topology. Given $\varepsilon > 0$ and $f_1, \dots, f_n \in (\ell^p)^*$, $f_i(y_{m,n})$ is of the form $\xi^i_n + n\xi^i_m$ where $\langle \xi^i_n \rangle \in \ell^q$. Choose n such that $|\xi^i_n| < \varepsilon/2$ for all i. Then choose $m > n$ such that $|\xi^i_m| < \varepsilon/2n$ for all i. Then $|f_i(y_{m,n})| \leq |\xi^i_n| + n|\xi^i_m| < \varepsilon$. Thus $F \cap \{x : |f_i(x)| < \varepsilon, i = 1, \dots, n\} \neq \emptyset$ and θ is a weak closure point of F.

Suppose $\langle z_k \rangle = \langle y_{m_k,n_k} \rangle = \langle x_{n_k} + n_k x_{m_k} \rangle$ is a sequence from F that converges weakly to zero. Given $\varepsilon > 0$ and $\langle \xi_n \rangle \in \ell^q$, there exists N such that $|\xi_{n_k} + n_k \xi_{m_k}| < \varepsilon$ for $k \geq N$. Suppose $\{m_k\}$ is bounded above. Then some m is repeated infinitely many times. Let $\xi_m = 1$ and $\xi_n = 0$ otherwise. For each N there exists $k \geq N$ such that $m_k = m$ so $|\xi_{n_k} + n_k \xi_{m_k}| = |n_k| \geq 1$. Thus $\{m_k\}$ is not bounded above and we may assume the sequence $\langle m_k \rangle$ is strictly increasing. Now suppose $\{n_k\}$ is bounded above. Then some n is repeated infinitely many times. Let $\xi_n = 1$ and $\xi_m = 0$ otherwise. For each N there exists $k \geq N$ such that $n_k = n$ so $|\xi_{n_k} + n_k \xi_{m_k}| = 1$. Thus $\{n_k\}$ is not bounded above and we may assume the sequence $\langle n_k \rangle$ is strictly increasing. Now let $\xi_{m_k} = 1/n_k$ for each k and $\xi_m = 0$ if $m \neq m_k$ for any k. Then $\langle \xi_n \rangle \in \ell^q$ and $|\xi_{n_k} + n_k \xi_{m_k}| \geq 1$ for all k. Contradiction. Hence there is no sequence $\langle z_k \rangle$ from F that converges weakly to zero.

38h. The weak topology of ℓ^1 is the weakest topology such that all functionals in $(\ell^1)^* = \ell^\infty$ are continuous. A base at θ is given by the sets $\{x \in \ell^1 : |f_i(x)| < \varepsilon, i = 1, \dots, n\}$ where $\varepsilon > 0$ and $f_1, \dots, f_n \in \ell^\infty$. A net $\langle (x^{(\alpha)}_n) \rangle$ in ℓ^1 converges weakly to $(x_n) \in \ell^1$ if and only if $\sum_n x^{(\alpha)}_n y_n$ converges to $\sum_n x_n y_n$ for each $(y_n) \in \ell^\infty$.

If a net $\langle (x^{(\alpha)}_n) \rangle$ in ℓ^1 converges weakly to $(x_n) \in \ell^1$, then for each n, taking $(y_n) \in \ell^\infty$ where $y_n = 1$ and $y_m = 0$ for $m \neq n$, we have $x^{(\alpha)}_n$ converging to x_n for each n.

If the net $\langle (x^{(\alpha)}_n) \rangle$ in ℓ^1 is bounded, say by M, and $x^{(\alpha)}_n$ converges to x_n for each n, then $\sum_n |x_n| \leq \sum_n |x^{(\alpha)}_n - x_n| + \sum_n |x^{(\alpha)}_n| \leq M$ so $(x_n) \in \ell^1$. If $(y_n) \in \ell^\infty$, then $|\sum_n (x^{(\alpha)}_n - x_n) y_n| \leq ||(y_n)||_\infty \sum_n |x^{(\alpha)}_n - x_n| \to 0$ so $\sum_n x^{(\alpha)}_n y_n$ converges to $\sum_n x_n y_n$ and $\langle (x^{(\alpha)}_n) \rangle$ converges weakly to (x_n).

For $k \in \mathbb{N}$, let $(x^{(k)}_n) \in \ell^1$ where $x^{(k)}_k = k$ and $x^{(k)}_n = 0$ if $n \neq k$. Then the sequence $\langle (x^{(k)}_n) \rangle$ is not bounded and $x^{(k)}_n$ converges to 0 for each n. However, taking $(y_n) \in \ell^\infty$ where $y_n = 1$ for all n, we have $\sum_n x^{(k)}_n y_n = k$, which does not converge to 0 so $\langle (x^{(k)}_n) \rangle$ does not converge weakly to θ.

The weak* topology on ℓ^1 as the dual of c_0 is the weakest topology such that all functionals in $\varphi[c_0] \subset \ell^\infty$ are continuous. A base at θ is given by the sets $\{f \in \ell^1 : |f(x_i)| < \varepsilon, i = 1, \dots, n\}$ where $\varepsilon > 0$ and $x_1, \dots, x_n \in c_0$. A net $\langle (x^{(\alpha)}_n) \rangle$ in ℓ^1 is weak* convergent to $(x_n) \in \ell^1$ if and only if $\sum_n x^{(\alpha)}_n y_n$ converges to $\sum_n x_n y_n$ for all $(y_n) \in c_0$.

Using the same arguments as above and replacing ℓ^∞ by c_0, we see that if a net $\langle (x_n^{(\alpha)}) \rangle$ in ℓ^1 is weak* convergent to $(x_n) \in \ell^1$, then $x_n^{(\alpha)}$ converges to x_n for each n. We also see that if the net is bounded and $x_n^{(\alpha)}$ converges to x_n, then the net is weak* convergent to (x_n).

For $k \in \mathbb{N}$, let $(x_n^{(k)}) \in \ell^1$ where $x_k^{(k)} = k$ and $x_n^{(k)} = 0$ if $n \neq k$. Then the sequence $\langle (x_n^{(k)}) \rangle$ is not bounded and $x_n^{(k)}$ converges to 0 for each n. However, taking $(y_n) \in c_0$ where $y_n = 1/n$ for all n, we have $\sum_n x_n^{(k)} y_n = 1$, which does not converge to 0 so $\langle (x_n^{(k)}) \rangle$ is not weak* convergent to θ.

39a. Let $X = c_0, \mathcal{F} = \ell^1$ and $\mathcal{F}_0$ the set of sequences with finitely many nonzero terms, which is dense in ℓ^1. Consider the sequence $\langle (x_n^{(k)}) \rangle$ in c_0 where $x_k^{(k)} = k^2$ and $x_n^{(k)} = 0$ if $n \neq k$. For any sequence $(y_n) \in \mathcal{F}_0$, we have $\sum_n x_n^{(k)} y_n = k^2 y_k \to 0$. Thus the sequence $\langle (x_n^{(k)}) \rangle$ converges to zero in the weak topology generated by $\mathcal{F}_0$. Now if the sequence $\langle (x_n^{(k)}) \rangle$ converges in the weak topology generated by $\mathcal{F}$, the weak limit must then be zero. Let (z_n) be the sequence in $\mathcal{F}$ where $z_n = 1/n^2$ for each n. Then $\sum_n x_n^{(k)} z_n = 1$. Thus the sequence $\langle (x_n^{(k)}) \rangle$ does not converge in the weak topology generated by $\mathcal{F}$. Hence $\mathcal{F}$ and $\mathcal{F}_0$ generate different weak topologies for X.

Now suppose S is a bounded subset of X. We may assume that S contains θ. Let $\mathcal{F}$ be a set of functionals in X^* and let $\mathcal{F}_0$ be a dense subset of $\mathcal{F}$ (in the norm topology on X^*). Note that in general, the weak topology generated by $\mathcal{F}_0$ is weaker than the weak topology generated by $\mathcal{F}$. A base at θ for the weak topology on S generated by $\mathcal{F}$ is given by the sets $\{x \in S : |f_i(x)| < \varepsilon, i = 1, \ldots, n\}$ where $\varepsilon > 0$ and $f_1, \ldots, f_n \in \mathcal{F}$. A set in a base at θ for the weak topology on S generated by $\mathcal{F}_0$ is also in a base at θ for the weak topology generated by $\mathcal{F}$. Suppose $x \in S$ so that $||x|| \leq M$ and $|f_i(x)| < \varepsilon$ for some $\varepsilon > 0$ and $f_1, \ldots, f_n \in \mathcal{F}$. For each i, there exists $g_i \in \mathcal{F}_0$ such that $||f_i - g_i|| < \varepsilon/2M$. If $|g_i(x)| < \varepsilon/2$ for $i = 1, \ldots, n$, then $|f_i(x)| \leq |f_i(x) - g_i(x)| + |g_i(x)| \leq ||f_i - g_i||\,||x|| + |g_i(x)| < \varepsilon$ for $i = 1, \ldots, n$. Thus any set in a base at θ for the weak topology on S generated by $\mathcal{F}$ contains a set in a base at θ for the weak topology generated by $\mathcal{F}_0$. Hence the two weak topologies are the same on S.

***39b.** Let S^* be the unit sphere in the dual X^* of a separable Banach space X. Let $\{x_n\}$ be a countable dense subset of X. Then $\{\varphi(x_n)\}$ is a countable dense subset of $\varphi[X]$. By part (a), $\{\varphi(x_n)\}$ generates the same weak topology on S^* as $\varphi[X]$. i.e. $\{\varphi(x_n)\}$ generates the weak* topology on S^*. Now define $\rho(f, g) = \sum 2^{-n} \frac{|f(x_n) - g(x_n)|}{1 + |f(x_n) - g(x_n)|}$. Then ρ is a metric on S^*. Furthermore $\rho(f_n, f) \to 0$ if and only if $|f_n(x_k) - f(x_k)| \to 0$ for each k (see Q7.24a) if and only if $|\varphi(x_k)(f_n) - \varphi(x_k)(f)| \to 0$ for each k if and only if $f_n \to f$ in the weak* topology. Hence S^* is metrizable.

40. Suppose X is a weakly compact set. Every $x^* \in X^*$ is continuous so $x^*[X]$ is compact in $\mathbb{R}$, and thus bounded, for each $x^* \in X^*$. For each $x \in X$ and $x^* \in X^*$, there is a constant M_{x^*} such that $|\varphi(x)(x^*)| = |x^*(x)| \leq M_{x^*}$. Thus $\{||\varphi(x)|| : x \in X\}$ is bounded. Since $||\varphi(x)|| = ||x||$ for each x, we have $\{||x|| : x \in X\}$ is bounded.

41a. Let S be the linear subspace of $C[0,1]$ given in Q28 (S is closed as a subspace of $L^2[0,1]$. Suppose $\langle f_n \rangle$ is a sequence in S such that $f_n \to f$ weakly in L^2. By Q28c, for each $y \in [0,1]$, there exists $k_y \in L^2$ such that for all $f \in S$ we have $f(y) = \int k_y f$. Now $\int f_n k_y \to \int f k_y$ for each $y \in [0,1]$ since $k_y \in L^2 = (L^2)^*$. Thus $f_n(y) \to f(y)$ for each $y \in [0,1]$.

41b. Suppose $\langle f_n \rangle$ is a sequence in S such that $f_n \to f$ weakly in L^2. By Q38a, $\langle ||f_n||_2 \rangle$ is bounded. By Q28b, there exists M such that $||f||_\infty \leq M||f||_2$ for all $f \in S$. In particular, $||f_n||_\infty \leq M||f_n||_2$ for all n. Hence $\langle ||f_n||_\infty \rangle$ is bounded. Now $\langle f_n^2 \rangle$ is a sequence of measurable functions with $|f_n|^2 \leq M'$ on $[0,1]$ and $f_n^2(y) \to f^2(y)$ for each $y \in [0,1]$ as a consequence of part (a). By the Lebesgue Convergence Theorem, $||f_n||_2^2 \to ||f||_2^2$ and so $||f_n||_2 \to ||f||_2$. By Q6.16, $f_n \to f$ strongly in L^2.

***41c.** Since L^2 is reflexive and S is a closed linear subspace, S is a reflexive Banach space by Q23d. By Alaoglu's Theorem, the unit ball of S^{**} is weak* compact. Then the unit ball of S is weakly compact since the weak* topology on S^{**} induces the weak topology on S when S is regarded as a subspace of S^{**}. By part (b), the unit ball of S is compact. Thus S is locally compact Hausdorff so by Q37, S is finite dimensional.

10.7 Convexity

42. Let A be a linear operator from the vector space X to the vector space Y. Let K be a convex set in X. If $x, y \in X$ and $0 \le \lambda \le 1$, then $\lambda Ax + (1-\lambda)Ay = A(\lambda x) + A((1-\lambda)y) = A(\lambda x + (1-\lambda)y) \in A[X]$. Thus $A[X]$ is a convex set in Y. Let K' be a convex set in Y. If $x, y \in A^{-1}[K']$, then $Ax, Ay \in K'$ so if $0 \le \lambda \le 1$, then $\lambda Ax + (1-\lambda)Ay \in K'$. Now $\lambda Ax + (1-\lambda)Ay = A(\lambda x + (1-\lambda)y)$ so $\lambda x + (1-\lambda)y \in A^{-1}[K']$. Thus $A^{-1}[K']$ is a convex set in X. By using the linearity of A, it can be shown that a similar result holds when "convex set" is replaced by "linear manifold".

Define the linear operator $A : \mathbb{R}^2 \to \mathbb{R}$ by $A(\langle x, y\rangle) = x + y$. Take the non-convex set $\{\langle 0, 1\rangle, \langle 1, 0\rangle\} \subset \mathbb{R}^2$. Its image under A is the conves set $\{1\} \subset \mathbb{R}$.

43. Let K be a convex set in a topological vector space. Let $x, y \in \bar{K}$ and $0 \le \lambda \le 1$. Let O be an open set containing $\lambda x + (1-\lambda)y$. Since addition and scalar multiplication are continuous, there are open sets U and V containing x and y respectively, as well as $\varepsilon > 0$, such that $\mu U + \eta V \subset O$ whenever $|\mu - \lambda| < \varepsilon$ and $|\eta - (1-\lambda)| < \varepsilon$. Now there exist $x' \in K \cap U$ and $y' \in K \cap V$. Then $\lambda x' + (1-\lambda)y' \in K \cap O$. Hence $\lambda x + (1-\lambda)y \in \bar{K}$ and $\bar{K}$ is convex.

44a. Let x_0 be an interior point of a subset K of a topological vector space X. There is an open set O such that $x_0 \in O \subset K$. Let $x \in X$. By continuity of addition at $\langle x_0, \theta\rangle$, there exists an open set U containing θ such that $x_0 + U \subset O$. By continuity of scalar multiplication at $\langle 0, x\rangle$, there exists $\varepsilon > 0$ such that $\lambda x \in U$ whenever $|\lambda| < \varepsilon$. Thus $x_0 + \lambda x \in O \subset K$ whenever $|\lambda| < \varepsilon$ and x_0 is an internal point of K.

44b. In $\mathbb{R}$, a convex set must be an interval and an internal point must not be an endpoint of the interval so it is an interior point. In $\mathbb{R}^n$ for $n \ge 2$, let $\{e_i : i = 1, \ldots, n\}$ be the standard basis. Suppose $x = \langle x_1, \ldots, x_n\rangle$ is an internal point of a convex set K. For each i, there exists $\varepsilon_i > 0$ such that $x + \lambda e_i \in K$ if $|\lambda| < \varepsilon_i$. Let $\varepsilon = \min_{1\le i\le n} \varepsilon_i$. If $|\lambda| < \varepsilon$, then $x + \lambda e_i \in K$ for all i. Now suppose $||y - x|| < \varepsilon/n$. Then $|y_i - x_i| < \varepsilon/n$ for all i so $x + n(y_i - x_i)e_i \in K$ for all i. Note that $y = \sum_{i=1}^n \frac{1}{n}(x + n(y_i - x_i)e_i)$, which belongs to K since K is convex. Thus there is an open ball centred at x and contained in K so x is an interior point of K.

***44c.** Consider $K = \overline{B_{\langle 0,1\rangle,1}} \cup \overline{B_{\langle 0,-1\rangle,1}} \cup \{\langle x, 0\rangle : |x| < 1\} \subset \mathbb{R}^2$. Then $\langle 0, 0\rangle$ is an internal point of K but not an interior point.

44d. Suppose a convex set K in a topological vector space has an interior point x. Let y be an internal point of K. There exists $\varepsilon > 0$ such that $y + \frac{\varepsilon}{2}(y - x) \in K$. Now there is an open set O such that $x \in O \subset K$. Let $\lambda = \varepsilon/2$. Then $y = \frac{\lambda}{1+\lambda}x + \frac{1}{1+\lambda}[y + \lambda(y - x)] \in \frac{\lambda}{1+\lambda}O + \frac{1}{1+\lambda}[y + \lambda(y - x)] \subset \frac{\lambda}{1+\lambda}K + \frac{1}{1+\lambda}K \subset K$. Since $\frac{\lambda}{1+\lambda}O + \frac{1}{1+\lambda}[y + \lambda(y - x)]$ is an open neighbourhood of y contained in K, y is an interior point of K.

***44e.** Let X be a topological vector space that is of second Baire category with respect to itself. Suppose a closed convex subset K of X has an internal point y. Let $X_n = \{x \in X : y + tx \in C \text{ for all } t \in [0, 1/n]\}$. Each X_n is a closed subset of X since addition and scalar multiplication are continuous and C is closed. Now $X = \bigcup X_n$ so some X_n has an interior point x. Then $x \in O \subset X_n$ for some open set O. Thus $y + \frac{1}{n}x \in y + \frac{1}{n}O \subset C$ so $y + \frac{1}{n}x$ is an interior point of C.

(*) Assumption of convexity not necessary?

45. Let K be a convex set containing θ and suppose that x is an internal point of K. Since x is an internal point, there exists $\varepsilon > 0$ such that $x + \mu x \in K$ for $|\mu| < \varepsilon$. Choose λ such that $0 < \lambda < 1$ and $(1-\lambda)^{-1}x \in K$. Note that $0 < 1 - \lambda < 1$. For $y \in K$, $x + \lambda y = (1-\lambda)[(1-\lambda)^{-1}x] + \lambda y \in K$ since K is convex.

46. Let $\mathcal{N}$ be a family of convex sets (containing θ) in a vector space X. Suppose $\mathcal{N}$ satisfies (i), (ii) and (iii). To show that the translates of sets in $\mathcal{N}$ form a base for a topology that makes X into a locally convex topological vector space, it suffices to show that $\mathcal{N}$ is a base at θ. If $N_1, N_2 \in \mathcal{N}$, there is an $N_3 \in \mathcal{N}$ with $N_3 \subset N_1 \cap N_2$. Thus condition (i) of Proposition 14 is satisfied. If $N \in \mathcal{N}$ and $x \in N$, then x is internal. By Q45, there exists λ with $0 < \lambda < 1$ and $x + \lambda N \subset N$. Note that $\lambda N \in \mathcal{N}$. Thus condition (ii) of Proposition 14 is satisfied. If $N \in \mathcal{N}$, then $\frac{1}{2}N + \frac{1}{2}N \subset N$. Note that $\frac{1}{2}N \in \mathcal{N}$. Thus condition (iii) of Proposition 14 is satisfied. If $N \in \mathcal{N}$ and $x \in X$, then $\lambda x \in N$ for some $\lambda \in \mathbb{R}$ since $\theta \in N$ and θ is internal. Now $x \in \frac{1}{\lambda}N$ so condition (iv) of Proposition 14 is satisfied. If $N \in \mathcal{N}$ and $0 < |\alpha| < 1$, then $\alpha N \subset N$ since $\theta in N$ and N is convex. Also, $\alpha N \in \mathcal{N}$. Thus condition (v) of

Proposition 14 is satisfied. Hence $\mathcal{N}$ is a base at θ for a topology that makes X into a (locally convex) topological vector space.

Conversely, if X is a locally convex topological vector space, then there is a base for the topology consisting of convex sets. Consider the family $\mathcal{N}$ consisting of all sets N in the base containing θ together with αN for $0 < |\alpha| < 1$. Then $\mathcal{N}$ is a base at θ. If $N \in \mathcal{N}$, then each point of N is interior and thus internal by Q. If $N_1, N_2 \in \mathcal{N}$, there is an $N_3 \in \mathcal{N}$ with $N_3 \subset N_1 \cap N_2$ since $N_1 \cap N_2$ is an open set containing θ. Condition (iii) is satisfied by the definition of $\mathcal{N}$.

(*) Proof of Proposition 21.

***47.**

48a. In $L^p[0,1], 1 < p < \infty$, suppose $||x|| = 1$. If $x = \lambda y + (1-\lambda)z$ with $||y|| \leq 1$ and $||z|| \leq 1$, then $1 = ||x|| = ||\lambda y + (1-\lambda)z|| \leq \lambda||y|| + (1-\lambda)||z|| \leq 1$ so equality holds in the Minkowski inequality. Thus λy and $(1-\lambda)z$ are collinear. i.e. $\lambda y = \alpha(1-\lambda)z$ for some α. Also, $||y|| = ||z|| = 1$. Then $y = z$. Hence x is an extreme point of the unit sphere $S = \{x : ||x|| \leq 1\}$.

(*) Note that an extreme point of the unit sphere of any normed space must have norm one. Otherwise if $0 < ||x|| < 1$, then $x = ||x||\frac{x}{||x||} + (1 - ||x||)\theta$ so x cannot be an extreme point of the unit sphere.

***48b.**

***48c.** Let $x \in L^1[0,1]$ with $||x|| = 1$. Choose $t \in [0,1]$ such that $\int_0^t |x(t)|\, dt = \frac{1}{2}$. Define $y(s) = 2x(s)$ if $s \leq t$ and $y(s) = 0$ otherwise. Also define $z(s) = 2x(s)$ if $s \geq t$ and $z(s) = 0$ otherwise. Now $y, z \in L^1[0,1]$, $||y|| = ||z|| = 1$ and $x = \frac{1}{2}(y+z)$ so x is not an extreme point of the unit sphere. Since any point with norm one in $L^1[0,1]$ is not an extreme point of the unit sphere, the unit sphere has no extreme points.

48d. If $L^1[0,1]$ is the dual of some normed space, then its unit sphere is weak* compact and convex so it has extreme points, contradicting part (c). Hence $L^1[0,1]$ is not the dual of any normed space.

48e. For $1 < p < \infty$, every $\langle x_i \rangle$ with $||\langle x_i \rangle|| = 1$ is an extreme point of the unit sphere of ℓ^p by a similar argument as in part (a).

Let $\langle x_i \rangle \in \ell^1$ with $||\langle x_i \rangle|| = 1$. We may assume $|x_1| \geq |x_2| > 0$. Let $y_1 = (\text{sgn } x_1)(|x_1| - |x_2|)$ and $y_2 = 2x_2$. Also, let $z_1 = x_1 + (\text{sgn } x_1)|x_2|$ and $z_2 = 0$. Then $y_1 + z_1 = 2x_1$ and $y_2 + z_2 = 2x_2$. Also, $|y_1| + |y_2| = |x_1| + |x_2|$ and $|z_1| + |z_2| \leq |x_1| + |x_2|$. For $i > 2$, define $y_i = z_i = x_i$. Then $\langle x_i \rangle = \frac{1}{2}(\langle y_i \rangle + \langle z_i \rangle)$ with $\langle y_i \rangle, \langle z_i \rangle \in S$. Thus the unit sphere in ℓ^1 has no extreme points.

Let $\langle x_i \rangle \in \ell^\infty$ with $|x_i| = 1$ for all i. If $1 = x_i = \lambda y_i + (1-\lambda)z_i$ with $|y_i| \leq 1$, $|z_i| \leq 1$ and $0 < \lambda < 1$, then $y_i = z_i = 1$. Similarly, if $-1 = x_i = \lambda y_i + (1-\lambda)z_i$ with $|y_i| \leq 1$, $|z_i| \leq 1$ and $0 < \lambda < 1$, then $y_i = z_i = -1$. Thus $\langle x_i \rangle$ is an extreme point of the unit sphere in ℓ^∞. If $|x_N| < 1$ for some N, then define $y_i = x_i + \frac{1-|x_i|}{2}$ and $z_i = x_i - \frac{1-|x_i|}{2}$ for all i so that $\langle x_i \rangle = \frac{1}{2}(\langle y_i \rangle + \langle z_i \rangle)$. Furthermore, $\langle y_i \rangle, \langle z_i \rangle \in S$. Hence the extreme points of the unit sphere in ℓ^∞ are those $\langle x_i \rangle$ with $|x_i| = 1$ for all i.

***48f.** The constant functions ± 1 are extreme points of the unit sphere in $C(X)$ where X is a compact Hausdorff space. Let $f \in C(X)$ with $||f|| = 1$ and suppose $|f(x_0)| < 1$ for some $x_0 \in X$. Fix $\varepsilon > 0$ such that $0 < |f(x_0)| - \varepsilon$ and $|f(x_0)| + \varepsilon < 1$. Let $A = \{x : |f(x)| = |f(x_0)|\}$ and $B = \{x : |f(x)| \in [0, f(x_0) - \varepsilon] \cup [f(x_0) + \varepsilon, 1]\}$. Then A and B are disjoint closed subsets of the compact Hausdorff (and thus normal) space X so by Urysohn's Lemma, there exists $g \in C(X)$ such that $0 \leq g \leq \varepsilon$, $g \equiv 0$ on B and $g \equiv \varepsilon$ on A. Now $||f + g|| \leq 1$, $||f - g|| \leq 1$ and $f = \frac{1}{2}[(f+g) + (f-g)]$. Thus f is not an extreme point of the unit sphere in $C(X)$. Hence the extreme points of the unit sphere are those f where $|f(x)| = 1$ for all $x \in X$ and by continuity, these are the constant functions ± 1.

The only extreme points of the unit sphere in $C[0,1]$ are the constant functions ± 1. If $C[0,1]$ is the dual of some normed space, then its unit sphere is a weak* compact convex set so by the Krein-Milman Theorem, it is the closed convex hull of its extreme points and contains only constant functions. Contradiction. Hence $C[0,1]$ is not the dual of any normed space.

49a. Let X be the vector space of all measurable real-valued functions on $[0,1]$ with addition and scalar multiplication defined in the usual way. Define $\sigma(x) = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\, dt$. Since $\frac{|x(t)+y(t)|}{1+|x(t)+y(t)|} = 1 - \frac{1}{1+|x(t)+y(t)|} \leq 1 - \frac{1}{1+|x(t)|+|y(t)|} = \frac{|x(t)|+|y(t)|}{1+|x(t)|+|y(t)|} \leq \frac{|x(t)|}{1+|x(t)|} + \frac{|y(t)|}{1+|y(t)|}$, we have $\sigma(x+y) \leq \sigma(x) + \sigma(y)$. By defining $\rho(x,y) = \sigma(x-y)$, we have a metric for X.

49b. Suppose $x_n \to x$ in measure. Given $\varepsilon > 0$, there exists N such that for all $n \geq N$ we have $m\{t :$

$|x_n(t)-x(t)|\geq\varepsilon/2\}<\varepsilon/2$. Then for $n\geq N$, $\rho(x_n,x)=\int_0^1\frac{|x_n(t)-x(t)|}{1+|x_n(t)-x(t)|}\,dt\leq\int_{\{t:|x_n(t)-x(t)|\geq\varepsilon/2\}}1\,dt+\int_{\{t:|x_n(t)-x(t)|<\varepsilon/2\}}|x_n(t)-x(t)|\,dt<\varepsilon$. Thus $x_n\to x$ in the metric ρ. Conversely, suppose $\langle x_n\rangle$ does not converge to x in measure. There exists $\varepsilon>0$ such that for all N there is $n\geq N$ with $m\{t:|x_n(t)-x(t)|\geq\varepsilon\}\geq\varepsilon$. Thus there is a subsequence $\langle x_{n_k}\rangle$ such that $m\{t:|x_{n_k}(t)-x(t)|\geq\varepsilon\}\geq\varepsilon$ for all k. Now $\rho(x_{n_k},x)=\int_0^1\frac{|x_{n_k}(t)-x(t)|}{1+|x_{n_k}(t)-x(t)|}\,dt\geq\int_{\{|x_{n_k}-x|\geq\varepsilon\}}\frac{|x_{n_k}(t)-x(t)|}{1+|x_{n_k}(t)-x(t)|}\,dt\geq\frac{\varepsilon^2}{1+\varepsilon}$ for all k so the subsequence $\langle x_{n_k}\rangle$ does not converge to x in the metric ρ. Hence $\langle x_n\rangle$ does not converge to x in the metric ρ.

49c. Let $\langle x_n\rangle$ be Cauchy in the metric ρ. Suppose $\langle x_n\rangle$ is not Cauchy in measure. There exists $\varepsilon>0$ such that for all N there is $n,m\geq N$ with $m\{t:|x_n(t)-x_m(t)|\geq\varepsilon\}\geq\varepsilon$. Thus there is a subsequence $\langle x_{n_k}\rangle$ such that $m\{t:|x_{n_{2k-1}}(t)-x_{n_{2k}}(t)|\geq\varepsilon\}\geq\varepsilon$ for all k. Now $\rho(x_{n_{2k-1}},x_{n_{2k}})=\int_0^1\frac{|x_{n_{2k-1}}(t)-x_{n_{2k}}(t)|}{1+|x_{n_{2k-1}}(t)-x_{n_{2k}}(t)|}\,dt\geq\int_{\{|x_{n_{2k-1}}-x_{n_{2k}}|\geq\varepsilon\}}\frac{|x_{n_{2k-1}}(t)-x_{n_{2k}}(t)|}{1+|x_{n_{2k-1}}(t)-x_{n_{2k}}(t)|}\,dt\geq\frac{\varepsilon^2}{1+\varepsilon}$ for all k so the subsequence $\langle x_{n_k}\rangle$ is not Cauchy in the metric ρ. Contradiction. Hence $\langle x_n\rangle$ is Cauchy in measure. By Q4.25, $\langle x_n\rangle$ converges in measure and by part (b), it converges in the metric ρ. Hence X is a complete metric space.

49d. Given $\varepsilon>0$, let $\delta=\varepsilon/2$. When $\tau(\langle x,y\rangle,\langle x',y'\rangle)<\delta$, we have $\rho(x,x')<\delta$ and $\rho(y,y')<\delta$. Now $\rho(x+y,x'+y')=\sigma(x-x'+y-y')\leq\sigma(x-x')+\sigma(y-y')=\rho(x,x')+\rho(y,y')<\varepsilon$. Hence addition is a continuous mapping of $X\times X$ into X.

49e. Given $a\in\mathbb{R}$, $x\in X$ and $\varepsilon>0$, by Q3.23a, there exists M such that $|x|\leq M$ except on a set of measure less than $\varepsilon/3$. Let $\delta=\min(1,\varepsilon/3(|a|+1),\varepsilon/3M)$. When $\tau(\langle a,x\rangle,\langle c,x'\rangle)<\delta$, we have $|c-a|<\delta$ and $\rho(x,x')<\delta$. Now $\rho(ax,cx')=\int_0^1\frac{|ax(t)-cx'(t)|}{1+|ax(t)-cx'(t)|}\,dt\leq\int_0^1\frac{|a-c|\,|x(t)|}{1+|a-c|\,|x(t)|}\,dt+\int_0^1\frac{|c|\,|x(t)-x'(t)|}{1+|c|\,|x(t)-x'(t)|}\,dt<M\delta+\varepsilon/3+(|a|+\delta)\delta<\varepsilon$. Hence scalar multiplication is a continuous mapping of $\mathbb{R}\times X$ to X.

49f. Given $x\in X$ and $\varepsilon>0$, there is a step function s such that $|x-s|<\varepsilon/2$ except on a set of measure less than $\varepsilon/2$. i.e. $m\{t:|x(t)-s(t)|\geq\varepsilon/2\}<\varepsilon/2$. Now $\rho(x,s)=\int_0^1\frac{|x(t)-s(t)|}{1+|x(t)-s(t)|}\,dt=\int_{\{|x-s|<\varepsilon/2\}}\frac{|x(t)-s(t)|}{1+|x(t)-s(t)|}\,dt+\int_{\{|x-s|\geq\varepsilon/2\}}\frac{|x(t)-s(t)|}{1+|x(t)-s(t)|}\,dt<\int_0^1\varepsilon/2\,dt+\int_{\{|x-s|\geq\varepsilon/2\}}1\,dt<\varepsilon$. Hence the set of step functions is dense in X.

***49g.**

49h. By parts (d) and (e), X is a topological vector space. Since there are no nonzero continuous linear functionals on X by part (g), X cannot be locally convex.

49i. Let s be the space of all sequences of real numbers and define $\sigma(\langle\xi_v\rangle)=\sum\frac{2^{-v}|\xi_v|}{1+|\xi_v|}$.

Analogues of parts (a) and (c) follow from Q7.24. The analogue of part (d) follows from the same argument as above. For the analogue of part (e), suppose $a\in\mathbb{R}$, $\langle\xi_v\rangle\in s$ and $\varepsilon>0$ are given. Let $\delta=\min(1,\varepsilon/2|a|,\varepsilon/2(\sigma(\langle\xi_v\rangle)+1))$. When $\tau(\langle a,\langle\xi_v\rangle\rangle,\langle c,\langle\eta_v\rangle\rangle)<\delta$, we have $|c-a|<\delta$ and $\sigma(\langle\xi_v\rangle-\langle\eta_v\rangle)<\delta$. Now $\sigma(a\langle\xi_v\rangle-c\langle\eta_v\rangle)=\sum\frac{2^{-v}|a\xi_v-c\eta_v|}{1+|a\xi_v-c\eta_v|}\leq\sum\frac{2^{-v}|a|\,|\xi_v-\eta_v|}{1+|a|\,|\xi_v-\eta_v|}+\sum\frac{2^{-v}|a-c|\,|\eta_v|}{1+|a-c|\,|\eta_v|}=|a|\sigma(\langle\xi_v\rangle-\langle\eta_v\rangle)+|a-c|\sigma(\langle\eta_v\rangle)<|a|\delta+(\sigma(\langle\xi_v\rangle)+\delta)\delta<\varepsilon$. Hence scalar multiplication is a continuous mapping of $\mathbb{R}\times s$ into s.

Let f be a continuous linear functional on s. For each v, let e_v be the sequence where the v-th entry is 1 and all other entries are 0. Now any sequence $\langle\xi_v\rangle\in s$ can be expressed as $\sum\xi_ve_v$ so $f(\langle\xi_v\rangle)=f(\sum\xi_ve_v)=\sum\xi_vf(e_v)$ since f is continuous and linear. Since the series converges for each sequence $\langle\xi_v\rangle$, there exists N such that $f(e_v)=0$ for $v>N$. Thus $f(\langle\xi_v\rangle)=\sum_{v=1}^N\xi_vf(e_v)$.

10.8 Hilbert space

50. Suppose $x_n\to x$ and $y_n\to y$. Then there exists M such that $||x_n||\leq M$ for all n. Given $\varepsilon>0$, choose N such that $||y_n-y||<\varepsilon/2M$ and $||x_n-x||<\varepsilon/2||y||$ for $n\geq N$. Now $|(x_n,y_n)-(x,y)|\leq|(x_n,y_n-y)|+|(x_n-x,y)|\leq||x_n||\,||y_n-y||+||x_n-x||\,||y||<\varepsilon$ for $n\geq N$. Hence $(x_n,y_n)\to(x,y)$.

***51a.** The use of trigonometric identities shows that $\{\frac{1}{\sqrt{2\pi}},\frac{\cos vt}{\sqrt{\pi}},\frac{\sin vt}{\sqrt{\pi}}\}$ is an orthonormal system for $L^2[0,2\pi]$. Suppose $x\in L^2[0,2\pi]$ such that $(x,\frac{\cos vt}{\sqrt{\pi}})=0$ and $(x,\frac{\sin vt}{\sqrt{\pi}})=0$ for all v. Let $\varepsilon>0$. By Proposition 6.8 and Q9.42, there is a finite Fourier series $\varphi=a_0+\sum_{n=1}^Na_n\cos nt+\sum_{n=1}^Nb_v\sin nt$ such that $||x-\varphi||<\varepsilon$. Now $||\varphi||^2=(\varphi,\varphi)=2\pi a_0^2+\pi\sum_{n=1}^N(a_n^2+b_n^2)$ and $\sum_v|(\varphi,\varphi_v)|^2=2\pi a_0^2+\sum_{n=1}^N((\frac{a_n}{\sqrt{\pi}}\pi)^2+(\frac{b_n}{\sqrt{\pi}}\pi)^2)$ so $||\varphi||^2=\sum_v|(\varphi,\varphi_v)|^2$. Thus there exists M such that $|\ ||\varphi||^2-\sum_{v=-n}^n|(\varphi,\varphi_v)|^2\ |<\varepsilon^2$

for $n \geq M$. Now $||\varphi - \sum_{v=-n}^{n}(\varphi,\varphi_v)\varphi_v||^2 = ||\varphi||^2 - \sum_{v=-n}^{n}|(\varphi,\varphi_v)|^2 < \varepsilon^2$ for $n \geq M$. Also, $||\sum_{v=-n}^{n}(\varphi,\varphi_v)\varphi_v - \sum_{v=-n}^{n}(x,\varphi_v)\varphi_v||^2 = \sum_{v=-n}^{n}|(\varphi - x,\varphi_v)|^2 \leq ||\varphi - x||^2 < \varepsilon^2$ for $n \geq M$. Thus $||x - \sum_{v=-n}^{n}(x,\varphi_v)\varphi_v|| \leq ||x-\varphi|| + ||\varphi - \sum_{v=-n}^{n}(\varphi,\varphi_v)\varphi_v|| + ||\sum_{v=-n}^{n}(\varphi - x,\varphi_v)\varphi_v|| < 3\varepsilon$ for $n \geq M$. Hence $x = \sum_v (x,\varphi_v)\varphi_v$ and $||x||^2 = \sum_v |(x,\varphi_v)|^2 = 0$ so $x = 0$. Hence $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos vt}{\sqrt{\pi}}, \frac{\sin vt}{\sqrt{\pi}}\}$ is a complete orthonormal system for $L^2[0,2\pi]$.

51b. By part (a) and Proposition 27, every function in $L^2[0,2\pi]$ is the limit in mean (of order 2) of its Fourier series. i.e. the Fourier series converges to the function in $L^2[0,2\pi]$.

52a. Let $x \in H$ and let $\{\varphi_v\}$ be an orthonormal system. Note that $\{v : a_v \neq 0\} = \bigcup_n \{v : a_v \geq 1/n\}$. By Bessel's inequality, we have $\sum_v |a_v|^2 \leq ||x||^2 < \infty$ so $\{v : a_v \geq 1/n\}$ is finite for each n. Hence $\{v : a_v \neq 0\}$ is countable.

***52b.** Let H be a Hilbert space and $\{\varphi_v\}$ a complete orthonormal system. By part (a), only countably many Fourier coefficients a_v are nonzero so we may list them as a sequence $\langle a_v\rangle$. By Bessel's inequality, $\sum_v |a_v|^2 \leq ||x||^2 < \infty$ so given $\varepsilon > 0$, there exists N such that $\sum_{v=N+1}^{\infty}|a_v|^2 < \varepsilon^2$. For $m > n \geq N$, we have $||\sum_{v=n}^{m} a_v\varphi_v||^2 = \sum_{v=n}^{m}|a_v^2 < \varepsilon^2$ so the sequence of partial sums is Cauchy in H and thus converges in H. i.e. $\sum_v a_v\varphi_v \in H$. Now $(x - \sum_v a_v\varphi_v, \varphi_v) = (x,\varphi_v) - a_v = 0$ for all v so $x - \sum_v a_v\varphi_v = 0$. i.e. $x = \sum_v a_v\varphi_v$.

If a complete orthonormal system in H is countable, say $\langle\varphi_v\rangle$, then $x = \sum_v a_v\varphi_v$ and x is a cluster point of the set of linear combinations of φ_v, which contains the countable dense set of linear combinations of φ_v with rational coefficients so H is separable. Thus a complete orthonormal system in a non-separable Hilbert space is uncountable.

***52c.** Let f be a bounded linear functional on H. Let $K = \ker f$. Since f is continuous, K is a closed linear subspace of H. We may assume $K^\perp \neq \{0\}$ so there exists $x_0 \in K^\perp$ with $f(x_0) = 1$. Define $y = x_0/||x_0||^2$. Then $0 = (x - f(x)x_0, x_0) = (x,x_0) - f(x)||x_0||^2$ for all $x \in H$. i.e. $f(x) = (x,y)$ for all $x \in H$. If $y' \in H$ such that $(x,y) = (x,y')$ for all $x \in H$, then $y - y' \in H^\perp$. In particular, $||y-y'||^2 = (y-y', y-y') = 0$ so $y = y'$. This proves the uniqueness of y. Since $|f(x)| = |(x,y)| \leq ||x||\,||y||$, we have $||f|| \leq ||y||$. Furthermore, since $f(y/||y||) = (y/||y||, y) = ||y||$, we have $||f|| = ||y||$.

52d. Let H be an infinite dimensional Hilbert space. If $\{\varphi_v\}$ is a complete orthonormal system in H, then the set of finite linear combinations of φ_v is a dense subset of H. Now if $|\{\varphi_v\}| = \mathbf{n}$, then $|\{\text{finite linear combinations of } \varphi_v\}| = \mathbf{n}\aleph_0 = \mathbf{n}$ so there is a dense subset of H with cardinality $\mathbf{n}$. If S is a dense subset of H, then for each φ_v, there exists $x_v \in S$ with $||x_v - \varphi_v|| < 1/\sqrt{2}$. If $v \neq u$, then $||x_v - x_u|| \geq ||\varphi_v - \varphi_u|| - ||x_v - \varphi_v|| - ||x_u - \varphi_u|| > \sqrt{2} - 1/\sqrt{2} - 1/\sqrt{2} = 0$ so $x_v \neq x_u$. Thus $|S| \geq |\{\varphi_v\}|$. Hence the number of elements in a complete orthonormal system in H is the smallest cardinal $\mathbf{n}$ such that there is a dense subset of H with $\mathbf{n}$ elements. Furthermore, this proves that every complete orthonormal system in H has the same number of elements.

***52e.** Suppose two Hilbert spaces H and H' are isomorphic with an isomorphism $\Phi : H \to H'$. If $\{\varphi_v\}$ is a complete orthonormal system in H, then $\{\Phi(\varphi_v)\}$ is a complete orthonormal system in H'. Similarly for Φ^{-1}. Thus $\dim H = \dim H'$. Conversely, suppose $\dim H = \dim H'$. Let $\mathcal{E}$ be a complete orthonormal system in H. Consider the Hilbert space $\ell^2(\mathcal{E}) = \{(f : \mathcal{E} \to \mathbb{R}) : \sum_{e\in\mathcal{E}}|f(e)|^2 < \infty\}$. If $x \in H$, define $\hat{x} : \mathcal{E} \to \mathbb{R}$ by $\hat{x}(e) = (x,e)$. Then $\sum_{e\in\mathcal{E}}|\hat{x}(e)|^2 = \sum_{e\in\mathcal{E}}|(x,e)|^2 = ||x||^2 < \infty$ so $\hat{x} \in \ell^2(\mathcal{E})$. Furthermore, $||x|| = ||\hat{x}||$. Define $\Phi : H \to \ell^2(\mathcal{E})$ by $\Phi(x) = \hat{x}$. Then Φ is a linear isometry. Now the range of Φ contains functions f such that $f(e) = 0$ for all but finitely many $e \in \mathcal{E}$ and is closed since Φ is an isometry. Hence Φ is an isomorphism. If $\mathcal{F}$ is a complete orthonormal system in H', then $|\mathcal{E}| = |\mathcal{F}|$ so $\ell^2(\mathcal{E})$ and $\ell^2(\mathcal{F})$ must be isomorphic. Hence H and H' are isomorphic.

***52f.** Let A be a nonempty set and let $\ell^2(A) = \{(f : A \to \mathbb{R}) : \sum_{a\in A}|f(a)|^2 < \infty\}$. Then $\ell^2(A)$ is a Hilbert space with $(f,g) = \sum_{a\in A} f(a)g(a)$. For each $a \in A$, let χ_a be the characteristic function of $\{a\}$. Then $\chi_a \in \ell^2(A)$ for each $a \in A$. Furthermore $\{\chi_a : a \in A\}$ is a complete orthonormal system in $\ell^2(A)$. Thus $\dim \ell^2(A) = |\{\chi_a : a \in A\}| = |A|$. Hence there is a Hilbert space of each dimension.

53a. Let P be a subset of H. Suppose $y_n \in P^\perp$ and $y_n \to y$. Then for each $x \in P$, we have $(x,y_n) \to (x,y)$. Since $(x,y_n) = 0$ for each n, we have $(x,y) = 0$ so $y \in P^\perp$. Thus $P^\perp$ is closed. If $a,b \in \mathbb{R}$ and $y,z \in P^\perp$, then for each $x \in P$, we have $(ay+bz, x) = a(y,x) + b(z,x) = 0$ so $ay + bz \in P^\perp$ and $P^\perp$ is a linear manifold.

***53b.** If $x \in P$, then $(x,y) = 0$ for all $y \in P^\perp$. Thus $P^{\perp\perp}$ is a closed linear manifold containing P. Suppose $P \subset Q$ where Q is a closed linear manifold. Then $Q^\perp \subset P^\perp$ and $P^{\perp\perp} \subset Q^{\perp\perp}$. Now if Q is a proper subset of $Q^{\perp\perp}$, pick $x \in Q^{\perp\perp} \setminus Q$. Then there exists $y_0 \in Q$ such that $||x - y_0|| = \inf\{||x - y|| : y \in Q\} = \delta$. Let $z = x - y_0$. For any $a \in \mathbb{R}$ and $y_1 \in Q$, we have $\delta^2 \leq ||z - ay_1||^2 = ||z||^2 - 2a(z, y_1) + a^2||y_1||^2 = \delta^2 - 2a(z, y_1) + a^2||y_1||^2$ so $a^2||y_1||^2 - 2a(z, y_1) \geq 0$. Thus $4(z, y_1)^2 \leq 0$ and it follows that $(z, y_1) = 0$. i.e. $z \perp Q$. Thus $z \in Q^{\perp\perp} \cap Q^{\perp}$ so $z = 0$ i.e. $x = y_0 \in Q$. Contradiction. Hence $Q = Q^{\perp\perp}$ so $P^{\perp\perp} \subset Q$.

53c. Let M be a closed linear manifold. Given $x \in H$, there exists $y \in M$ such that $x - y \perp M$. Then $x = y + z$ where $z = x - y \in M^{\perp}$. If $x = y + z = y' + z'$ where $y, y' \in M$ and $z, z' \in M^{\perp}$, then $y - y' = z - z'$ so $y - y', z - z' \in M \cap M^{\perp}$. Hence $y = y'$ and $z = z'$. Furthermore $||x||^2 = (y + z, y + z) = ||y||^2 + ||z||^2 + 2(y, z) = ||y||^2 + ||z||^2$ since $z \perp y$.

54. Let $\langle x_n \rangle$ be a bounded sequence of elements in a separable Hilbert space H. Suppose $||x_n|| \leq M$ for all n and let $\langle \varphi_n \rangle$ be a complete orthonormal system in H. Now $|(x_n, \varphi_1)| \leq M$ for all n so there is a subsequence $\langle (x_{n_k}, \varphi_1) \rangle$ that converges. Then $|(x_{n_k}, \varphi_2)| \leq M$ for all k so there is a subsequence $\langle (x_{n_{k_l}}, \varphi_2) \rangle$ that converges. Furthermore, $\langle (x_{n_{k_l}}, \varphi_1) \rangle$ also converges. Continuing the process, we obtain the diagonal sequence $\langle x_{n_n} \rangle$ such that $\langle (x_{n_n}, \varphi_k) \rangle$ converges for any k. For any bounded linear functional f on H, there is a unique $y \in H$ such that $f(x) = (x, y)$ for all $x \in H$. Furthermore, $y = \sum_k (y, \varphi_k)\varphi_k$. Now $(x_{n_n}, y) = (x_{n_n}, \sum_k (y, \varphi_k)\varphi_k) = \sum_k (x_{n_n}, \varphi_k)(y, \varphi_k)$. Since $\langle (x_{n_n}, \varphi_k) \rangle$ converges for each k, $\langle (x_{n_n}, y) \rangle$ converges. i.e. $\langle x_n \rangle$ has a subsequence which converges weakly.

55. Let S be a subspace of $L^2[0, 1]$ and suppose that there is a constant K such that $|f(x)| \leq K||f||$ for all $x \in [0, 1]$. If $\langle f_1, \ldots, f_n \rangle$ is any finite orthonormal sequence in S, then for any $a_1, \ldots, a_n \in \mathbb{R}$, we have $(\sum_{i=1}^n a_i f_i(x))^2 \leq K^2 || \sum_{i=1}^n a_i f_i ||^2 = K^2 (\sum_{i=1}^n a_i f_i, \sum_{i=1}^n a_i f_i) = K^2 \sum_{i=1}^n a_i^2$ for all $x \in [0, 1]$. Fix $x \in [0, 1]$. For each i, let $a_i = \frac{f_i(x)}{\sqrt{\sum_{i=1}^n f_i(x)^2}}$. Then $(\sum_{i=1}^n a_i f_i(x))^2 = \sum_{i=1}^n f_i(x)^2$ and $\sum_{i=1}^n a_i^2 = 1$. Thus $\sum_{i=1}^n f_i(x)^2 \leq K^2$ for all $x \in [0, 1]$ and $n = \sum_{i=1}^n ||f_i||^2 = \int_0^1 (\sum_{i=1}^n f_i(x)^2) \leq \int_0^1 K^2 = K^2$. Hence the dimension of S is at most K^2.

11 Measure and Integration

11.1 Measure spaces

1. Let $\{A_n\}$ be a collection of measurable sets. Let $B_1 = A_1$ and $B_k = A_k \setminus \bigcup_{n=1}^{k-1} A_n$ for $k > 1$. Then $\{B_n\}$ is a collection of pairwise disjoint measurable sets such that $\bigcup A_n = \bigcup B_n$. Now $\mu(\bigcup A_k) = \mu(\bigcup B_k) = \sum \mu(B_k) = \lim_n \sum_{k=1}^n \mu(B_k) = \lim_n \mu(\bigcup_{k=1}^n B_k) = \lim_n \mu(\bigcup_{k=1}^n A_k)$.

2a. Let $\{(X_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)\}$ be a collection of measure spaces, and suppose that the sets $\{X_\alpha\}$ are disjoint. Define $X = \bigcup X_\alpha$, $\mathfrak{B} = \{B : \forall \alpha\ B \cap X_\alpha \in \mathfrak{B}_\alpha\}$ and $\mu(B) = \sum \mu_\alpha(B \cap X_\alpha)$. Now $\emptyset \in \mathfrak{B}$ since $\emptyset \cap X_\alpha = \emptyset \in \mathfrak{B}_\alpha$ for all α. If $B \in \mathfrak{B}$, then $B \cap X_\alpha \in \mathfrak{B}_\alpha$ for all α so $B^c \cap X_\alpha = X_\alpha \setminus (B \cap X_\alpha) \in \mathfrak{B}_\alpha$ for all α. Thus $B^c \in \mathfrak{B}$. If $\langle B_n \rangle$ is a sequence in $\mathfrak{B}$, then for each n, $B_n \cap X_\alpha \in \mathfrak{B}_\alpha$ for all α so $\bigcup B_n \cap X_\alpha = \bigcup (B_n \cap X_\alpha) \in \mathfrak{B}_\alpha$ for each α. Thus $\bigcup B_n \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a σ-algebra.

2b. $\mu(\emptyset) = \sum \mu_\alpha(\emptyset \cap X_\alpha) = \sum \mu_\alpha(\emptyset) = 0$. For any sequence of disjoint sets $B_i \in \mathfrak{B}$, we have $\mu(\bigcup B_i) = \sum \mu_\alpha(\bigcup B_i \cap X_\alpha) = \sum_\alpha \sum_i \mu_\alpha(B_i \cap X_\alpha) = \sum_i \sum_\alpha \mu_\alpha(B_i \cap X_\alpha) = \sum_i \mu(B_i)$. Hence μ is a measure.

2c. Suppose that all but a countable number of the μ_α are zero and the remainder are σ-finite. Let $\langle \mu_n \rangle$ be the countably many μ_α that are nonzero and σ-finite. Then for each n, $X_n = \bigcup_k X_{n,k}$ where $X_{n,k} \in \mathfrak{B}_n$ and $\mu_n(X_{n,k}) < \infty$. In particular, $X_{n,k} \in \mathfrak{B}$ for all n, k. Let $A = \{\alpha : \mu_\alpha \text{ is zero}\}$. Then $X = \bigcup_n X_n \cup \bigcup_{\alpha \in A} X_\alpha = \bigcup_{n,k} X_{n,k} \cup \bigcup_{\alpha \in A} X_\alpha$. Since $\bigcup_{\alpha \in A} X_\alpha \cap X_\alpha = X_\alpha \in \mathfrak{B}_\alpha$ if $\alpha \in A$ and $\bigcup_{\alpha \in A} X_\alpha \cap X_\alpha = \emptyset \in \mathfrak{B}_\alpha$ if $\alpha \notin A$, we have $\bigcup_{\alpha \in A} X_\alpha \in \mathfrak{B}$. Also, $\mu(\bigcup_{\alpha \in A} X_\alpha) = \sum \mu_\alpha(\bigcup_{\alpha \in A} \cap X_\alpha) = \sum_{\alpha \in A} \mu_\alpha X_\alpha = 0 < \infty$. Hence μ is σ-finite.

Conversely, suppose μ is σ-finite. Then $X = \bigcup_n Y_n$ where $Y_n \in \mathfrak{B}$ and $\mu(Y_n) < \infty$ for each n. We may assume that the Y_n are disjoint. Now for each n, $\sum \mu_\alpha(Y_n \cap X_\alpha) = \mu(Y_n \cap \bigcup_\alpha X_\alpha) = \mu(Y_n) < \infty$ so $\{\alpha : \mu_\alpha(Y_n \cap X_\alpha) > 0\}$ is countable. Thus $\bigcup_n \{\alpha : \mu_\alpha(Y_n \cap X_\alpha) > 0\}$ is countable. For each α, $\mu_\alpha(X_\alpha) = \mu_\alpha(X_\alpha \cap \bigcup_n Y_n) = \sum_n \mu_\alpha(X_\alpha \cap Y_n)$. If $\mu_\alpha(X_\alpha) > 0$, then $\mu_\alpha(X_\alpha \cap Y_n) > 0$ for some n. Hence $\{\alpha : \mu_\alpha(X_\alpha) > 0\}$ is countable. i.e. all but a countable number of the μ_α are zero. Furthermore, $X_\alpha = \bigcup_n (X_\alpha \cap Y_n)$ where $X_\alpha \cap Y_n \in \mathfrak{B}_\alpha$ and $\mu_\alpha(X_\alpha \cap Y_n) \leq \sum \mu_\alpha(X_\alpha \cap Y_n) = \mu(Y_n) < \infty$ for each n. Hence the remaining μ_α are σ-finite.

(*) Union of measure spaces

3a. Suppose $E_1, E_2 \in \mathfrak{B}$ and $\mu(E_1 \Delta E_2) = 0$. Then $\mu(E_1 \setminus E_2) = \mu(E_2 \setminus E_1) = 0$. Hence $\mu(E_1) = \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) = \mu(E_2 \setminus E_1) + \mu(E_1 \cap E_2) = \mu(E_2)$.

3b. Suppose μ is complete, $E_1 \in \mathfrak{B}$ and $\mu(E_1 \Delta E_2) = 0$. Then $E_2 \setminus E_1 \in \mathfrak{B}$ since $E_2 \setminus E_1 \subset E_1 \Delta E_2$. Also, $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2) \in \mathfrak{B}$. Hence $E_2 = (E_2 \setminus E_1) \cup (E_1 \cap E_2) \in \mathfrak{B}$.

4. Let $(X, \mathfrak{B}, \mu)$ be a measure space and $Y \in \mathfrak{B}$. Let $\mathfrak{B}_Y$ consist of those sets of $\mathfrak{B}$ that are contained in Y. Set $\mu_Y E = \mu E$ if $E \in \mathfrak{B}_Y$. Clearly $\emptyset \in \mathfrak{B}_Y$. If $B \in \mathfrak{B}_Y$, then $B \in \mathfrak{B}$ and $B \subset Y$. Thus $Y \setminus B \in \mathfrak{B}$ and $Y \setminus B \subset Y$ so $Y \setminus B \in \mathfrak{B}_Y$. If $\langle B_n \rangle$ is a sequence of sets in $\mathfrak{B}_Y$, then $B_n \in \mathfrak{B}$ and $B_n \subset Y$ for all n. Thus $\bigcup B_n \in \mathfrak{B}$ and $\bigcup B_n \subset Y$ so $\bigcup B_n \in \mathfrak{B}_Y$. Furthermore, $\mu_Y(\emptyset) = \mu(\emptyset) = 0$ and if $\langle B_n \rangle$ is a sequence of disjoint sets in $\mathfrak{B}_Y$, then $\mu_Y(\bigcup B_n) = \mu(\bigcup B_n) = \sum \mu(B_n) = \sum \mu_Y(B_n)$. Hence $(Y, \mathfrak{B}_Y, \mu_Y)$ is a measure space.

(*) Restriction of measure to measurable subset

5a. Let $(X, \mathfrak{B})$ be a measurable space. Suppose μ and ν are measures defined on $\mathfrak{B}$ and define the set function $\lambda E = \mu E + \nu E$ on $\mathfrak{B}$. Then $\lambda(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0$. Also, if $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{B}$, then $\lambda(\bigcup E_n) = \mu(\bigcup E_n) + \nu(\bigcup E_n) = \sum \mu E_n + \sum \nu E_n = \sum \lambda E_n$. Hence λ is also a measure.

***5b.** Suppose μ and ν are measures on $\mathfrak{B}$ and $\mu \geq \nu$. Note that $\mu - \nu$ is a measure when restricted to measurable sets with finite ν-measure. Define $\lambda(E) = \sup_{F \subset E, \nu(F) < \infty}(\mu(F) - \nu(F))$. Clearly $\lambda(\emptyset) = 0$. If $E_1, E_2 \in \mathfrak{B}$ with $E_1 \cap E_2 = \emptyset$, then for any $F \subset E_1 \cup E_2$ with $\nu(F) < \infty$, we have $\mu(F) - \nu(F) = (\mu - \nu)(F \cap E_1) + (\mu - \nu)(F \cap E_2) \leq \lambda(E_1) + \lambda(E_2)$. Thus $\lambda(E_1 \cup E_2) \leq \lambda(E_1) + \lambda(E_2)$. On the other hand, if $F_1 \subset E_1$ and $F_2 \subset E_2$ with $\nu(F_1), \nu(F_2) < \infty$, we have $(\mu - \nu)(F_1) + (\mu - \nu)(F_2) = (\mu - \nu)(F_1 \cup F_2) \leq \lambda(E_1 \cup E_2)$. Thus $\lambda(E_1) + \lambda(E_2) \leq \lambda(E_1 \cup E_2)$. Hence λ is finitely additive.

Now suppose $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{B}$. If $F \subset \bigcup E_n$ with $\nu(F) < \infty$, then $(\mu - \nu)(F) = \sum(\mu - \nu)(F \cap E_n) \leq \sum \lambda(E_n)$. Thus $\lambda(\bigcup E_n) \leq \sum \lambda(E_n)$. On the other hand, for any N, we have $\sum_{n=1}^{N} \lambda(E_n) = \lambda(\bigcup_{n=1}^{N} E_n) \leq \lambda(\bigcup E_n)$. Thus $\sum \lambda E_n \leq \lambda(\bigcup E_n)$. Hence λ is countably additive.

If $\nu(E) = \infty$, then $\mu(E) = \infty$ so $\nu(E) + \lambda(E) = \mu(E)$ and if $\nu(E) < \infty$, then $\nu(E) + \lambda(E) = \nu(E) + (\mu(E) - \nu(E)) = \mu(E)$. Hence $\mu = \nu + \lambda$.

5c. Suppose $\mu = \nu + \eta$ for some measure η. If ν is finite, then $\eta = \mu - \nu = \lambda$. Suppose ν is σ-finite. Then $X = \bigcup X_n$ where $X_n \in \mathfrak{B}$ and $\nu(X_n) < \infty$ for each n. We may assume that the X_n are disjoint. Now for any set $E \in \mathfrak{B}$, we have $\eta(E \cap X_n) = \lambda(E \cap X_n)$ since $\nu(E \cap X_n) < \infty$. Then $\eta(E) = \eta(\bigcup(E \cap X_n)) = \sum \eta(E \cap X_n) = \sum \lambda(E \cap X_n) = \lambda(\bigcup(E \cap X_n)) = \lambda(E)$. Hence λ in part (b) is the unique such measure.

***5d.** If $\mu = \nu + \lambda = \nu + \lambda'$, then $\lambda(F) = \lambda'(F)$ for any set F with $\nu(F) < \infty$. Given $E \in \mathfrak{B}$, for any $F \subset E$ with $\nu(F) < \infty$, we have $\mu(F) - \nu(F) = \lambda(F) = \lambda'(F) \leq \lambda'(E)$. Hence $\lambda(E) \leq \lambda'(E)$ and λ in part (b) is the smallest such measure.

6a. Let μ be a σ-finite measure so $X = \bigcup X_n$ where $\mu(X_n) < \infty$ for each n. For any measurable set E of infinite measure, we have $\infty = \mu(\bigcup(E \cap X_n)) = \lim_k \mu(\bigcup_{n=1}^{k}(E \cap X_n))$. Thus for any N, there exists k such that $N < \mu(\bigcup_{n=1}^{k}(E \cap X_n)) < \infty$. Hence μ is semifinite.

6b. Given a measure μ, define $\mu_1(E) = \sup_{F \subset E, \mu(F) < \infty} \mu(F)$. Then $\mu_1(\emptyset) = \mu(\emptyset) = 0$. If $\langle E_n \rangle$ is a sequence of disjoint measurable sets, then for any $F \subset \bigcup E_n$ with $\mu(F) < \infty$, we have $\mu(F) = \sum \mu(F \cap E_n) \leq \sum \mu_1(E_n)$. Thus $\mu_1(\bigcup E_n) \leq \sum \mu_1(E_n)$. Conversely, if $F_1 \subset E_1$ and $F_2 \subset E_2$ with $\mu(F_1), \mu(F_2) < \infty$, then $\mu(F_1) + \mu(F_2) = \mu(F_1 \cup F_2) \leq \mu_1(E_1 \cup E_2)$ so $\mu_1(E_1) + \mu_1(E_2) \leq \mu_1(E_1 \cup E_2)$. It follows that for any k, $\sum_{n=1}^{k} \mu_1(E_n) \leq \mu_1(\bigcup_{n=1}^{k} E_n) \leq \mu_1(\bigcup E_n)$. Thus $\sum \mu_1(E_n) \leq \mu_1(\bigcup E_n)$. Hence μ_1 is a measure. Furthermore, if $\mu_1(E) = \infty$, then for any N, there exists $F \subset E$ with $N < \mu(F) < \infty$. Then $N < \mu_1(F) < \infty$. Hence μ_1 is semifinite.

Now define $\mu_2(E) = \sup_{F \subset E, \mu_1(F) < \infty}(\mu(F) - \mu_1(F))$. Then μ_2 is a measure (c.f. Q5b). If $\mu(F) < \infty$ for all $F \subset E$ with $\mu_1(F) < \infty$, then $\mu(F) = \mu_1(F)$ for all $F \subset E$ with $\mu_1(F) < \infty$ so $\mu_2(E) = 0$. If $\mu(F) = \infty$ for some $F \subset E$ with $\mu_1(F) < \infty$, then $\mu(F) - \mu_1(F) = \infty$ so $\mu_2(E) = \infty$. Hence μ_2 only assumes the values 0 and ∞.

***6c.**

7. Given a measure space $(X, \mathfrak{B}, \mu)$, let $\mathfrak{B}_0$ be the collection of sets $A \cup B$ where $A \in \mathfrak{B}$ and $B \subset C$ for some $C \in \mathfrak{B}$ with $\mu(C) = 0$. Clearly $\emptyset \in \mathfrak{B}_0$. If $\langle E_n \rangle$ is a sequence in $\mathfrak{B}_0$, then $E_n = A_n \cup B_n$ for each

n where $A_n \in \mathfrak{B}$ and $B_n \subset C_n$ for some $C_n \in \mathfrak{B}$ with $\mu(C_n) = 0$. Now $\bigcup E_n = \bigcup A_n \cup \bigcup B_n$ where $\bigcup A_n \in \mathfrak{B}$ and $\bigcup B_n \subset \bigcup C_n$ with $\mu(\bigcup C_n) = 0$. If $E \in \mathfrak{B}_0$, then $E = A \cup B$ where $A \in \mathfrak{B}$ and $B \subset C$ for some $C \in \mathfrak{B}$ with $\mu(C) = 0$. Then $E^c = A^c \cap B^c = A^c \cap (C \setminus (C \setminus B))^c = A^c \cap (C^c \cup (C \setminus B)) = (A^c \cap C^c) \cup (A^c \cap (C \setminus B))$ where $A^c \cap C^c \in \mathfrak{B}$ and $A^c \cap (C \setminus B) \subset C$ with $\mu(C) = 0$. Hence $\mathfrak{B}_0$ is a σ-algebra. Furthermore $\mathfrak{B} \subset \mathfrak{B}_0$.

If $E \in \mathfrak{B}_0$ with $E = A \cup B = A' \cup B'$ with $A, A' \in \mathfrak{B}$, $B \subset C$ and $B' \subset C'$ for some $C, C' \in \mathfrak{B}$ with $\mu(C) = \mu(C') = 0$, then $A \cup B \cup C \cup C' = A' \cup B' \cup C \cup C'$ so $A \cup C \cup C' = A' \cup C \cup C'$. Now $\mu(A) \leq \mu(A \cup C \cup C') \leq \mu(A) + \mu(C) + \mu(C') = \mu(A)$. Thus $\mu(A) = \mu(A \cup C \cup C')$. Similarly, $\mu(A') = \mu(A' \cup C \cup C')$. Hence $\mu(A) = \mu(A')$.

For $E \in \mathfrak{B}_0$, define $\mu_0(E) = \mu(A)$. Then μ_0 is well-defined. Clearly, $\mu_0(\emptyset) = 0$. If $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{B}_0$, then $\bigcup E_n = \bigcup A_n \cup \bigcup B_n$ so $\mu_0(\bigcup E_n) = \mu(\bigcup A_n) = \sum \mu(A_n) = \sum \mu_0(E_n)$. Thus μ_0 is a measure. Furthermore, if $E \in \mathfrak{B}$, then $\mu_0(E) = \mu(E)$.

If $\mu_0(A \cup B) = 0$ and $E \subset A \cup B$, then $E = A \cup (E \cap A)$ where $E \cap A \subset B \subset C$ with $\mu(C) = 0$ so $E \in \mathfrak{B}_0$. Hence $(X, \mathfrak{B}_0, \mu_0)$ is a complete measure space extending $(X, \mathfrak{B}, \mu)$.

8a. Let μ be a σ-finite measure. If E is locally measurable, then $E \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$. Now $X = \bigcup X_n$ with $X_n \in \mathfrak{B}$ and $\mu(X_n) < \infty$ for all n. Thus $E = \bigcup (E \cap X_n)$ where $E \cap X_n \in \mathfrak{B}$ for all n. Thus $E \in \mathfrak{B}$. i.e. E is measurable. Hence μ is saturated.

8b. Let $\mathfrak{C}$ be the collection of locally measurable sets. Clearly $\emptyset \in \mathfrak{C}$. If $C \in \mathfrak{C}$, then $C \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$. Now for any such set B, $C^c \cap B = B \setminus (C \cap B) \in \mathfrak{B}$. Thus $C^c \in \mathfrak{C}$. If $\langle C_n \rangle$ is a sequence of sets in $\mathfrak{C}$, then $C_n \cap B \in \mathfrak{B}$ for each $B \in \mathfrak{B}$ with $\mu(B) < \infty$. Now for any such set B, $\bigcup C_n \cap B = \bigcup (C_n \cap B) \in \mathfrak{B}$. Thus $\bigcup C_n \in \mathfrak{C}$. Hence $\mathfrak{C}$ is a σ-algebra.

8c. Let $(X, \mathfrak{B}, \mu)$ be a measure space and $\mathfrak{C}$ the σ-algebra of locally measurable sets. For $E \in \mathfrak{C}$, set $\bar{\mu}E = \mu E$ if $E \in \mathfrak{B}$ and $\bar{\mu}E = \infty$ if $E \notin \mathfrak{B}$. Clearly $\bar{\mu}(\emptyset) = 0$. If $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{C}$, then we consider a few cases. If $\bigcup E_n \notin \mathfrak{B}$, then $\bar{\mu}(\bigcup E_n) = \infty$. Furthermore, $E_n \notin \mathfrak{B}$ for some n so $\sum \bar{\mu}(E_n) = \infty = \bar{\mu}(\bigcup E_n)$. Now suppose $\bigcup E_n \in \mathfrak{B}$. If $\mu(\bigcup E_n) < \infty$, then $E_n = E_n \cap \bigcup E_n \in \mathfrak{B}$ for each n so $\bar{\mu}(\bigcup E_n) = \mu(\bigcup E_n) = \sum \mu(E_n) = \sum \bar{\mu}(E_n)$. If $\mu(\bigcup E_n) = \infty$, then either $E_n \in \mathfrak{B}$ for all n, so $\sum \bar{\mu}(E_n) = \sum \mu(E_n) = \mu(\bigcup E_n) = \infty = \bar{\mu}(\bigcup E_n)$, or $E_n \notin \mathfrak{B}$ for some n, so $\sum \bar{\mu}(E_n) = \infty = \bar{\mu}(\bigcup E_n)$. Hence $\bar{\mu}$ is a measure on $\mathfrak{C}$.

Let E be a locally measurable set in $(X, \mathfrak{C}, \bar{\mu})$. Then $E \cap C \in \mathfrak{C}$ for any $C \in \mathfrak{C}$ with $\bar{\mu}(C) < \infty$. i.e. $E \cap C \in \mathfrak{C}$ for any $C \in \mathfrak{B}$ with $\mu(C) < \infty$. Thus $E \cap C \in \mathfrak{B}$ for any $C \in \mathfrak{B}$ with $\mu(C) < \infty$ and E is a locally measurable set in $(X, \mathfrak{B}, \mu)$ so $E \in \mathfrak{C}$. Hence $(X, \mathfrak{C}, \bar{\mu})$ is a saturated measure space.

***8d.** Suppose μ is semifinite and $E \in \mathfrak{C}$. Set $\underline{\mu}(E) = \sup\{\mu(B) : B \in \mathfrak{B}, B \subset E\}$. Clearly $\bar{\mu}(\emptyset) = 0$. If $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{C}$, for any $B \in \mathfrak{B}$ with $B \subset \bigcup E_n$, provided $\mu(B) < \infty$, we have $\mu(B) = \mu(\bigcup (B \cap E_n)) = \sum \mu(B \cap E_n) \leq \sum \underline{\mu}(E_n)$. On the other hand, if $\mu(B) = \infty$, then $\underline{\mu}(\bigcup E_n) = \infty$. Thus $\underline{\mu}(\bigcup E_n) \leq \sum \underline{\mu}(E_n)$. If $E_1, E_2 \in \mathfrak{C}$ with $E_1 \cap E_2 = \emptyset$ and $B_1, B_2 \in \mathfrak{B}$ with $B_1 \subset E_1$ and $B_2 \subset E_2$, then $\underline{\mu}(E_1) + \underline{\mu}(E_2) \leq \mu(B_1) + \mu(B_2) = \mu(B_1 \cup B_2) \leq \underline{\mu}(E_1 \cup E_2)$. Now for any N, we have $\sum_{n=1}^N \underline{\mu}(E_n) \leq \underline{\mu}(\bigcup_{n=1}^N E_n) \leq \underline{\mu}(\bigcup E_n)$. Thus $\sum \underline{\mu}(E_n) \leq \underline{\mu}(\bigcup E_n)$. Hence $\underline{\mu}$ is a measure on $\mathfrak{C}$.

Let E be a locally measurable set in $(X, \mathfrak{C}, \underline{\mu})$. Then $E \cap C \in \mathfrak{C}$ for any $C \in \mathfrak{C}$ with $\underline{\mu}(C) < \infty$. If $B \in \mathfrak{B}$ with $\mu(B) < \infty$, then $B \in \mathfrak{C}$ with $\underline{\mu}(B) = \mu(B) < \infty$. Thus $E \cap B \in \mathfrak{C}$. It follows that $E \cap B \in \mathfrak{B}$ so $E \in \mathfrak{C}$. Hence $(X, \mathfrak{C}, \underline{\mu})$ is a saturated measure space. Furthermore, $\underline{\mu}$ is an extension of μ.

9a. Let $\mathfrak{R}$ be a σ-ring that is not a σ-algebra. Let $\mathfrak{B}$ be the smallest σ-algebra containing $\mathfrak{R}$ and set $\mathfrak{R}' = \{E : E^c \in \mathfrak{R}\}$. Note that $\emptyset \in \mathfrak{R} \subset \mathfrak{R} \cup \mathfrak{R}'$. If $A \in \mathfrak{R} \cup \mathfrak{R}'$, then either $A \in \mathfrak{R}$ so $A^c \in \mathfrak{R}' \subset \mathfrak{R} \cup \mathfrak{R}'$ or $A \in \mathfrak{R}'$ so $A^c \in \mathfrak{R} \subset \mathfrak{R} \cup \mathfrak{R}'$. If $\langle A_n \rangle$ is a sequence in $\mathfrak{R} \cup \mathfrak{R}'$, then $\bigcup A_n = \bigcup \{A_n : A_n \in \mathfrak{R}\} \cup \bigcup \{A_n : A_n \in \mathfrak{R}'\} \in \mathfrak{R} \cup \mathfrak{R}'$. If $\mathfrak{A}$ is a σ-algebra containing $\mathfrak{R}$, then $\mathfrak{A}$ also contains $\mathfrak{R}'$. Thus $\mathfrak{A} \supset \mathfrak{R} \cup \mathfrak{R}'$. Hence $\mathfrak{R} \cup \mathfrak{R}' = \mathfrak{B}$. Furthermore, if $E \in \mathfrak{R} \cap \mathfrak{R}'$, then $E, E^c \in \mathfrak{R}$ so $X = E \cup E^c \in \mathfrak{R}$ and $\mathfrak{R}$ is a σ-algebra. Contradiction. Hence $\mathfrak{R} \cap \mathfrak{R}' = \emptyset$.

9b. If μ is a measure on $\mathfrak{R}$, define $\bar{\mu}$ on $\mathfrak{B}$ by $\bar{\mu}(E) = \mu(E)$ if $E \in \mathfrak{R}$ and $\bar{\mu}(E) = \infty$ if $E \in \mathfrak{R}'$. Clearly $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$. Let $\langle E_n \rangle$ be a sequence of disjoint sets in $\mathfrak{B}$. If $E_1 \in \mathfrak{R}$ and $E_2 \in \mathfrak{R}'$, then $(E^2)^c \in \mathfrak{R}$ so $(E_1)^c \cap (E_2)^c = (E_2)^c \setminus E_1 \in \mathfrak{R}$. Thus $E_1 \cup E_2 \in \mathfrak{R}'$ so $\bar{\mu}(E_1 \cup E_2) = \infty = \bar{\mu}(E_1) + \bar{\mu}(E_2)$. If $E_n \in \mathfrak{R}$ for all n, then $\bigcup E_n \in \mathfrak{R}$. Similarly, if $E_n \in \mathfrak{R}'$ for all n, then $\bigcup E_n \in \mathfrak{R}'$. In both cases we have $\bar{\mu}(\bigcup E_n) = \sum \bar{\mu}(E_n)$. Thus in general, $\bar{\mu}(\bigcup E_n) = \bar{\mu}(\bigcup \{E_n : E_n \in \mathfrak{R}\} \cup \bigcup \{E_n : E_n \in \mathfrak{R}'\}) = \bar{\mu}(\bigcup \{E_n : E_n \in \mathfrak{R}\}) + \bar{\mu}(\bigcup \{E_n : E_n \in \mathfrak{R}'\}) = \sum_{E_n \in \mathfrak{R}} \bar{\mu}(E_n) + \sum_{E_n \in \mathfrak{R}'} \bar{\mu}(E_n) = \sum \bar{\mu}(E_n)$. Hence $\bar{\mu}$ is a measure on $\mathfrak{B}$.

9c. Define $\underline{\mu}$ on $\mathfrak{B}$ by $\underline{\mu}(E) = \mu(E)$ if $E \in \mathfrak{R}$ and $\underline{\mu}(E) = \sup\{\mu(A) : A \subset E, A \in \mathfrak{R}\}$ if $E \in \mathfrak{R}'$. Clearly $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$. Let $\langle E_n \rangle$ be a sequence of disjoint sets in $\mathfrak{B}$. Note that if $E_n \in \mathfrak{R}$ for all n or $E_n \in \mathfrak{R}'$ for all n, then $\underline{\mu}(\bigcup E_n) = \sum \underline{\mu}(E_n)$. It follows that in general $\underline{\mu}(\bigcup E_n) = \sum \underline{\mu}(E_n)$. Hence $\underline{\mu}$ is a measure on $\mathfrak{B}$.

***9d.**

***9e.**

11.2 Measurable functions

10. Let f be a nonnegative measurable function. For each pair $\langle n, k \rangle$ of integers, set $E_{n,k} = \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\}$ and set $\varphi_n = 2^{-n} \sum_{k=0}^{2^{2n}} k\chi_{E_{n,k}} + (2^{2n}+1)2^{-n}\chi_{\{f \geq (2^{2n}+1)2^{-n}\}}$. Then each $E_{n,k}$ is measurable and each φ_n is a simple function. Note that $E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}$ for all $\langle n, k \rangle$. Suppose $x \in E_{n,k}$. Then $\varphi_n(x) = k2^{-n}$. If $x \in E_{n+1,2k}$, then $\varphi_{n+1}(x) = (2k)2^{-(n+1)} = k2^{-n} = \varphi_n(x)$. If $x \in E_{n+1,2k+1}$, then $\varphi_{n+1}(x) = (2k+1)2^{-(n+1)} > (2k)2^{-(n+1)}\varphi_n(x)$. Also, if $f(x) \geq (2^{2n}+1)2^{-n}$, then $\varphi_n(x) = (2^{2n}+1)2^{-n} \leq \varphi_{n+1}(x)$. Thus $\varphi_n \leq \varphi_{n+1}$. For $x \in X$, if $f(x) < \infty$, then for sufficiently large n, $f(x) - \varphi_n(x) \leq 2^{-n}$. If $f(x) = \infty$, then $\varphi_n(x) = (2^{2n}+1)2^{-n} > 2^n \to \infty$. Thus $f = \lim \varphi_n$ at each point of X.

If f is defined on a σ-finite measure space $(X, \mathfrak{B}, \mu)$, then $X = \bigcup X_n$ where $\mu(X_n) < \infty$ for each n. Define $\varphi_n == 2^{-n} \sum_{k=0}^{2^{2n}} k\chi_{E_{n,k} \cap \bigcup_{m=1}^n X_m} + (2^{2n}+1)2^{-n}\chi_{\{f \geq (2^{2n}+1)2^{-n}\} \cap \bigcup_{m=1}^n X_m}$. Then $\varphi_{n+1} \geq \varphi_n$ and $f = \lim \varphi_n$. Furthermore, each φ_n vanishes outside $\bigcup_{m=1}^n X_m$, a set of finite measure.

(*) Proof of Proposition 7

11. Suppose μ is a complete measure and f be a measurable function. Suppose $f = g$ a.e. Then $\{x : g(x) < \alpha\} = \{x : f = g, f(x) < \alpha\} \cup \{x : f \neq g, g(x) < \alpha\} = (\{x : f(x) < \alpha\} \setminus \{x : f \neq g, f(x) < \alpha\}) \cup \{x : f \neq g, g(x) < \alpha\}$. Now $\{x : f \neq g, f(x) < \alpha\}$ and $\{x : f \neq g, g(x) < \alpha\}$ are subsets of a set of measure zero so they are measurable. Also, $\{x : f(x) < \alpha\}$ is measurable since f is measurable. Hence $\{x : g(x) < \alpha\}$ is measurable for any α and g is measurable.

Q3.28 gives an example of a set A of Lebesgue measure zero that is not a Borel set so Lebesgue measures restricted to the σ-algebra of Borel sets is not complete. Now χ_A is not measurable since $\{x : \chi_A(x) > 1/2\} = A$ but $m(A) = 0$ so $\chi_A = 0$ a.e. and the constant zero function is measurable.

(*) Proof of Proposition 8

12. Let $\langle f_n \rangle$ be a sequence of measurable functions that converge to a function f except at the points of a set E of measure zero. Suppose μ is complete. Given α, $\{x : f(x) > \alpha\} = \{x \notin E : \overline{\lim} f_n(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$. Since $\mu(E) = 0$, $\{x \in E : f(x) > \alpha\}$ is measurable. Also, $\{x \notin E : \overline{\lim} f_n(x) > \alpha\} = E^c \cap \{x : \overline{\lim} f_n(x) > \alpha\}$ so it is measurable. Hence $\{x : f(x) > \alpha\}$ is measurable and f is measurable.

Note: If $f_n(x) \to f(x)$ for all x, then completeness is not required.

13a. Let $\langle f_n \rangle$ be a sequence of measurable real-valued functions that converge to f in measure. For any k, there exists n_k and a measurable set E_k with $\mu(E_k) < 2^{-k}$ such that $|f_{n_k}(x) - f(x)| < 2^{-k}$ for $x \notin E_k$. If $x \notin \bigcup_{k=m}^\infty E_k$, then $|f_{n_k}(x) - f(x)| < 2^{-k}$ for $k \geq m$. Thus if $x \notin \bigcap_m \bigcup_{k=m}^\infty E_k$, then for some m, we have $|f_{n_k}(x) - f(x)| < 2^{-k}$ for $k \geq m$ so $\langle f_{n_k} \rangle$ converges to f. Furthermore, $\mu(\bigcap_m \bigcup_{k=m}^\infty E_k) \leq \mu(\bigcup_{k=m}^\infty E_k) \leq \sum_{k=m}^\infty \mu(E_k) < \sum_{k=m}^\infty 2^{-k} = 2^{-m+1}$ for all m so $\mu(\bigcap_m \bigcup_{k=m}^\infty E_k) = 0$ and $\langle f_{n_k} \rangle$ converges to f a.e.

(c.f. Proposition 4.18)

13b. Let $\langle f_n \rangle$ be a sequence of measurable functions each of which vanishes outside a fixed measurable set A with $\mu(A) < \infty$. Suppose that $f_n(x) \to f(x)$ for almost all x, say except on a set B of measure zero. If $x \in A^c \backslash B$, then $f_n(x) = 0$ for all n so $f(x) = 0$. Given $\varepsilon > 0$, let $G_n = \{x \in A \backslash B : |f_n(x) - f(x)| \geq \varepsilon\}$ and let $E_N = \bigcup_{n=N}^\infty G_n$. Note that $E_{N+1} \subset E_N$. If $x \in A \backslash B$, then there exists N such that $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$. i.e. $x \notin E_N$ for some N. Thus $\bigcap E_N = \emptyset$ and $\lim \mu(E_N) = 0$ so there exists N such that $\mu(E_N) < \varepsilon$. Furthermore, $\mu(E_N \cup B) < \varepsilon$ and if $x \notin E_N \cup B$, then either $x \in A$ and $x \notin G_n^c$ for all $n \geq N$ so $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ or $x \notin A$ so $|f_n(x) - f(x)| = 0$ for all n. Hence $\langle f_n \rangle$ converges to f in measure.

13c. Let $\langle f_n \rangle$ be a sequence that is Cauchy in measure. We may choose $n_{k+1} > n_k$ such that $\mu\{x : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} < 2^{-k}$. Let $E_k = \{x : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\}$ and let $F_m = \bigcup_{k=m}^{\infty} E_k$. Then $\mu(\bigcap F_m) \leq \mu(\bigcup_{k=m}^{\infty} E_k) \leq 2^{-k+1}$ for all k so $\mu(\bigcap F_m) = 0$. If $x \notin \bigcap F_m$, then $x \notin F_m$ for some m so $|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}$ for all $k \geq m$ and $|f_{n_l}(x) - f_{n_k}(x)| < 2^{-k+1}$ for $l \geq k \geq m$. Thus the series $\sum(f_{n_{k+1}} - f_{n_k})$ converges a.e. to a function g. Let $f = g + f_{n_1}$. Then $f_{n_k} \to f$ in measure since the partial sums are of the form $f_{n_k} - f_{n_1}$. Given $\varepsilon > 0$, choose N such that $\mu\{x : |f_n(x) - f_m(x)| \geq \varepsilon/2\} < \varepsilon/2$ for $n > m \geq N$ and $\mu\{x : |f_{n_k}(x) - f(x)| \geq \varepsilon/2\} < \varepsilon/2$ for $k \geq N$. Then $\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subset \{x : |f_n(x) - f_{n_k}(x)| \geq \varepsilon/2\} \cup \{x : |f_{n_k}(x) - f(x)| \geq \varepsilon/2\}$ for all $n, k \geq N$. Thus $\mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon$ for all $n \geq N$ and $\langle f_n \rangle$ converges to f in measure.

(c.f. Q4.25)

14. Let $(X, \mathfrak{B}, \mu)$ be a measure space and $(X, \mathfrak{B}_0, \mu_0)$ its completion. Suppose g is measurable with respect to $\mathfrak{B}$ and there is a set $E \in \mathfrak{B}$ with $\mu(E) = 0$ and $f = g$ on $X \setminus E$. For any α, $\{x : g(x) < \alpha\} \in \mathfrak{B}$. Now $\{x : f(x) < \alpha\} = \{x \in X \setminus E : g(x) < \alpha\} \cup \{x \in E : f(x) < \alpha\}$ where $\{x \in X \setminus E : g(x) < \alpha\} \in \mathfrak{B}$ and $\{x \in E : f(x) < \alpha\} \subset E$. Thus $\{x : f(x) < \alpha\} \in \mathfrak{B}_0$ and f is measurable with respect to $\mathfrak{B}_0$.

For the converse, first consider the case of characteristic functions. Suppose χ_A is measurable with respect to $\mathfrak{B}_0$. Then $A \in \mathfrak{B}_0$ so $A = A' \cup B'$ where $A' \in \mathfrak{B}$ and $B' \subset C'$ with $C' \in \mathfrak{B}$ and $\mu(C') = 0$. Define $f(x) = \chi_A(x)$ if $x \notin C'$ and $f(x) = 1$ if $x \in C'$. If $\alpha < 0$, then $\{x : f(x) > \alpha\} = X$. If $\alpha \geq 1$, then $\{x : f(x) > \alpha\} = \emptyset$. If $0 \leq \alpha < 1$, then $\{x : f(x) > \alpha\} = \{x : f(x) = 1\} = A \cup C' = A' \cup C'$. Thus $\{x : f(x) > \alpha\} \in \mathfrak{B}$ for all α and f is measurable with respect to $\mathfrak{B}$. Next consider the case of simple functions. Suppose $g = \sum_{i=1}^{n} c_i \chi_{A_i}$ is measurable with respect to $\mathfrak{B}_0$. Then each χ_{A_i} is measurable with respect to $\mathfrak{B}_0$. For each i, there is a function h_i measurable with respect to $\mathfrak{B}$ and a set $E_i \in \mathfrak{B}$ with $\mu(E_i) = 0$ and $\chi_{A_i} = h_i$ on $X \setminus E_i$. Then $g = \sum_{i=1}^{n} c_i h_i$ on $X \setminus \bigcup_{i=1}^{n} E_i$ where $\bigcup_{i=1}^{n} E_i \in \mathfrak{B}$ and $\mu(\bigcup_{i=1}^{n} E_i) = 0$. Furthermore, $\sum_{i=1}^{n} c_i h_i$ is measurable with respect to $\mathfrak{B}$. Now for a nonnegative measurable function f, there is a sequence of simple functions $\langle \varphi_n \rangle$ converging pointwise to f on X. For each n, there is a function ψ_n measurable with respect to $\mathfrak{B}$ and a set $E_n \in \mathfrak{B}$ with $\mu(E_n) = 0$ and $\varphi_n = \psi_n$ on $X \setminus E_n$. Then $f = \lim \psi_n$ on $X \setminus \bigcup E_n$ where $\bigcup E_n \in \mathfrak{B}$ and $\mu(\bigcup E_n) = 0$. Furthermore, $\lim \psi_n$ is measurable with respect to $\mathfrak{B}$. Finally, for general a measurable function f, we have $f = f^+ - f^-$ where f^+ and f^- are nonnegative measurable functions and the result follows from the previous case.

15. Let D be the rationals. Suppose that to each $\beta \in D$ there is assigned a $B_\beta \in \mathfrak{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable function f on X such that $f \leq \beta$ on B_β and $f \geq \beta$ on $X \setminus B_\beta$. Now $\{x : f(x) < \alpha\} = \bigcup_{\beta<\alpha} B_\beta$ and $\{x : f(x) \leq \alpha\} = \bigcap_{\gamma>\alpha} \bigcup_{\beta<\gamma} B_\beta$. Similarly, $\{x : f(x) > \alpha\} = X \setminus \bigcap_{\beta>\alpha} B_\beta$ and $\{x : f(x) \geq \alpha\} = \bigcap_{\gamma<\alpha}(X \setminus \bigcap_{\beta>\gamma} B_\beta)$. Also, $\{x : f(x) = \alpha\} = \bigcap_{\gamma>\alpha} \bigcup_{\beta<\gamma} B_\beta \cap \bigcap_{\gamma<\alpha}(X \setminus \bigcap_{\beta>\gamma} B_\beta)$.

16. Egoroff's Theorem: Let $\langle f_n \rangle$ be a sequence of measurable functions each of which vanishes outside a fixed measurable set A of finite measure. Suppose that $f_n(x) \to f(x)$ for almost all x. By Q13b, $\langle f_n \rangle$ converges to f in measure. Given $\varepsilon > 0$ and n, there exist N_n and a measurable set E_n with $\mu(E_n) < \varepsilon 2^{-n}$ such that $|f_n(x) - f(x)| < 1/n$ for $n \geq N_n$ and $x \notin E_n$. Let $E = \bigcup E_n$. Then $\mu(\bigcup E_n) \leq \sum \mu(E_n) < \varepsilon$. Choose n_0 such that $1/n_0 < \varepsilon$. If $x \notin E$ and $n \geq N_{n_0}$, we have $|f_n(x) - f(x)| < 1/n_0 < \varepsilon$. Thus f_n converges uniformly to f on $X \setminus E$.

11.3 Integration

17. Let f and g be measurable functions and E a measurable set.

(i) For a constant c_1, note that if $c_1 \geq 0$, then $(c_1 f)^+ = c_1 f^+$ and $(c_1 f)^- = c_1 f^-$ so $\int_E c_1 f = \int_E c_1 f^+ - \int_E c_1 f^- = c_1 \int_E f^+ - c_1 \int_E f^- = c_1 \int_E f$. On the other hand, if $c_1 < 0$, then $(c_1 f)^+ = -c_1 f^-$ and $(c_1 f)^- = -c_1 f^+$ so $\int_E c_1 f = \int_E (-c_1 f^-) - \int_E (-c_1 f^+) = -c_1 \int_E f^- - (-c_1) \int_E f^+ = c_1 \int_E f$. Note that if $f = f_1 - f_2$ where f_1, f_2 are nonnegative integrable functions, then $f^+ + f_2 = f^- + f_1$ so $\int f^+ + \int f_2 = \int f^- + \int f_1$ and $\int f = \int f^+ - \int f^- = \int f_1 - \int f_2$. Now since f and g are integrable, so are $f^+ + g^+$ and $f^- + g^-$. Furthermore, $f + g = (f^+ + g^+) - (f^- + g^-)$. Thus $\int(f + g) = \int(f^+ + g^+) - \int(f^- + g^-) = \int f^+ + \int g^+ - \int f^- - \int g^- = \int f + \int g$. Together, we have $\int_E (c_1 f + c_2 g) = c_1 \int_E f + c_2 \int_E g$.

(ii) Suppose $|h| \leq |f|$ and h is measurable. Since f is integrable, so are f^+ and f^-. Thus $|f| = f^+ + f^-$ is integrable. By (iii), we have $\int h \leq \int |h| \leq \int |f| < \infty$. Thus h is integrable.

(iii) Suppose $f \geq g$ a.e. Then $f - g \geq 0$ a.e. and $\int(f-g) \geq 0$. By (i), we have $\int(f-g) = \int f - \int g$ so $\int f - \int g \geq 0$. i.e. $\int f \geq \int g$.

(*) Proof of Proposition 15

18. Suppose that μ is not complete. Define a bounded function f to be integrable over a set E of finite measure if $\sup_{\varphi \leq f} \int_E \varphi \, d\mu = \inf_{\psi \geq f} \int_E \psi \, d\mu$ for all simple functions φ and ψ. If f is integrable, then given n, there are simple functions φ_n and ψ_n such that $\varphi_n \leq f \leq \psi_n$ and $\int \psi_n \, d\mu - \int \varphi_n \, d\mu < 1/n$. Then the functions $\psi^* = \inf \psi_n$ and $\varphi^* = \sup \varphi_n$ are measurable and $\varphi^* \leq f \leq \psi^*$. Now the set $\Delta = \{x : \varphi^*(x) < \psi^*(x)\}$ is the union of the sets $\Delta_v = \{x : \varphi^*(x) < \psi^*(x) - 1/v\}$. Each Δ_v is contained in the set $\{x : \varphi_n(x) < \psi_n(x) - 1/v\}$, which has measure less than v/n. Since n is arbitrary, $\mu(\Delta_v) = 0$ so $\mu(\Delta) = 0$. Thus $\varphi^* = f = \psi^*$ except on a set of measure zero. Hence f is measurable with respect to the completion of μ by Q14.

Conversely, if f is measurable with respect to the completion of μ, then by Q14, $f = g$ on $X \setminus E$ where g is measurable with respect to μ and $\mu(E) = 0$. We may assume that g is bounded by M. By considering the sets $E_k = \{x : kM/n \geq g(x) \geq (k-1)M/n\}$, $-n \leq k \leq n$, and the simple functions $\psi_n = (M/n)\sum_{k=-n}^{n} k\chi_{E_k}$ and $\varphi_n = (M/n)\sum_{k=-n}^{n}(k-1)\chi_{E_k}$, we see that g is integrable. Note that $\sup_{\varphi \leq g} \int_E \varphi \, d\mu = \sup_{\varphi \leq f} \int_E \varphi \, d\mu$ and $\inf_{\psi \geq g} \int_E \psi \, d\mu = \inf_{\psi \geq f} \int_E \psi \, d\mu$. It follows that f is integrable.

19. Let f be an integrable function on the measure space $(X, \mathfrak{B}, \mu)$. Let $\varepsilon > 0$ be given. Suppose f is nonnegative and bounded by M. If $\mu(E) < \varepsilon/M$, then $\int_E f < \varepsilon$. In particular, the result holds for all simple functions. If f is nonnegative, there is an increasing sequence $\langle \varphi_n \rangle$ of nonnegative simple functions converging to f on X. By the Monotone Convergence Theorem, we have $\int f = \lim \int \varphi_n$ so there exists N such that $\int(f - f_N) < \varepsilon/2$. Choose $\delta < \varepsilon/2 \sup|\varphi_N|$. If $\mu(E) < \delta$, then $\int_E f = \int_E(f - f_N) + \int_E f_N < \varepsilon/2 + \varepsilon/2 = \varepsilon$ so the result holds for nonnegative measurable functions. For an integrable function f, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then $\int_E |f| < \varepsilon$. Thus $|\int_E f| \leq \int_E |f| < \varepsilon$.

20. Fatou's Lemma: Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions that converge in measure on a set E to a function f. There is a subsequence $\langle f_{n_k} \rangle$ such that $\lim \int_E f_{n_k} = \underline{\lim} \int_E f_n$. Now $\langle f_{n_k} \rangle$ converges to f in measure on E so by Q13a it has a subsequence $\langle f_{n_{k_j}} \rangle$ that converges to f almost everywhere on E. Thus $\int_E f \leq \underline{\lim} \int_E f_{n_{k_j}} = \lim \int_E f_{n_k} = \underline{\lim} \int_E f_n$.

Monotone Convergence Theorem: Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions which converge in measure to a function f and suppose that $f_n \leq f$ for all n. Since $f_n \leq f$, we have $\int f_n \leq \int f$. By Fatou's Lemma, we have $\int f \leq \underline{\lim} \int f_n \leq \overline{\lim} \int f_n \leq \int f$ so equality holds and $\int f = \lim \int f_n$.

Lebesgue Convergence Theorem: Let g be integrable over E and suppose that $\langle f_n \rangle$ is a sequence of measurable functions such that on E we have $|f_n(x)| \leq g(x)$ and such that $\langle f_n \rangle$ converges in measure to f almost everywhere on E. The functions $g - f_n$ are nonnegative so by Fatou's Lemma, $\int_E(g - f) \leq \underline{\lim} \int_E(g - f_n)$. There is a subsequence $\langle f_{n_k} \rangle$ that converges to f almost everywhere on E so $|f| \leq |g|$ on E and f integrable. Thus $\int_E g - \int_E f \leq \int_E g - \overline{\lim} \int_E f_n$ and $\overline{\lim} \int_E f_n \leq \int_E f$. Similarly, by considering the functions $g + f_n$, we have $\int_E f \leq \underline{\lim} \int_E f_n$. Thus equality holds and $\int_E f = \lim \int_E f_n$.

21a. Suppose f is integrable. Then $|f|$ is integrable. Now $\{x : f(x) \neq 0\} = \{x : |f(x)| > 0\} = \bigcup\{x : |f(x)| \geq 1/n\}$. Each of the sets $\{x : |f(x)| \geq 1/n\}$ is measurable. If $\mu\{x : |f(x)| \geq 1/n\} = \infty$ for some n, then $\int |f| \geq \int_{\{x:|f(x)|\geq 1/n\}} |f| \geq (1/n)\mu\{x : |f(x)| \geq 1/n\} = \infty$. Contradiction. Thus $\mu\{x : |f(x)| \geq 1/n\} < \infty$ for all n and $\{x : f(x) \neq 0\}$ is of σ-finite measure.

21b. Suppose f is integrable and $f \geq 0$. There is an increasing sequence $\langle \varphi_n \rangle$ of simple functions such that $f = \lim \varphi_n$. We may redefine each φ_n to be zero on $\{x : f(x) = 0\}$. Since $\{x : f(x) \neq 0\}$ is σ-finite, we may further redefine each φ_n to vanish outside a set of finite measure by Proposition 7 (c.f. Q10).

21c. Suppose f is integrable with respect to μ. Then f^+ and f^- are nonnegative integrable functions so by part (b), there are increasing sequences $\langle \varphi_n \rangle$ and $\langle \psi_n \rangle$ of simple functions each of which vanishes outside a set of finite measure such that $\lim \varphi_n = f^+$ and $\lim \psi_n = f^-$. By the Monotone Convergence Theorem, $\int f^+ \, d\mu = \lim \int \varphi_n \, d\mu$ and $\int f^- \, d\mu = \lim \int \psi_n \, d\mu$. Given $\varepsilon > 0$, there is a simple function φ_N such that $\int f^+ \, d\mu - \int \varphi_n \, d\mu < \varepsilon/2$ and there is a simple function $\psi_{N'}$ such that $\int f^- \, d\mu - \int \psi_{N'} \, d\mu < \varepsilon/2$. Let $\varphi = \varphi_N - \psi_{N'}$. Then φ is a simple function and $\int |f - \varphi| \, d\mu = \int |f^+ - \varphi_N - f^- + \psi_{N'}| \, d\mu \leq \int |f^+ - \varphi_N| \, d\mu + \int |f^- - \psi_{N'}| \, d\mu = (\int f^+ \, d\mu - \int \varphi_N \, d\mu) + (\int f^- \, d\mu - \int \psi_{N'} \, d\mu) < \varepsilon$.

22a. Let $(X, \mathfrak{B}, \mu)$ be a measure space and g a nonnegative measurable function on X. Set $\nu(E) = \int_E g \, d\mu$. Clearly $\nu(\emptyset) = 0$. Let $\langle E_n \rangle$ be a sequence of disjoint sets in $\mathfrak{B}$. Then $\nu(\bigcup E_n) = \int_{\bigcup E_n} g \, d\mu =$

$\int g\chi_{\bigcup E_n}\ d\mu = \int \sum g\chi_{E_n}\ d\mu = \sum \int g\chi_{E_n}\ d\mu = \sum \int_{E_n} g\ d\mu = \sum \nu(E_n)$. Hence ν is a measure on $\mathfrak{B}$.

22b. Let f be a nonnegative measurable function on X. If φ is a simple function given by $\sum_{i=1}^n c_i\chi_{E_i}$. Then $\int \varphi\ d\nu = \sum_{i=1}^n c_i\nu(E_i) = \sum_{i=1}^n c_i \int_{E_i} g\ d\mu = \sum_{i=1}^n c_i \int g\chi_{E_i}\ d\mu = \int \sum_{i=1}^n c_i g\chi_{E_i}\ d\mu = \int \varphi g\ d\mu$. Now if f is a nonnegative measurable function, there is an increasing sequence $\langle\varphi_n\rangle$ of simple functions such that $f = \lim \varphi_n$. Then $\langle\varphi_n g\rangle$ is an increasing sequence of nonnegative measurable functions such that $fg = \lim \varphi_n g$. By the Monotone Convergence Theorem, we have $\int fg\ d\mu = \lim \int \varphi_n g\ d\mu = \lim \int \varphi_n\ d\nu = \int f\ d\nu$.

23a. Let $(X, \mathfrak{B}, \mu)$ be a measure space. Suppose f is locally measurable. For any α and any $E \in \mathfrak{B}$ with $\mu(E) < \infty$, $\{x : f(x) > \alpha\} \cap E = \{x : f\chi_E(x) > \alpha\}$ if $\alpha \geq 0$ and $\{x : f(x) > \alpha\} \cap E = \{x : f\chi_E(x) > 0\} \cup (\{x : f\chi_E(x) = 0\} \cap E) \cup \{x : 0 > f\chi_E(x) > \alpha\}$ if $\alpha < 0$. Thus $\{x : f(x) > \alpha\} \cap E$ is measurable so $\{x : f(x) > \alpha\}$ is locally measurable and f is measurable with respect to the σ-algebra of locally measurable sets.

Conversely, suppose f is measurable with respect to the σ-algebra of locally measurable sets. For any α and any $E \in \mathfrak{B}$ with $\mu(E) < \infty$, $\{x : f\chi_E(x) > \alpha\} = \{x : f(x) > \alpha\} \cap E$ if $\alpha \geq 0$ and $\{x : f\chi_E(x) > \alpha\} = (\{x : f(x) > \alpha\} \cap E) \cup E^c$ if $\alpha < 0$. Thus $\{x : f\chi_E(x) > \alpha\}$ is measurable and f is locally measurable.

23b. Let μ be a σ-finite measure. Define integration for nonnegative locally measurable functions f by taking $\int f$ to be the supremum of $\int \varphi$ as φ ranges over all simple functions less than f. For a simple function $\varphi = \sum_{i=1}^n c_i\chi_{E_i}$, we have $\int \varphi = \sum_{i=1}^n c_i\mu(E_i) = \sum_{i=1}^n c_i\underline{\mu}(E_i) = \int \varphi\ d\underline{\mu}$.

Let $X = \bigcup X_n$ where each X_n is measurable and $\mu(X_n) < \infty$. Then $\int f = \int f\chi_{\bigcup X_n} = \int \sum_n f\chi_{X_n}$. Now $f\chi_{X_n}$ is measurable for each n so there is an increasing sequence $\langle\varphi_k^{(n)}\rangle$ of simple functions converging to $f\chi_{X_n}$. Thus $\int f = \int \sum_n f\chi_{X_n} = \sum_n \int f\chi_{X_n} = \sum_n \lim_k \int \varphi_k^{(n)} = \sum_n \lim_k \int \varphi_k^{(n)}\ d\underline{\mu} = \sum_n \int f\chi_{X_n}\ d\underline{\mu} = \int \sum_n f\chi_{X_n}\ d\underline{\mu} = \int f\chi_{\bigcup X_n}\ d\underline{\mu} = \int f\ d\underline{\mu}$.

11.4 General convergence theorems

24. Let $(X, \mathfrak{B})$ be a measurable space and $\langle\mu_n\rangle$ a sequence of measures on $\mathfrak{B}$ such that for each $E \in \mathfrak{B}$, $\mu_{n+1}(E) \geq \mu_n(E)$. Let $\mu(E) = \lim \mu_n(E)$. Clearly $\mu(\emptyset) = 0$. Let $\langle E_k\rangle$ be a sequence of disjoint sets in $\mathfrak{B}$. Then $\mu(\bigcup_k E_k) = \lim_n \mu_n(\bigcup_k E_k) = \lim_n \sum_k \mu_n(E_k) = \sum_k \lim_n \mu_n(E_k) = \sum_k \mu(E_k)$ where the interchanging of the limit and the summation is valid because $\mu_n(E_k)$ is increasing in n for each k. Hence μ is a measure.

***25.** Let m be Lebesgue measure. For each n, define μ_n by $\mu_n(E) = m(E)2^{-n}$. Then $\langle\mu_n\rangle$ is a decreasing sequence of measures. Note that $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}[k, k+1)$. Now $\mu(\bigcup_k[k, k+1) = \lim_n \mu_n(\bigcup_k[k, k+1)) = \lim_n m(\mathbb{R})2^{-n} = \infty$ but $\sum_k \mu([k, k+1)) = \sum_k \lim_n \mu_n([k, k+1)) = \sum_k \lim_n m([k, k+1))2^{-n} = \sum_k \lim_n 2^{-n} = 0$. Hence μ is not a measure.

***26.** Let $(X, \mathfrak{B})$ be a measurable space and $\langle\mu_n\rangle$ a sequence of measures on $\mathfrak{B}$ that converge setwise to to a set function μ. Clearly $\mu(\emptyset) = 0$. If $E_1, E_2 \in \mathfrak{B}$ and $E_1 \cap E_2 = \emptyset$, then $\mu(E_1 \cup E_2) = \lim \mu_n(E_1 \cup E_2) = \lim(\mu_n(E_1) + \mu_n(E_2)) = \lim \mu_n(E_1) + \lim \mu_n(E_2) = \mu(E_1) + \mu(E_2)$. Thus μ is finitely additive. If μ is not a measure, then it is not $\emptyset$-continuous. i.e. there is a decreasing sequence $\langle E_n\rangle$ of set in $\mathfrak{B}$ with $\bigcap E_n = \emptyset$ and $\lim \mu(E_n) = \varepsilon > 0$. Define $\alpha_1 = \beta_1 = 1$. If α_j and β_j have been defined for $j \leq k$, let $\alpha_{k+1} > \alpha_k$ such that $\mu_{\alpha_{k+1}}(E_{\beta_k}) \geq 7\varepsilon/8$. Then let $\beta_{k+1} > \beta_k$ such that $\varepsilon/8 \geq \mu_{\alpha_{k+1}}(E_{\beta_{k+1}})$. Define $F_n = E_{\beta_n} \setminus E_{\beta_{n+1}}$. Then $\mu_{\alpha_{n+1}}(F_n) \geq 3\varepsilon/4$. Now for j odd and $1 \leq k < j$, we have $\mu_{\alpha_j}(\sum_{n \text{ even}, n\geq k} F_n) \geq 3\varepsilon/4$ so for $k \geq 1$, we have $\mu(\sum_{n \text{ even}, n\geq k} F_n) \geq 3\varepsilon/4$. This inequality is also true for odd n. Thus for $k \geq 1$, we have $\mu(E_{\beta_k}) = \mu(\sum F_n) \geq 3\varepsilon/2$. Contradiction. Hence μ is a measure.

11.5 Signed measures

27a. Let ν be a signed measure on a measurable space $(X, \mathfrak{B})$ and let $\{A, B\}$ be a Hahn decomposition for ν. Let N be a null set and consider $\{A \cup N, B \setminus N\}$. Note that $(A \cup N) \cup (B \setminus N) = A \cup B = X$ and $(A \cup N) \cap (B \setminus N) = A \cap (B \setminus N) = \emptyset$. For any measurable subset E of $A \cup N$, we have $\nu(E) = \nu(E \cap A) + \nu(E \cap (N \setminus A)) = \nu(E \cap A) \geq 0$. For any measurable subset E' of $B \setminus N$, we have $\nu(E') = \nu(E' \cap B) - \nu(E' \cap B \cap N) = \nu(E' \cap B) \leq 0$. Thus $\{A \cup N, B \setminus N\}$ is also a Hahn decomposition for ν.

Consider a set $\{a, b, c\}$ with the σ-algebra being its power set. Define $\nu(\{a\}) = -1, \nu(\{b\}) = 0, \nu(\{c\}) = 1$ and extend it to a signed measure on $\{a, b, c\}$ in the natural way. Then $\{\{a, b\}, \{c\}\}$ and $\{\{a\}, \{b, c\}\}$ are both Hahn decompositions for ν.

27b. Suppose $\{A, B\}$ and $\{A', B'\}$ are Hahn decompositions for ν. Then $A \setminus A' = A \cap B'$ and $A' \setminus A = A' \cap B$. Thus $A \setminus A'$ and $A' \setminus A$ are null sets. Since $A\Delta A' = (A \setminus A') \cup (A' \setminus A)$, $A\Delta A'$ is also a null set. Similarly, $B\Delta B'$ is a null set. Hence the Hahn decomposition is unique except for null sets.

28. Let $\{A, B\}$ be a Hahn decomposition for ν. Suppose $\nu = \nu^+ - \nu^- = \mu_1 - \mu_2$ where $\mu_1 \perp \mu_2$. There are disjoint measurable sets A', B' such that $X = A' \cup B'$ and $\mu_1(B') = \mu_2(A') = 0$. If $E \subset A'$, then $\nu(E) = \mu_1(E) - \mu_2(E) = \mu_1(E) \geq 0$. If $E \subset B'$, then $\nu(E) = -\mu_2(E) \leq 0$. Thus $\{A', B'\}$ is also a Hahn decomposition for ν. Now for any measurable set E, we have $\nu^+(E) = \nu(E \cap A) = \nu(E \cap A \cap A') + \nu(E \cap (A \setminus A')) = \nu(E \cap A \cap A') + \nu(E \cap (A' \setminus A)) = \mu_1(E \cap A \cap A') + \mu_1(E \cap (A' \setminus A)) = \mu_1(E \cap A') + \mu_1(E \cap B') = \mu_1(E)$. Similarly, $\nu^-(E) = \mu_2(E)$.

29. Let $\{A, B\}$ be a Hahn decomposition for ν and let E be any measurable set. Since $\nu_2(E) \geq 0$, we have $\nu(E) = \nu^+(E) - \nu^-(E) \leq \nu^+(E)$. Since $\nu_1(E) \geq 0$, we have $-\nu^-(E) \leq \nu^+(E) - \nu^-(E) = \nu(E)$. Thus $-\nu^-(E) \leq \nu(E) \leq \nu^+(E)$.

Now $\nu(E) \leq \nu^+(E) \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$ and $-\nu(E) \leq \nu^-(E) \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$. Hence $|\nu(E)| \leq |\nu|(E)$.

30. Let ν_1 and ν_2 be finite signed measures and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha\nu_1 + \beta\nu_2$ is a signed measure. Then $\alpha\nu_1 + \beta\nu_2$ is a finite signed measure since $(\alpha\nu_1 + \beta\nu_2)(\bigcup E_n) = \alpha\nu_1(\bigcup E_n) + \beta\nu_2(\bigcup E_n) = \alpha \sum \nu_1(E_n) + \beta \sum \nu_2(E_n) = \sum(\alpha\nu_1(E_n) + \beta\nu_2(E_n))$ for any sequence $\langle E_n \rangle$ of disjoint measurable sets where the last series converges absolutely.

Let $\{A, B\}$ be a Hahn decomposition for ν. Suppose $\alpha \geq 0$. Then $|\alpha\nu|(E) = (\alpha\nu)^+(E) + (\alpha\nu)^-(E) = \alpha\nu^+(E) + \alpha\nu^-(E) = |\alpha|\,|\nu|(E)$ since $\alpha\nu = \alpha\nu^+ - \alpha\nu^-$ and $\alpha\nu^+ \perp \alpha\nu^-$ as $\alpha\nu^+(B) = 0 = \alpha\nu^-(A)$. Suppose $\alpha < 0$. Then $|\alpha\nu|(E) = (\alpha\nu)^+(E) + (\alpha\nu)^-(E) = -\alpha\nu^-(E) - \alpha\nu^+(E) = |\alpha|\,|\nu|(E)$ since $\alpha\nu = \alpha\nu^+ - \alpha\nu^- = -\alpha\nu^- - (-\alpha\nu^+)$ and $-\alpha\nu^+ \perp -\alpha\nu^-$ as $-\alpha\nu^+(B) = 0 = -\alpha\nu^-(A)$. Hence $|\alpha\nu| = |\alpha|\,|\nu|$.

Also, $|\nu_1 + \nu_2|(E) = |\nu_1(E) + \nu_2(E)| \leq |\nu_1(E)| + |\nu_2(E)| = |\nu_1|(E) + |\nu_2|(E)$. Hence $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

31. Define integration with respect to a signed measure ν by defining $\int f\, d\nu = \int f\, d\nu^+ - \int f\, d\nu^-$. Suppose $|f| \leq M$. Then $|\int_E f\, d\nu| = |\int_E f\, d\nu^+ - \int_E f\, d\nu^-| \leq |\int_E f\, d\nu^+| + |\int_E f\, d\nu^-| \leq \int_E |f|\, d\nu^+ + \int_E |f|\, d\nu^- \leq M\nu^+(E) + M\nu^-(E) = M|\nu|(E)$.

Let $\{A, B\}$ be a Hahn decomposition for ν. Then $\int_E (\chi_{E\cap A} - \chi_{E\cap B})\, d\nu = \int_{E\cap A} 1\, d\nu - \int_{E\cap B} 1\, d\nu = [\nu^+(E\cap A) - \nu^-(E\cap A)] - [\nu^+(E\cap B) - \nu^-(E\cap B)] = \nu^+(E\cap A) + \nu^-(E\cap A) + \nu^+(E\cap B) + \nu^-(E\cap B) = \nu^+(E) + \nu^-(E) = |\nu|(E)$ where $\chi_{E\cap A} - \chi_{E\cap B}$ is measurable and $|\chi_{E\cap A} - \chi_{E\cap B}| \leq 1$.

32a. Let μ and ν be finite signed measures. Define $\mu \wedge \nu$ by $(\mu \wedge \nu)(E) = \min(\mu(E), \nu(E))$. Note that $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ so $\mu \wedge \nu$ is a finite signed measure by Q30 and it is smaller than μ and ν. Furthermore, if η is a signed measure smaller than μ and ν, then $\eta \leq \mu \wedge \nu$.

32b. Define $\mu \vee \nu$ by $(\mu \vee \nu)(E) = \max(\mu(E), \nu(E))$. Note that $\mu \vee \nu = \frac{1}{2}(\mu + \nu + |\mu - \nu|)$ so $\mu \vee \nu$ is a finite signed measure by Q30 and it is larger than μ and ν. Furthermore, if η is a signed measure larger than μ and ν, then $\eta \geq \mu \vee \nu$. Also, $\mu \vee \nu + \mu \wedge \nu = \max(\mu, \nu) + \min(\mu, \nu) = \mu + \nu$.

32c. Suppose μ and ν are positive measures. If $\mu \perp \nu$, then there are disjoint measurable sets A and B such that $X = A \cup B$ and $\mu(B) = 0 = \nu(A)$. For any measurable set E, we have $(\mu \wedge \nu)(E) = (\mu \wedge \nu(E \cap A) + (\mu \wedge \nu)(E \cap B) = \min(\mu(E \cap A), \nu(E \cap A)) + \min(\mu(E \cap B), \nu(E \cap B)) = 0$. Conversely, suppose $\mu \wedge \nu = 0$. If $\mu(E) = \nu(E) = 0$ for all measurable sets, then $\mu = \nu = 0$ and $\mu \perp \nu$. Thus we may assume that $\mu(E) = 0 < \nu(E)$ for some E. If $\nu(E^c) = 0$, it follows that $\mu \perp \nu$. On the other hand, if $\nu(E^c) > 0$, then $\mu(E^c) = 0$ so $\mu(X) = \mu(E) + \mu(E^c) = 0$. Thus $\mu = 0$ and we still have $\mu \perp \nu$.

11.6 The Radon-Nikodym Theorem

33a. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and let ν be a measure on $\mathfrak{B}$ which is absolutely continuous with respect to μ. Let $X = \bigcup X_i$ with $\mu(X_i) < \infty$ for each i. We may assume the X_i are pairwise disjoint. For each i, let $\mathfrak{B}_i = \{E \in \mathfrak{B} : E \subset E_i\}$, $\mu_i = \mu|_{\mathfrak{B}_i}$ and $\nu_i = \nu|_{\mathfrak{B}_i}$. Then $(X_i, \mathfrak{B}_i, \mu_i)$ is a finite measure space and $\nu_i << \mu_i$. Thus for each i there is a nonnegative μ_i-measurable function f_i such that $\nu_i(E) = \int_E f_i \, d\mu_i$ for all $E \in \mathfrak{B}_i$. Define f by $f(x) = f_i(x)$ if $x \in X_i$. If $E \subset X$, then $E \cap X_i \in \mathfrak{B}_i$ for each i. Thus $\nu(E) = \sum \nu(E \cap X_i) = \sum \nu_i(E \cap X_i) = \sum \int_{E \cap X_i} f_i \, d\mu_i = \sum \int_{E \cap X_i} f \, d\mu = \int_E f \, d\mu$.

33b. If f and g are nonnegative measurable functions such that $\nu(E) = \int_E f \, d\mu = \int_E g \, d\mu$ for any measurable set E, then $\int f \, d\mu = \int g \, d\mu$ so $f = g$ a.e.

34a. Radon-Nikodym derivatives: Suppose $\nu << \mu$ and f is a nonnegative measurable function. For a nonnegative simple function $\varphi = \sum_{i=1}^n c_i \chi_{E_i}$, we have $\int \varphi \, d\nu = \sum_{i=1}^n c_i \nu(E_i) = \sum_{i=1}^n c_i (\int_{E_i} \left[\frac{d\nu}{d\mu}\right] \, d\mu) = \int \sum_{i=1}^n c_i \chi_{E_i} \left[\frac{d\nu}{d\mu}\right] \, d\mu = \int \varphi \left[\frac{d\nu}{d\mu}\right] \, d\mu$. For a nonnegative measurable function f, let $\langle \varphi_i \rangle$ be an increasing sequence of nonnegative simple functions converging pointwise to f. Then $\int f \, d\nu = \lim \int \varphi_i \, d\nu = \lim \int \varphi_i \left[\frac{d\nu}{d\mu}\right] \, d\mu = \int f \left[\frac{d\nu}{d\mu}\right] \, d\mu$.

34b. If $\nu_1 << \mu$ and $\nu_2 << \mu$, then $\nu_1 + \nu_2 << \mu$. For any measurable set E, we have $\nu_1(E) = \int_E \left[\frac{d\nu_1}{d\mu}\right] \, d\mu$ and $\nu_2(E) = \int_E \left[\frac{d\nu_2}{d\mu}\right] \, d\mu$. Thus $(\nu_1 + \nu_2)(E) = \int_E \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\mu}\right] \, d\mu$. By uniqueness of the Radon-Nikodym derivative, we have $\left[\frac{d(\nu_1+\nu_2)}{d\mu}\right] = \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\mu}\right]$.

34c. Suppose $\nu << \mu << \lambda$. For any measurable set E, we have $\nu(E) = \int_E \left[\frac{d\nu}{d\mu}\right] \, d\mu = \int_E \left[\frac{d\nu}{d\mu}\right] \left[\frac{d\mu}{d\lambda}\right] \, d\lambda$ where the last equality follows from part (a). Hence $\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right] \left[\frac{d\mu}{d\lambda}\right]$.

34d. Suppose $\nu << \mu$ and $\mu << \nu$. Then $\left[\frac{d\nu}{d\nu}\right] = \left[\frac{d\nu}{d\mu}\right] \left[\frac{d\mu}{d\nu}\right]$ by part (c). But $\left[\frac{d\nu}{d\nu}\right] \equiv 1$ so $\left[\frac{d\nu}{d\mu}\right] = \left[\frac{d\mu}{d\nu}\right]^{-1}$.

35a. Suppose ν is a signed measure such that $\nu \perp \mu$ and $\nu << \mu$. There are disjoint measurable sets A and B such that $X = A \cup B$ and $|\nu|(B) = 0 = |\mu|(A)$. Then $|\nu(A)| = 0$ so $|\nu|(X) = |\nu|(A) + |\nu|(B) = 0$. Hence $\nu^+ = 0 = \nu^-$ so $\nu = 0$.

35b. Suppose ν_1 and ν_2 are singular with respect to μ. There are disjoint measurable sets A_1 and B_1 such that $X = A_1 \cup B_1$ and $\mu(B_1) = 0 = \nu_1(A_1)$. Similarly, there are disjoint measurable sets A_2 and B_2 such that $X = A_2 \cup B_2$ and $\mu(B_2) = 0 = \nu_2(A_2)$. Now $X = (A_1 \cap A_2) \cup (B_1 \cup B_2)$, $(A_1 \cap A_2) \cap (B_1 \cup B_2) = \emptyset$ and $(c_1\nu_1 + c_2\nu_2)(A_1 \cap A_2) = 0 = \mu(B_1 \cup B_2)$. Hence $c_1\nu_1 + c_2\nu_2 \perp \mu$.

35c. Suppose $\nu_1 << \mu$ and $\nu_2 << \mu$. If $\mu(E) = 0$, then $\nu_1(E) = 0 = \nu_2(E)$. Thus $(c_1\nu_1 + c_2\nu_2)(E) = 0$ and $c_1\nu_1 + c_2\nu_2 << \mu$.

35d. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and let ν be a σ-finite measure on $\mathfrak{B}$. Suppose $\nu = \nu_0 + \nu_1 = \nu_0' + \nu_1'$ where $\nu_0 \perp \mu$, $\nu_0' \perp \mu$, $\nu_1 << \mu$ and $\nu_2 << \mu$. Now $\nu_0 - \nu_0' = \nu_1' - \nu_1$ where $\nu_0 - \nu_0'$ and $\nu_1' - \nu_1$ are signed measures such that $\nu_0 - \nu_0' \perp \mu$ and $\nu_1' - \nu_1 << \mu$ by parts (b) and (c). Then by part (a), we have $\nu_0 - \nu_0' = 0 = \nu_1' - \nu_1$ so $\nu_0 = \nu_0'$ and $\nu_1 = \nu_1'$.

36. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and suppose ν is a signed measure on $\mathfrak{B}$ which is absolutely continuous with respect to μ. Consider the Jordan decomposition $\nu = \nu_1 - \nu_2$ where either ν^+ or ν^- must be finite. There are measurable functions f and g such that $\nu^+(E) = \int_E f \, d\mu$ and $\nu^-(E) = \int_E g \, d\mu$. Then $\nu(E) = \nu^+(E) - \nu^-(E) = \int_E (f - g) \, d\mu$.

37a. Complex measures: Given a complex measure ν, its real and imaginary parts are finite signed measures and so have Jordan decompositions $\mu_1 - \mu_2$ and $\mu_3 - \mu_4$ respectively. Hence $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures.

***37b.**

***37c.**

***37d.**

***37e.**

(*) Polar decomposition for complex measures

38a. Let μ and ν be finite measures on a measurable space $(X, \mathfrak{B})$ and set $\lambda = \mu + \nu$. Define $F(f) = \int f \, d\mu$. Note that $L^2(\lambda) \subset L^1(\lambda) \subset L^1(\mu)$ since $\int |f| \, d\lambda \leq (\lambda(X))^{1/2} (\int |f|^2 \, d\lambda)^{1/2}$ and $\int |f| \, d\mu \leq \int |f| \, d\lambda$. Thus F is well-defined. Clearly F is linear. Furthermore $|F(f)| \leq (\mu(X))^{1/2} ||f||_2$. Hence F is a bounded linear functional on $L^2(\lambda)$.

38b. There exists a unique function $g \in L^2(\lambda)$ such that $F(f) = (f, g)$. Note that $\{g > 1\} = \bigcup \{g \geq 1 + 1/n\}$. Let $E_n = \{g \geq 1 + 1/n\}$. Then $\mu(E_n) = F(\chi_{E_n}) = \int_{E_n} g \, d\lambda \geq (1 + 1/n)\lambda(E_n)$ so

$(1/n)\mu(E_n) + \nu(E_n) \leq 0$. It follows that $\mu(E_n) = 0 = \nu(E_n)$ and $\lambda(E_n) = 0$. Thus $\lambda(\{g > 1\}) = 0$. Similarly, $\lambda(\{g < 0\}=0$. Now $\mu(E) = F(\chi_E) = (\chi_E, g) = \int_E g \; d\lambda$ and $\nu(E) = \lambda(E) - \mu(E) = \int_E (1-g) \; d\lambda$.

38c. Suppose $\nu << \mu$. If $\mu(E) = 0$, then $\nu(E) = 0$ so $\lambda(E) = 0$. Thus $\lambda << \mu$. Let $E = \{x : g(x) = 0\}$. Then $\mu(E) = \int_E g \; d\lambda = 0$. Thus $g = 0$ only on a set of μ-measure zero. In this case, we have $\lambda(E) = \int_E 1 \; d\lambda = \int_E gg^{-1} \; d\lambda = \int_E g^{-1} \; d\mu$ by Q22.

38d. Suppose $\nu << \mu$. Then $\int_E (1-g)g^{-1} \; d\mu = \int_E g^{-1} \; d\mu - \int_E gg^{-1} \; d\mu = \lambda(E) - \mu(E) = \nu(E)$. In particular, $(1-g)g^{-1}$ is integrable with respect to μ.

(*) Alternate proof of the Radon-Nikodym Theorem

39. Let $X = [0,1]$, $\mathfrak{B}$ the class of Lebesgue measurable subsets of $[0,1]$ and take ν to be Lebesgue measure and μ to be the counting measure on $\mathfrak{B}$. Then ν is finite and absolutely continuous with respect to μ. Suppose there is a function f such that $\nu(E) = \int_E f \; d\mu$ for all $E \in \mathfrak{B}$. Since ν is finite, f is integrable with respect to μ. Thus $E_0 = \{x : f(x) \neq 0\}$ is countable. Now $0 = \nu(E_0) = \int_{E_0} f \; d\mu$. Contradiction. Hence there is no such function f.

40a. Decomposable measures: Let $\{X_\alpha\}$ be a decomposition for a measure μ and E a measurable set. Note that $\mu(\bigcup\{X_\alpha \cap E : \mu(X_\alpha \cap E) = 0\}) = 0$ and $\mu(E \setminus \bigcup X_\alpha) = 0$. Thus $\mu(E) = \mu(\bigcup\{X_\alpha \cap E : \mu(X_\alpha \cap E) > 0\})$. If $\mu(X_\alpha \cap E) > 0$ for countably many α, then we have a countable union and it follows that $\mu(E) = \sum \mu(X_\alpha \cap E)$. If $\mu(X_\alpha \cap E) > 0$ for uncountably many α, then $\sum \mu(X_\alpha \cap E) = \infty$. If $\mu(E) < \infty$, then for any finite union $\bigcup_{k=1}^n (X_k \cap E)$, we have $\mu(\bigcup_{k=1}^n (X_k \cap E)) \leq \mu(\bigcup(X_\alpha \cap E)) \leq \mu(\bigcup\{X_\alpha \cap E : \mu(X_\alpha \cap E) > 0\}) = \mu(E)$ so $\sum \mu(X_\alpha \cap E) \leq \mu(E) < \infty$. Contradiction. Thus $\mu(E) = \infty = \sum \mu(X_\alpha \cap E)$.

40b. Let $\{X_\alpha\}$ be a decomposition for a complete measure μ. If f is locally measurable, then since $\mu(X_\alpha) < \infty$ for each α, the restriction of f to each X_α is measurable. Conversely, suppose the restriction of f to each X_α is measurable. Given a measurable set E with $\mu(E) < \infty$, let $A = \bigcup\{X_\alpha \cap E : \mu(X_\alpha \cap E) > 0\}$, $B = \bigcup\{X_\alpha \cap E : \mu(X_\alpha \cap E) = 0\}$ and $C = E \setminus \bigcup X_\alpha)$. For $\beta \geq 0$, $\{x : f\chi_E(x) > \beta\} = \{x \in A : f(x) > \beta\} \cup \{x \in B : f(x) > \beta\} \cup \{x \in C : f(x) > \beta\}$. The last two sets are measurable since they are subsets of sets of measure zero. The first set is measurable since it is simply $\{x : f\chi_{X_\alpha}(x) > \beta\} \cap E$ for some α. For $\beta < 0$, $\{x : f\chi_E(x) > \beta\} = E^c \cup (E \cap \{x : f(x) > \beta\})$. By a similar argument as before, $\{x : f\chi_E(x) > \beta\}$ is measurable. Hence f is locally measurable.

Let f be a nonnegative locally measurable function of X. Now $\int_X f \; d\mu \geq \int_{\bigcup_{i=1}^n X_i} f \; d\mu = \sum_{i=1}^n \int_{X_i} f \; d\mu$ for any finite set $\{X_1, \ldots, X_n\}$. Thus $\int_X f \; d\mu \geq \sum_\alpha \int_{X_\alpha} f \; d\mu$. On the other hand, for any simple function $\varphi = \sum_{i=1}^n c_i \chi_{E_i}$ with $\varphi \leq f$, we have $\int_X \varphi \; d\mu = \sum_{i=1}^n c_i \mu(E_i \cap \bigcup X_\alpha) = \sum_{i=1}^n c_i (\sum_\alpha \mu(E_i \cap X_\alpha)) = \sum_\alpha (\sum_{i=1}^n c_i \mu(E_i \cap X_\alpha)) = \sum_\alpha \int_{X_\alpha} \varphi \; d\mu$. Thus $\int_X f \; d\mu \leq \sum_\alpha \int_{X_\alpha} f \; d\mu$.

***40c.** Let ν be absolutely continuous with respect to μ and suppose that there is a collection $\{X_\alpha\}$ which is a decomposition for both μ and ν. Let ν_α be defined by $\nu_\alpha(E) = \nu(X_\alpha \cap E)$ for each α. Then there is a nonnegative measurable function f_α such that $\nu_\alpha(E) = \int_E f_\alpha \; d\mu$. The function $f = \sum_\alpha f_\alpha$ is locally measurable and $\nu(E) = \sum_\alpha \nu(X_\alpha \cap E) = \sum_\alpha \nu_\alpha(E) = \sum_\alpha \int_E f_\alpha \; d\mu = \sum_\alpha \int_{X_\alpha \cap E} f \; d\mu = \int_E f \; d\mu$.

40d. If instead of assuming $\{X_\alpha\}$ to be a decomposition for ν, we merely assume that if $E \in \mathfrak{B}$ and $\nu(E \cap X_\alpha) = 0$ for all α, then $\nu(E) = 0$, the reverse implication in part (b) remains valid although the forward implication may not be true. Thus the argument in part (c) remains valid and so does the conclusion.

11.7 The L^p spaces

41. Let $f \in L^p(\mu), 1 \leq p < \infty$, and $\varepsilon > 0$. First assume that $f \geq 0$. Let $\langle \varphi_n \rangle$ be an increasing sequence of nonnegative simple functions converging pointwise to f, each of which vanishes outside a set of finite measure. Then the sequence $\langle |f - \varphi_n|^p \rangle$ converges pointwise to zero and is bounded by $2f^p$. By the Lebesgue Convergence Theorem, $\lim \int |f - \varphi_n|^p \; d\mu = 0$. i.e. $\lim ||f - \varphi_n||_p = 0$. Thus $||f - \varphi||_p < \varepsilon$ for some φ.

Now for a general f, there are simple functions φ and ψ vanishing outside sets of finite measure such that $||f^+ - \varphi||_p < \varepsilon/2$ and $||f^- - \psi||_p < \varepsilon/2$. Then $||f - (\varphi + \psi)||_p < \varepsilon$.

(*) Proof of Proposition 26

42. Let $(X, \mathfrak{B}, \mu)$ be a finite measure space and g an integrable function such that for some constant

M, $|\int g\varphi\, d\mu| \leq M||\varphi||_1$ for all simple functions φ. Let $E = \{x : |g(x)| \geq M + \varepsilon\}$ and let $\varphi = (sgng)\chi_E$. Then $(M+\varepsilon)\mu(E) \leq \int_E |g|\, d\mu = |\int g\varphi\, d\mu| \leq M||\varphi||_1 = M\mu(E)$. Thus $\mu(E) = 0$ and $g \in L^\infty$.

(*) Proof of Lemma 27 for $p = 1$

43. Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and g an integrable function such that for some constant M, $|\int g\varphi\, d\mu| \leq M||\varphi||_p$ for all simple functions φ. Now $X = \bigcup X_n$ where $\mu(X_n) < \infty$ for each n. Let $g_n = g\chi_{\bigcup_{i=1}^n X_i}$. Then $g_n \in L^q$, $g_n \to g$ and $|g_n| \leq |g|$ for each n. Thus $\lim |g - g_n|^q\, d\mu = 0$. It follows that $\lim ||g - g_n||_q = 0$ and $g \in L^q$.

44. Let $\langle E_n \rangle$ be a sequence of disjoint measurable sets and for each n let f_n be a function in $L^p, 1 \leq p < \infty$, that vanishes outside E_n. Set $f = \sum f_n$. Then $\int |f|^p = \int |\sum f_n|^p = \int \sum |f_n|^p = \sum \int |f_n|^p = \sum ||f_n||^p$. Hence $f \in L^p$ if and only if $\sum ||f_n||^p < \infty$. In this case, $||f||^p = \sum ||f_n||^p$. Also, since the norm is continuous, we have $||\sum_{i=1}^n f_i||_p \to ||f||_p$ so $||f - \sum_{i=1}^n f_i||_p \to 0$ (c.f. Q6.16).

45. For $g \in L^q$, let F be the linear functional on L^p defined by $F(f) = \int fg\, d\mu$. By the Hölder inequality, we have $|F(f)| = |\int fg\, d\mu| \leq \int |fg|\, d\mu \leq ||f||_p||g||_q$ so $||F|| \leq ||g||_q$. For $1 < p < \infty$, let $f = |g|^{q/p}(sgng)$. Then $|f|^p = |g|^q = fg$ so $f \in L^p$ and $||f||_p = ||g||_q^{p/q}$. Now $|F(f)| = |\int fg\, d\mu| = \int |g|^q\, d\mu = ||g||_q^q = ||g||_q\, ||f||_p$. Thus $||F|| \geq ||g||_q$. If $q = 1$ and $p = \infty$, we may assume $||g||_1 > 0$. Let $f = sgng$. Then $f \in L^\infty$, $||f||_\infty = 1$ and $|F(f)| = |\int fg\, d\mu| = \int |g|\, d\mu = ||g||_1 = ||f||_\infty ||g||_1$ so $||F|| \geq ||g||_1$. If $q = \infty$ and $p = 1$, given $\varepsilon > 0$, let $E = \{x : g(x) > ||g||_\infty - \varepsilon\}$ and let $f = \chi_E$. Then $f \in L^1$, $||f||_1 = \int |f|\, d\mu = \mu(E)$ and $|F(f)| = |\int fg\, d\mu| = |\int_E g\, d\mu| \geq (||g||_\infty - \varepsilon)||f||_1$ so $||F|| \geq ||g||_\infty$.

46a. Let μ be the counting measure on a countable set X. We may enumerate the elements of X by $\langle x_n \rangle$. By considering simple functions, we see that $|f|^p$ is integrable if and only if $\sum |f(x_n)|^p < \infty$. Hence $L^p(\mu) = \ell^p$.

***46b.**

***47a.**

***47b.**

***48.** Let A and B be uncountable sets with different numbers of elements and let $X = A \times B$. Let $\mathfrak{B}$ be the collection of subsets E of X such that for every horizontal or vertical line L either $E \cap L$ or $E^c \cap L$ is countable. Clearly $\emptyset \in \mathfrak{B}$. Suppose $E \in \mathfrak{B}$. By the symmetry in the definition, $E^c \in \mathfrak{B}$. Suppose $\langle E_n \rangle$ is a sequence of sets in $\mathfrak{B}$. For every horizontal or vertical line L, $(\bigcup E_n) \cap L = \bigcup (E_n \cap L)$ and $(\bigcup E_n)^c \cap L = \bigcap E_n^c \cap L$. If $E_n \cap L$ is countable for all n, then $(\bigcup E_n) \cap L$ is countable. Otherwise, $E_n^c \cap L$ is countable for some n and $(\bigcup E_n)^c \cap L$ is countable. Thus $\bigcup E_n \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a σ-algebra. Let $\mu(E)$ be the number of horizontal and vertical lines L for which $E^c \cap L$ is countable and $\nu(E)$ be the number of horizontal lines with $E^c \cap L$ countable. Clearly $\mu(\emptyset) = 0 = \nu(\emptyset)$ since A and B are uncountable. Suppose $\langle E_n \rangle$ is a sequence of disjoint sets in $\mathfrak{B}$. Then $\mu(\bigcup E_n)$ is the number of horizontal and vertical lines L for which $\bigcap E_n^c \cap L$ is countable. Note that if $E_n^c \cap L$ is countable, then $E_m \cap L$ is countable for $m \neq n$ since $E_m \subset E_n^c$. If $E_n^c \cap L$ is countable, then $\bigcap E_n^c \cap L$ is countable. On the other hand, if $\bigcap E_n^c \cap L$ is countable, then $E_n^c \cap L$ is countable for some n, for otherwise we have $(\bigcup E_n) \cap L$ being countable. It follows that $\mu(\bigcup E_n) = \sum \mu(E_n)$. Thus μ is a measure on $\mathfrak{B}$ and similarly, ν is a measure on $\mathfrak{B}$.

Define a bounded linear functional F on $L^1(\mu)$ by setting $F(f) = \int f\, d\nu$.

12 Measure and Outer Measure

12.1 Outer measure and measurability

1. Suppose $\bar{\mu}(E) = \mu^*(E) = 0$. For any set A, we have $\mu^*(A \cap E) = 0$ since $A \cap E \subset E$. Also, $A \cap E^c \subset E$ so $\mu^*(E) \geq \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Thus E is measurable. If $F \subset E$, then $\mu^*(F) = 0$ so F is measurable. Hence $\bar{\mu}$ is complete.

2. Suppose that $\langle E_i \rangle$ is a sequence of disjoint measurable sets and $E = \bigcup E_i$. For any set A, we have $\mu^*(A \cap E) \leq \sum \mu^*(A \cap E_i)$ by countable subadditivity. Now $\mu^*(A \cap (E_1 \cup E_2)) \geq \mu^*(A \cap (E_1 \cup E_2) \cap E_1) + \mu^*(A \cap (E_1 \cup E_2) \cap E_1^c) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$ by measurability of E_1. By induction we have $\mu^*(A \cap \bigcup_{i=1}^n E_i) \geq \sum_{i=1}^n \mu^*(A \cap E_i)$ for all n. Thus $\mu^*(A \cap E) \geq \mu^*(A \cap \bigcup_{i=1}^n E_i) \geq \sum_{i=1}^n \mu^*(A \cap E_i)$ for all n so $\mu^*(A \cap E) \geq \sum \mu^*(A \cap E_i)$. Hence $\mu^*(A \cap E) = \sum \mu^*(A \cap E_i)$.

12.2 The extension theorem

***3.** Let X be the set of rational numbers and $\mathfrak{A}$ be the algebra of finite unions of intervals of the form $(a,b]$ with $\mu(a,b]=\infty$ and $\mu(\emptyset)=0$. Note that the smallest σ-algebra containing $\mathfrak{A}$ will contain one-point sets. Let k be a positive number. Define $\mu_k(A)=k|A|$. Then μ_k is a measure on the smallest σ-algebra containing $\mathfrak{A}$ and extends μ.

4a. If A is the union of each of two finite disjoint collections $\{C_i\}$ and $\{D_j\}$ of sets in $\mathcal{C}$, then for each i, we have $\mu(C_i)=\sum_j \mu(C_i\cap D_j)$. Similarly, for each j, we have $\mu(D_j)=\sum_i \mu(C_i\cap D_j)$. Thus $\sum_i \mu(C_i)=\sum_i\sum_j \mu(C_i\cap D_j)=\sum_j\sum_i \mu(C_i\cap D_j)=\sum_j \mu(D_j)$.

4b. Defining $\mu(A)=\sum_{i=1}^n \mu(C_i)$ whenever A is the disjoint union of the sets $C_i\in\mathcal{C}$, we have a finitely additive set function on $\mathfrak{A}$. Thus $\mu(\bigcup C_i)\geq \mu(\bigcup_{i=1}^n C_i)=\sum_{i=1}^n \mu(C_i)$ for all n so $\mu(\bigcup C_i)\geq \sum \mu(C_i)$. Condition (ii) gives the reverse inequality $\mu(\bigcup C_i)\leq \sum\mu(C_i)$ so μ is countably additive.

(*) Proof of Proposition 9

5a. Let $\mathcal{C}$ be a semialgebra of sets and $\mathfrak{A}$ the smallest algebra of sets containing $\mathcal{C}$. The union of two finite unions of sets in $\mathcal{C}$ is still a finite union of sets in $\mathcal{C}$. Also, $(\bigcup_{i=1}^n C_i)^c=\bigcap_{i=1}^n C_i^c$ is a finite intersection of finite unions of sets in $\mathcal{C}$, which is then a finite union of sets in $\mathcal{C}$. Thus the collection of finite unions of sets in $\mathcal{C}$ is an algebra containing $\mathcal{C}$. Furthermore, if $\mathfrak{A}'$ is an algebra containing $\mathcal{C}$, then it contains all finite unions of sets in $\mathcal{C}$. Hence $\mathfrak{A}$ is comprised of sets of the form $A=\bigcup_{i=1}^n C_i$.

5b. Clearly $\mathcal{C}_\sigma\subset\mathfrak{A}_\sigma$. On the other hand, since each set in $\mathfrak{A}$ is a finite union of sets in $\mathcal{C}$, we have $\mathfrak{A}_\sigma\subset\mathcal{C}_\sigma$. Hence $\mathfrak{A}_\sigma=\mathcal{C}_\sigma$.

6a. Let $\mathfrak{A}$ be a collection of sets which is closed under finite unions and finite intersections. Countable unions of sets in $\mathfrak{A}_\sigma$ are still countable unions of sets in $\mathfrak{A}$. Also, if $A_i,B_j\in\mathfrak{A}$, then $(\bigcup_i A_i)\cap(\bigcup_j B_j)=\bigcup_{i,j}(A_i\cap B_j)$, which is a countable union of sets in $\mathfrak{A}$. Hence $\mathfrak{A}_\sigma$ is closed under countable unions and finite intersections.

6b. If $\bigcap B_i\in\mathfrak{A}_{\sigma\delta}$, then $\bigcap B_i=\bigcap_n\bigcap_{i=1}^n B_i$ where $\bigcap_{i=1}^n B_i\in\mathfrak{A}_\sigma$ and $\bigcap_{i=1}^n B_i\supset\bigcap_{i=1}^{n+1} B_i$ for each n. Hence each set in $\mathfrak{A}_{\sigma\delta}$ is the intersection of a decreasing sequence of sets in $\mathfrak{A}_\sigma$.

7. Let μ be a finite measure on an algebra $\mathfrak{A}$, and μ^* the induced outer measure. Suppose that for each $\varepsilon>0$ there is a set $A\in\mathfrak{A}_{\sigma\delta}$, $A\subset E$, such that $\mu^*(E\setminus A)<\varepsilon$. Note that A is measurable. Thus for any set B, we have $\mu^*(B)=\mu^*(B\cap A)+\mu^*(B\cap A^c)\geq\mu^*(B\cap E)-\mu^*(B\cap(E\setminus A))+\mu^*(B\cap E^c)>\mu^*(B\cap E)+\mu^*(B\cap E^c)-\varepsilon$. Thus $\mu^*(B)\geq\mu^*(B\cap E)+\mu^*(B\cap E^c)$ and E is measurable.

Conversely, suppose E is measurable. Given $\varepsilon>0$, there is a set $B\in\mathfrak{A}_\sigma$ such that $E^c\subset B$ and $\mu^*(B)\leq\mu^*(E^c)+\varepsilon$. Let $A=B^c$. Then $A\in\mathfrak{A}_\delta$ and $A\subset E$. Furthermore, $\mu^*(E\setminus A)=\mu^*(E\cap B)=\mu^*(B)-\mu^*(B\cap E^c)=\mu^*(B)-\mu^*(E^c)\leq\varepsilon$.

8a. If we start with an outer measure μ^* on X and form the induced measure $\bar\mu$ on the μ^*-measurable sets, we can use $\bar\mu$ to induce an outer measure μ^+. For each set E, we have $\mu^*(E)\leq\sum\mu^*(A_i)$ for any sequence of μ^*-measurable sets A_i with $E\subset\bigcup A_i$. Taking the infimum over all such sequences, we have $\mu^*(E)\leq\inf\sum\mu^*(A_i)=\inf\sum\bar\mu(A_i)=\mu^+(E)$.

8b. Suppose there is a μ^*-measurable set $A\supset E$ with $\mu^*(A)=\mu^*(E)$. Then $\mu^+(E)\geq\bar\mu(A)=\mu^*(A)=\mu^*(E)$. Thus by part (a), we have $\mu^+(E)=\mu^*(E)$.

Conversely, suppose $\mu^+(E)=\mu^*(E)$. For each n, there is a μ^*-measurable set (a countable union of μ^*-measurable sets) A_n with $E\subset A_n$ and $\mu^*(A_n)=\bar\mu(A_n)\leq\mu^+(E)+1/n=\mu^*(E)+1/n$. Let $A=\bigcap A_n$. Then A is μ^*-measurable, $E\subset A$ and $\mu^*(A)\leq\mu^*(E)+1/n$ for all n so $\mu^*(A)=\mu^*(E)$.

8c. If $\mu^+(E)=\mu^*(E)$ for each set E, then by part (b), for each set E, there is a μ^*-measurable set A with $A\supset E$ and $\mu^*(A)=\mu^*(E)$. In particular, $\mu^*(A)\leq\mu^*(E)+\varepsilon$ so μ^* is regular.

Conversely, if μ^* is regular, then for each set E and each n, there is a μ^*-measurable set A_n with $A_n\supset E$ and $\mu^*(A_n)\leq\mu^*(E)+1/n$. Let $A=\bigcap A_n$. Then A is μ^*-measurable, $A\supset E$ and $\mu^*(A)=\mu^*(E)$. By part (b), $\mu^+(E)=\mu^*(E)$ for each set E.

8d. If μ^* is regular, then $\mu^+(E)=\mu^*(E)$ for every E by part (c). In particular, μ^* is induced by the measure $\bar\mu$ on the σ-algebra of μ^*-measurable sets.

Conversely, suppose μ^* is induced by a measure μ on an algebra $\mathfrak{A}$. For each set E and any $\varepsilon>0$, there is a sequence $\langle A_i\rangle$ of sets in $\mathfrak{A}$ with $E\subset\bigcup A_i$ and $\mu^*(\bigcup A_i)\leq\mu^*(E)+\varepsilon$. Each A_i is μ^*-measurable so $\bigcup A_i$ is μ^*-measurable. Hence μ^* is regular.

8e. Let X be a set consisting of two points a and b. Define $\mu^*(\emptyset) = \mu^*(\{a\}) = 0$ and $\mu^*(\{b\}) = \mu^*(X) = 1$. Then μ^* is an outer measure on X. The set X is the only μ^*-measurable set containing $\{a\}$ and $\mu^*(X) = 1 > \mu^*(\{a\}) + 1/2$. Hence μ^* is not regular.

9a. Let μ^* be a regular outer measure. The measure $\bar{\mu}$ induced by μ^* is complete by Q1. Let E be locally $\bar{\mu}$-measurable. Then $E \cap B$ is μ^*-measurable for any μ^*-measurable set B with $\mu^*(B) = \bar{\mu}(B) < \infty$. We may assume $\mu^*(E) < \infty$. Since μ^* is regular, it is induced by a measure on an algebra $\mathfrak{A}$ so there is a set $B \in \mathfrak{A}_\sigma$ with $E \subset B$ and $\mu^*(B) \leq \mu^*(E) + 1 < \infty$. Thus $E = E \cap B$ is μ^*-measurable.

***9b.**

10. Let μ be a measure on an algebra $\mathfrak{A}$ and $\bar{\mu}$ the extension of it given by the Carathéodory process. Let E be measurable with respect to $\bar{\mu}$ and $\bar{\mu}(E) < \infty$. Given $\varepsilon > 0$, there is a countable collection $\{A_n\}$ of sets in $\mathfrak{A}$ such that $E \subset \bigcup A_n$ and $\sum \mu(A_n) \leq \mu^*(E) + \varepsilon/2$. There exists N such that $\sum_{n=N+1}^{\infty} \mu(A_n) < \varepsilon/2$. Let $A = \bigcup_{n=1}^{N} \mu(A_n)$. Then $A \in \mathfrak{A}$ and $\bar{\mu}(A \Delta E) = \bar{\mu}(A \setminus E) + \bar{\mu}(E \setminus A) \leq \sum \mu(A_n) - \mu^*(E) + \sum_{n=N+1}^{\infty} \mu(A_n) < \varepsilon$.

11a. Let μ be a measure on $\mathfrak{A}$ and $\bar{\mu}$ its extension. Let $\varepsilon > 0$. If f is $\bar{\mu}$-integrable, then there is a simple function $\sum_{i=1}^{n} c_i \chi_{E_i}$ where each E_i is μ^*-measurable and $\int |f - \sum_{i=1}^{n} c_i \chi_{E_i}|\, d\bar{\mu} < \varepsilon/2$. The simple function may be taken to vanish outside a set of finite measure so we may assume each E_i has finite μ^*-measure. For each E_i, there exists $A_i \in \mathfrak{A}$ such that $\bar{\mu}(A_i \Delta E_i) < \varepsilon/2n$. Consider the $\mathfrak{A}$-simple function $\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$. Then $\int |f - \varphi|\, d\bar{\mu} < \varepsilon$.

***11b.**

12.3 The Lebesgue-Stieltjes integral

12. Let F be a monotone increasing function continuous on the right. Suppose $(a, b] \subset \bigcup_{i=1}^{\infty}(a_i, b_i]$. Let $\varepsilon > 0$. There exists $\eta_i > 0$ such that $F(b_i + \eta_i) < F(b_i) + \varepsilon 2^{-i}$. There exists $\delta > 0$ such that $F(a + \delta) < F(a) + \varepsilon$. Then the open intervals $(a_i, b_i + \eta_i)$ cover the closed interval $[a + \delta, b]$. By the Heine-Borel Theorem, a finite subcollection of the open intervals covers $[a + \delta, b]$. Pick an open interval $(a_1, b_1 + \eta_1)$ containing $a + \delta$. If $b_1 + \eta_1 \leq b$, then there is an interval $(a_2, b_2 + \eta_2)$ containing $b_1 + \eta_1$. Continuing in this fashion, we obtain a sequence $(a_1, b_1 + \eta_1), \ldots, (a_k, b_k + \eta_k)$ from the finite subcollection such that $a_i < b_{i-1} + \eta_{i-1} < b_i + eta_i$. The process must terminate with some interval $(a_k, b_k + \eta_k$ but it terminates only if $b \in (a_k, b_k + \eta_k)$. Thus $\sum_{i=1}^{\infty} F(b_i) - F(a_i) \geq (F(b_k + \eta_k) - \varepsilon 2^{-i} - F(a_k)) + (F(b_{k-1} + \eta_{k-1}) - \varepsilon 2^{-i+1} - F(a_{k-1})) + \cdots + (F(b_1 + \eta_1) - \varepsilon 2^{-1} - F(a_1)) > F(b_k + \eta_k) - F(a_1) > F(b) - F(a + \delta)$. Now $F(b) - F(a) = (F(b) - F(a + \delta)) + (F(a + \delta) - F(a)) < \sum_{i=1}^{\infty} F(b_i) - F(a_i) + \varepsilon$. Hence $F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i)$. If $(a, b]$ is unbounded, we may approximate it by a bounded interval by considering limits.

(*) Proof of Lemma 11

13. Let F be a monotone increasing function and define $F^*(x) = \lim_{y \to x^+} F(y)$. Clearly F^* is monotone increasing since F is. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $F(y) - F^*(x) < \varepsilon$ whenever $y \in (x, x + \delta)$. Now when $z \in (y, y + \delta')$ where $0 < \delta' < x + \delta - y$, we have $F(z) - F^*(x) < \varepsilon$. Thus $F^*(y) - F^*(x) \leq F(z) - F^*(x) < \varepsilon$. Hence F^* is continuous on the right. If F is continuous on the right at x, then $F^*(x) = \lim_{y \to x^+} F(y) = F(x)$. Thus since F^* is continuous on the right, we have $(F^*)^* = F^*$. Suppose F and G are monotone increasing functions which agree wherever they are both continuous. Then F^* and G^* agree wherever both F and G are continuous. Since they are monotone, their points of continuity are dense. It follows that $F^* = G^*$ since F^* and G^* are continuous on the right.

14a. Let F be a bounded function of bounded variation. Then $F = G - H$ where G and H are monotone increasing functions. There are unique Baire measures μ_G and μ_H such that $\mu_G(a, b] = G^*(b) - G^*(a)$ and $\mu_H = H^*(b) - H^*(a)$. Let $\nu = \mu_G - \mu_H$. Then ν is a signed Baire measure and $\nu(a, b] = \mu_G(a, b] - \mu_H(a, b] = (G^*(b) - G^*(a)) - (H^*(b) - H^*(a)) = (G(b+) - G(a+)) - (H(b+) - H(a+)) = F(b+) - F(a+)$.

14b. The signed Baire measure in part (a) has a Jordan decomposition $\nu = \nu^+ - \nu^-$. Now $F = G - H$ where G corresponds to the positive variation of F and H corresponds to the negative variation of F. Then G and H give rise to Baire measures μ_G and μ_H with $\nu = \mu_G - \mu_H$. By the uniqueness of the Jordan decomposition, $\nu^+ = \mu_G$ and $\nu^- = \mu_H$ so ν^+ and ν^- correspond to the positive and negative variations of F.

14c. If F is of bounded variation, define $\int \varphi\, dF = \int \varphi\, d\nu = \int \varphi\, d\nu^+ - \int \varphi\, d\nu^-$ where ν is the signed

Baire measure in part (a).

14d. Suppose $|\varphi| \leq M$ and the total variation of F is T. Then $|\int \varphi\, dF| = |\int \varphi\, d\nu| \leq MT$ since ν^+ and ν^- correspond to the positive and negative variations of F and $T = P + N$.

15a. Let F be the cumulative distribution function of the Baire measure ν and assume that F is continuous. Suppose the interval (a, b) is in the range of F. Since F is monotone, $F^{-1}[(a, b)] = (c, d)$ where $F(c) = a$ and $F(d) = b$. Thus $m(a, b) = b - a = F(d) - F(c) = \nu[F^{-1}[(a, b)]]$. Since the class of Borel sets is the smallest σ-algebra containing the algebra of open intervals, the uniqueness of the extension in Theorem 8 gives the result for general Borel sets.

15b. For a discontinuous cumulative distribution function F, note that the set C of points at which F is continuous is a G_δ and thus a Borel set. Similarly, the set D of points at which F is discontinuous is a Borel set. Furthermore, D is at most countable since F is monotone. Thus $F^{-1}[D]$ is also at most countable. Now for a Borel set E, we have $m(E) = m(E \cap C) + m(E \cap D) = m(E \cap C) = \nu[F^{-1}[E \cap C]] = \nu[F^{-1}[E \cap C]] + \nu[F^{-1}[E \cap D]] = \nu[F^{-1}[E]]$.

16. Let F be a continuous increasing function on $[a, b]$ with $F(a) = c$, $F(b) = d$ and let φ be a nonnegative Borel measurable function on $[c, d]$. Now F is the cumulative distribution function of a finite Baire measure ν. First assume that φ is a characteristic function χ_E of a Borel set. Then $\int_a^b \varphi(F(x))\, dF(x) = \int_a^b \chi_E(F(x))\, dF(x) = \int_a^b \chi_E(F(x))\, d\nu = \nu[F^{-1}[E]] = m(E) = \int_c^d \chi_E(y)\, dy = \int_c^d \varphi(y)\, dy$. Since simple functions are finite linear combinations of characteristic functions of Borel sets, the result follows from the linearity of the integrals. For a general φ, there is an increasing sequence of nonnegative simple functions converging pointwise to φ and the result follows from the Monotone Convergence Theorem.

***17a.** Suppose a measure μ is absolutely continuous with respect to Lebesgue measure and let F be its cumulative distribution function. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever $m(E) < \delta$. For any finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$, we have $\mu(\bigcup_{i=1}^n (x_i, x_i')) < \varepsilon$. i.e. $\sum_{i=1}^n |F(x_i') - F(x_i)| < \varepsilon$. Thus F is absolutely continuous.

Conversely, suppose F is absolutely continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon$ for any finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$. Let E be a measurable set with $m(E) < \delta/2$. There is a sequence of open intervals $\langle I_n \rangle$ such that $E \subset \bigcup I_n$ and $\sum m(I_n) < \delta$. We may assume the intervals are nonoverlapping. Now $\mu(E) \leq \mu(\bigcup I_n) = \sum \mu(I_n) = \sum (F(b_n) - F(a_n))$ where $I_n = (a_n, b_n)$. Since $\sum_{n=1}^k (b_n - a_n) = \sum_{n=1}^k m(I_n) < \delta$ for each n, we have $\sum_{n=1}^k (F(b_n) - F(a_n)) < \varepsilon$ for each n. Thus $\mu(E) \leq \sum (F(b_n) - F(a_n)) < \varepsilon$ and $\mu << m$.

17b. If μ is absolutely continuous with respect to Lebesgue measure, then its cumulative distribution function F is absolutely continuous so by Theorem 5.14, we have $F(x) = \int_a^x F'(t)\ dt + F(a)$. i.e. $\mu(a, x] = \int_a^x F'(t)\ dt$. It follows that $\mu(E) = \int_E F'\ dt$ for any measurable set E. By uniqueness of the Radon-Nikodym derivative, $[\frac{d\mu}{dm}] = F'$ a.e.

17c. If F is absolutely continuous, then the Baire measure for which $\mu(a, b] = F(b) - F(a)$ is absolutely continuous with respect to Lebesgue measure and F' is its Radon-Nikodym derivative. By Q11.34a, we have $\int f\ dF = \int f\ d\mu = \int f F'\ dx$.

***18. Riemann's Convergence Criterion**: Let f be a nonnegative monotone decreasing function on $(0, \infty)$, g a nonnegative increasing function on $(0, \infty)$ and $\langle a_n \rangle$ a nonnegative sequence. Suppose that for each $x \in (0, \infty)$ the number of n such that $a_n \geq f(x)$ is at most $g(x)$.

12.4 Product measures

19. Let $X = Y$ be the set of positive integers, $\mathfrak{A} = \mathfrak{B} = \mathcal{P}(X)$, and let $\nu = \mu$ be the counting measure.

Fubini's Theorem: Let f be a function on $X \times Y$ such that $\sum_{x,y} |f(x, y)| < \infty$.

(i) For (almost) all x, the function $f_x(y) = f(x, y)$ satisfies $\sum_y |f_x(y)| < \infty$.

(i') For (almost) all y, the function $f^y(x) = f(x, y)$ satisfies $\sum_x |f^y(x)| < \infty$.

(ii) $\sum_x \sum_y f(x, y) < \infty$

(ii') $\sum_y \sum_x f(x, y) < \infty$

(iii) $\sum_x \sum_y f(x, y) = \sum_{x,y} f(x, y) = \sum_y \sum_x f(x, y)$

Tonelli's Theorem: Let f be a nonnegative function on $X \times Y$.

Parts (i), (i'), (ii) and (ii') are trivial.

(iii) $\sum_x \sum_y f(x,y) = \sum_{x,y} f(x,y) = \sum_y \sum_x f(x,y)$

20. Let $(X, \mathfrak{B}, \mu)$ be any σ-finite measure space and Y the set of positive integers with ν the counting measure. Let f be a nonnegative measurable function on $X \times Y$. For each $n \in Y$, the function $f_n(x) = f(x,n)$ is a nonnegative measurable function on X. Both Theorem 20 and Corollary 11.14 give the result that $\int_X \sum_n f(x,n) = \sum_n \int_X f(x,n)$. Note that Corollary 11.14 is valid even if μ is not σ-finite so the Tonelli Theorem is true without σ-finiteness if $(Y, \mathfrak{B}, \nu)$ is this special measure space.

21. Let $X = Y = [0,1]$ and let $\mu = \nu$ be the Lebesgue measure. Note that $X \times Y$ satisfies the second axiom of countability and has a countable basis consisting of measurable rectangles. Thus each open set in $X \times Y$ is a countable union of measurable rectangles and is itself measurable. Since each open set is measurable and the collection of Borel sets is the smallest σ-algebra containing all the open sets, each Borel set in $X \times Y$ is measurable.

22. Let h and g be integrable functions on X and Y, and define $f(x,y) = h(x)g(y)$. If $h = \chi_A$ and $g = \chi_B$ where $A \subset X$ and $B \subset Y$ are measurable sets, then $f = \chi_{A\times B}$ where $A \times B$ is a measurable rectangle. Thus f is integrable on $X \times Y$ and $\int_{X\times Y} f \; d(\mu\times\nu) = (\mu\times\nu)(A\times B) = \mu(A)\nu(B) = \int_X h \; d\mu \int_Y g \; d\nu$. It follows that the result holds for simple functions and thus nonnegative integrable functions. For general integrable functions h and g, note that $f^+ = h^+g^+ + h^-g^-$ and $f^- = h^+g^- + h^-g^+$. Thus f is integrable on $X\times Y$ and $\int_{X\times Y} f \; d(\mu\times\nu) = \int_{X\times Y} f^+ \; d(\mu\times\nu) - \int_{X\times Y} f^- \; d(\mu\times\nu) = \int_X h^+ \; d\mu \int_Y g^+ \; d\nu + \int_X h^- \; d\mu \int_Y g^- \; d\nu - \int_X h^+ \; d\mu \int_Y g^- \; d\nu - \int_X h^- \; d\mu \int_Y g^+ \; d\nu = \int_X h^+ \; d\mu \int_Y g \; d\nu - \int_X h^- \; d\mu \int_Y g \; d\nu = \int_X h \; d\mu \int_Y g \; d\nu$.

23. Suppose that instead of assuming μ and ν to be σ-finite, we merely assume that $\{\langle x,y\rangle : f(x,y) \neq 0\}$ is a set of σ-finite measure. Then there is still an increasing sequence of simple functions each vanishing outside a set of finite measure and converging pointwise to f. Thus the proof of Tonelli's Theorem is still valid and the theorem is still true.

24. Let $X = Y$ be the positive integers and $\mu = \nu$ be the counting measure. Let

$$f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y, \\ -2 + 2^{-x} & \text{if } x = y+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_x \sum_y f(x,y) = f(1,1) = \frac{3}{2}$ but $\sum_y \sum_x f(x,y) = \sum_y (2-2^{-y}) + (-2+2^{-y-1}) = \sum_y 2^{-y-1} - 2^{-y} = -\sum_y 2^{-y-1} = -\frac{1}{2}$.

(*) We cannot remove the hypothesis that f be nonnegative from Tonelli's Theorem or that f be integrable from Fubini's Theorem.

25. Let $X = Y$ be the interval $[0,1]$, with $\mathfrak{A} = \mathfrak{B}$ the class of Borel sets. Let μ be Lebesgue measure and ν the counting measure. Let $\Delta = \{\langle x,y\rangle \in X\times Y : x = y\}$ be the diagonal. Let $I_{j,n} = [\frac{j-1}{n}, \frac{j}{n}]$ for $j = 1, \ldots, n$ and let $I_n = \bigcup_{j=1}^n I_{j,n}$. Note that $\Delta = \bigcap_n I_n$. Hence Δ is measurable (and in fact an $\mathfrak{R}_{\sigma\delta}$). Now $\int_X [\int_Y \chi_\Delta \; d\nu] \; d\mu = \int_X \nu(\{y : y = x\}) \; d\mu = \int_X 1 \; d\mu = 1$ but $\int_Y [\int_X \chi_\Delta \; d\mu] \; d\nu = \int_Y \mu(\{x : x = y\}) \; d\nu = \int_Y 0 \; d\nu = 0$. Let $\Delta \subset \bigcup (A_n \times B_n)$ where $A_n \in \mathfrak{A}$ and $B_n \in \mathfrak{B}$. Then some B_n must be infinite so that $(\mu\times\nu)(A_n\times B_n) = \infty$. Thus $\sum (\mu\times\nu)(A_n \times B_n) = \infty$. By definition of outer measure, it follows that $\int_{X\times Y} \chi_\Delta \; d(\mu\times\nu) = (\mu\times\nu)(\Delta) = \infty$.

(*) We cannot remove the hypothesis that f be integrable from Fubini's Theorem or that μ and ν be σ-finite from Tonelli's Theorem.

26. Let $X = Y$ be the set of ordinals less than or equal to the first uncountable ordinal Ω. Let $\mathfrak{A} = \mathfrak{B}$ be the σ-algebra consisting of all countable sets and their complements. Define $\mu = \nu$ by setting $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ otherwise. Define a subset S of $X \times Y$ by $S = \{\langle x,y\rangle : x < y\}$. Now $S_x = \{y : y > x\}$ and $S_y = \{x : x < y\}$ are measurable for each x and y. Let f be the characteristic function of S. Then $\int_Y [\int_X f \; d\mu(x)] \; d\nu(y) = \int_Y \mu(S_y) \; d\nu(Y) = \int_{\{\Omega\}} 1 \; d\nu(y) = 0$ and $\int_X [\int_Y f \; d\nu(y)] \; d\mu(x) = \int_X \nu(S_x) \; d\mu(x) = \int_X 1 \; d\mu(x) = 1$.

If we assume the continuum hypothesis, i.e. that X can be put in one-one correspondence with $[0,1]$, then we can take f to be a function on the unit square such that f_x and f^y are bounded and measurable for each x and y but such that the conclusions of the Fubini and Tonelli Theorems do not hold.

(*) The hypothesis that f be measurable with respect to the product measure cannot be omitted from the Fubini and Tonelli Theorems even if we assume the measurability of f^y and f_x and the integrability of $\int f(x,y)\,d\nu(y)$ and $\int f(x,y)\,d\mu(x)$.

***27.** Suppose $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ are two σ-finite measure spaces. Then the extension of the nonnegative set function $\lambda(A \times B) = \mu(A)\nu(B)$ on the semialgebra of measurable rectangles to $\mathfrak{A} \times \mathfrak{B}$ is σ-finite. It follows from the Carathéodory extension theorem that the product measure is the only measure on $\mathfrak{A} \times \mathfrak{B}$ which assigns the value $\mu(A)\nu(B)$ to each measurable rectangle $A \times B$.

28a. Suppose $E \in \mathfrak{A}\times\mathfrak{B}$. If μ and ν are σ-finite, then so is $\mu\times\nu$ so $E = \bigcup(E \cap F_i)$ where $(\mu\times\nu)(F_i) < \infty$ for each i. By Proposition 18, $(E \cap F_i)_x$ is measurable for almost all x so $E_x = \bigcup(E \cap F_i)_x \in \mathfrak{B}$ for almost all x.

28b. Suppose f is measurable with respect to $\mathfrak{A} \times \mathfrak{B}$. For any α, we have $E = \{\langle x, y\rangle : f(x, y) > \alpha\} \in \mathfrak{A}\times\mathfrak{B}$ so $E_x \in \mathfrak{B}$ for almost all x. Now $E_x = \{y : f(x, y) > \alpha\} = \{y : f_x(y) > \alpha\}$. Thus f_x is measurable with respect to $\mathfrak{B}$ for almost all x.

29a. Let $X = Y = \mathbb{R}$ and let $\mu = \nu$ be Lebesgue measure. Then $\mu \times \nu$ is two-dimensional Lebesgue measure on $X \times Y = \mathbb{R}^2$. For each measurable subset E of $\mathbb{R}$, let $\sigma(E) = \{\langle x, y\rangle : x - y \in E\}$. If E is an open set, then $\sigma(E)$ is open and thus measurable. If E is a G_δ with $E = \bigcap E_i$ where each E_i is open, then $\sigma(E) = \bigcap \sigma(E_i)$, which is measurable. If E is a set of measure zero, then $\sigma(E)$ is a set of measure zero and is thus measurable. A general measurable set E is thhe difference of a G_δ set A and a set B of measure zero and $\sigma(E) = \sigma(A) \setminus \sigma(B)$ so $\sigma(E)$ is measurable.

29b. Let f be a measurable function on $\mathbb{R}$ and define the function F by $F(x, y) = f(x-y)$. For any α, we have $\{\langle x, y\rangle : F(x, y) > \alpha\} = \{\langle x, y\rangle : f(x - y) > \alpha\} = \{\langle x, y\rangle : x - y \in f^{-1}[(\alpha,\infty)]\} = \sigma(f^{-1}[(\alpha,\infty)])$. The interval (α,∞) is a Borel set so $f^{-1}[(\alpha,\infty)]$ is measurable. It follows from part (a) that $\{\langle x, y\rangle : F(x, y) > \alpha\}$ is measurable. Hence F is a measurable function on $\mathbb{R}^2$.

29c. Let f and g be integrable functions on $\mathbb{R}$ and define the function φ by $\varphi(y) = f(x - y)g(y)$. By Tonelli's Theorem, $\int_{X\times Y} |f(x - y)g(y)|\,dxdy = \int_Y [\int_X |f(x - y)g(y)|\,dx]\,dy = \int_Y |g(y)|[\int_X |f(x - y)|\,dx]\,dy = \int_Y |g(y)|[\int_X |f(x)|\,dx]\,dy = \int |f| \int |g|$. Thus the function $|f(x - y)g(y)|$ is integrable. By Fubini's Theorem, for almost all x, the function φ is integrable. Let $h = \int_Y \varphi$. Then $\int |h| = \int_X |\int_Y \varphi| \leq \int_X \int_Y |\varphi| = \int_{X\times Y} |\varphi| \leq \int |f| \int |g|$.

30a. Let f and g be functions in $L^1(-\infty,\infty)$ and define $f * g$ to be the function defined by $\int f(y - x)g(x)\,dx$. If $f(y - x)g(x)$ is integrable at y, then define $F(x) = f(y - x)g(x)$ and $G(x) = F(y - x)$. Then G is integrable and $\int G(x)\,dx = \int F(x)\,dx$. i.e. $\int f(x)g(y - x)\,dx = \int f(y - x)g(x)\,dx$. Thus for $y \in \mathbb{R}$, $f(y - x)g(x)$ is integrable if and only if $f(x)g(y - x)$ is integrable and their integrals are the same in this case. When $f(y - x)g(x)$ is not integrable, $(f * g)(y) = (g * f)(y) = 0$ since the function is integrable for almost all y. Hence $f * g = g * f$.

30b. For $x, y \in \mathbb{R}$ such that $f(y - x - u)g(u)$ is integrable, define $F(u) = f(y - x - u)g(u)$. Consider $G(u) = F(u-x)$. Then G is integrable and $\int f(y-u)g(u-x)\,du = \int G(u)\,du = \int F(u)\,du = (f*g)(y-x)$. The function $H(u, x) = f(y-u)g(u-x)h(x)$ is integrable. Then $((f*g)*h)(y) = \int (f*g)(y-x)h(x)\,dx = \int[\int f(y - u)g(u - x)\,du]h(x)\,dx = \int f(y - u)g(u - x)h(x)\,dudx = \int[\int f(y - u)g(u - x)h(x)\,dx]\,du = \int f(y - u)(g * h)(u)\,du = (f * (g * h))(y)$.

30c. For $f \in L^1$, define $\hat{f}$ by $\hat{f}(s) = \int e^{ist} f(t)\,dt$. Then $|\hat{f}| \leq \int |f|$ so $\hat{f}$ is a bounded complex function. Furthermore, for any $s \in \mathbb{R}$, we have $\widehat{f * g}(s) = \int e^{ist}(f*g)(t)\,dt = \int e^{ist}[\int f(t-x)g(x)\,dx]\,dt = \int[\int f(t-x)e^{is(t-x)}g(x)e^{isx}\,dx]\,dt = \int[\int f(t - x)e^{is(t-x)}g(x)e^{isx}\,dt]\,dx = \int[\int f(t - x)e^{is(t-x)}\,dt]g(x)e^{isx}\,dx = \int[\int f(u)e^{isu}\,du]g(x)e^{isx}\,dx = \hat{f}(s)\hat{g}(s)$.

31. Let f be a nonnegative integrable function on $(-\infty,\infty)$ and let m_2 be two-dimensional Lebesgue measure on $\mathbb{R}^2$. There is an increasing sequence of nonnegative simple functions $\langle\varphi_n\rangle$ converging pointwise to f. Let $E_n = \{\langle x, y\rangle : 0 < y < \varphi_n(x)\}$. Then $\{\langle x, y\rangle : 0 < y < f(x)\} = \bigcup E_n$. Write $\varphi_n = \sum_{i=1}^{k_n} a_i^{(n)} \chi_{K_i^{(n)}}$. We may assume the $K_i^{(n)}$ are disjoint. Also, $a_i^{(n)} > 0$ for each i. Then $E_n = (F_1^{(n)} \times (0, a_1^{(n)})) \cup \cdots \cup (F_{k_n}^{(n)} \times (0, a_{k_n}^{(n)}))$. Hence E_n is measurable so $\{\langle x, y\rangle : 0 < y < f(x)\}$ is measurable. Furthermore, $m_2(E_n) \to m_2\{\langle x, y\rangle : 0 < y < f(x)\}$. On the other hand, $m_2(E_n) = \sum_{i=1}^{k_n} m(F_i^{(n)})m(0, a_i^{(n)}) = \sum_{i=1}^{n} a_i^{(n)} m(F_i^{(n)}) = \int \varphi_n\,dx$. By Monotone Convergence Theorem, $\int \varphi_n\,dx \to \int f\,dx$ so $m_2\{\langle x, y\rangle : 0 < y < f(x)\} = \int f\,dx$. Now $\{\langle x, y\rangle : 0 \leq y \leq f(x)\} = \{\langle x, y\rangle : 0 < y < f(x)\} \cup \{\langle x, 0\rangle : x \in \mathbb{R}\} \cup \{\langle x, f(x)\rangle : x \in \mathbb{R}\}$. Note that $m_2\{\langle x, 0\rangle : x \in \mathbb{R}\} = m(\mathbb{R})m\{0\} = \infty \cdot 0 = 0$. Also, $m_2\{\langle x, f(x)\rangle : x \in \mathbb{R}\} = 0$ by considering a covering of the set by

$A_n \cup [(n-1)\varepsilon, n\varepsilon)$ where $A_n = \{x \in \mathbb{R} : f(x) \in [(n-1)\varepsilon, n\varepsilon)\}$. Thus $\{\langle x, y\rangle : 0 \le y \le f(x)\}$ is measurable and $m_2\{\langle x, y\rangle : 0 \le y \le f(x)\} = \int f\ dx$.

Let $\varphi(t) = m\{x : f(x) \ge t\}$. Since $\{x : f(x) \ge t\} \subset \{x : f(x) \ge t'\}$ when $t \ge t'$, φ is a decreasing function. Now $\int_0^\infty \varphi(t)\ dt = \int_0^\infty [\int \chi_{\{x : f(x) \ge t\}}(x)\ dx]\ dt = \int [\int \chi_{\{x:f(x)\ge t\}}(x)\chi_{[0,\infty)}(t)\ dx]\ dt = \int \chi_{\{\langle x,y\rangle : 0 \le t \le f(x)\}} = m_2\{\langle x, y\rangle : 0 \le t \le f(x)\} = \int f\ dx$.

***32.** Let $\langle (X_i, \mathfrak{A}_i, \mu_i)\rangle_{i=1}^n$ be a finite collection of measure spaces. We can form the product measure $\mu_1 \times \cdots \times \mu_n$ on the space $X_1 \times \cdots \times X_n$ by starting with the semialgebra of rectangles of the form $R = A_1 \times \cdots \times A_n$ and $\mu(R) = \prod \mu_i(A_i)$, and using the Carathéodory extension procedure. If $E \subset X_1 \times \cdots \times X_n$ is covered by a sequence of measurable rectangles $R_k \subset X_1 \times \cdots \times X_n$, then $R_k = R_k^1 \cup R_k^2$ where $R_k^1 \subset X_1 \times \cdots \times X_p$ and $R_k^2 \subset X_{p+1} \times \cdots \times X_n$ are measurable rectangles. Then $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))^*(E) \le \sum (\mu_1 \times \cdots \times \mu_p)(R_k^1)(\mu_{p+1} \times \cdots \times \mu_n)(R_k^2) = \sum (\mu_1 \times \cdots \times \mu_n)(R_k)$ so $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))^*(E) \le (\mu_1 \times \cdots \times \mu_n)^*(E)$.

On the other hand, if $R = F \times G$ is a measurable rectangle in $(X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_n)$, then $F \subset \bigcup F_k$ and $G \subset \bigcup G_j$ where F_k is a measurable rectangle in $X_1 \times \cdots \times X_p$ and G_j is a measurable rectangle in $X_{p+1} \times \cdots \times X_n$. Now $R \subset \bigcup_{k,j}(F_k \times G_j)$ and $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))(R) + \varepsilon = (\mu_1 \times \cdots \times \mu_p)(F)(\mu_{p+1} \times \cdots \times \mu_n)(G) + \varepsilon > \sum_{k,j}(\mu_1 \times \cdots \times \mu_n)(F_k \times G_j)$. Now if $E \subset (X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_n)$, then $E \subset \bigcup R_i$ and $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))^*(E) + \varepsilon > \sum_i ((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))(R_i) > \sum_i \sum_{k,j} (\mu_1 \times \cdots \times \mu_n)(F_k \times G_j) - \varepsilon' \ge (\mu_1 \times \cdots \mu_n)^*(E) - \varepsilon$. Thus $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))^*(E) \ge (\mu_1 \times \cdots \times \mu_n)^*(E)$.

Hence $((\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n))^* = \mu_1 \times \cdots \times \mu_n{}^*$. It then follows that $(\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n) = \mu_1 \times \cdots \times \mu_n$.

***33.** Let $\{(X_\lambda, \mathfrak{A}_\lambda, \mu_\lambda)\}$ be a collection of probability measure spaces.

12.5 Integral operators

***34.** By Proposition 21, we have $||T|| \le M_{\alpha,\beta}$.

35. Let $k(x, y)$ be a measurable function on $X \times Y$ of absolute operator type (p, q) and $g \in L^q(\nu)$. Then $|k|$ is of operator type (p, q) so by Proposition 21, for almost all x, the integral $\int_Y |k(x, y)||g(y)|\ d\nu$ exists. Thus for almost all x, the integral $f(x) = \int_Y k(x, y)g(y)\ d\nu$ exists. Furthermore, the function f belongs to $L^p(\mu)$ an $||f||_p \le ||\int_Y |k(x, y)||g(y)|\ d\nu||_p \le ||\,|k|\,||_{p,q}||g||_q$.

(*) Proof of Corollary 22

36. Let g, h and k be functions on $\mathbb{R}^n$ of class L^q, L^p and L^r respectively, with $1/p + 1/q + 1/r = 2$. We may write $1/p = 1 - (1-\lambda)/r$ and $1/q = 1 - \lambda/r$ for some $0 \le \lambda \le 1$. Then k is of covariant type (p, q) so the integral $f(x) = \int_{\mathbb{R}^n} k(x, y)g(y)\ dy$ exists for almost all x and the function f belongs to L^{p^*} with $||f||_{p^*} \le ||k||_r||g||_q$ by Proposition 21. Now $\int\int_{\mathbb{R}^{2n}} |h(x)k(x-y)g(y)|\ dxdy = \int_{\mathbb{R}^n} |h(x)f(x)|\ dx \le ||h||_p||k||_r||g||_q$.

(*) Proof of Proposition 25

37. Let $g \in L^q$ and $k \in L^r$, with $1/q + 1/r > 1$. Let $1/p = 1/q + 1/r - 1$. We may write $1/q = 1 - \lambda/r$ and $1/p = 1 - (1-\lambda)/r$ where $0 \le \lambda \le 1$. Then k is of covariant type (p, q) so the function $f(x) = \int_{\mathbb{R}^n} k(x-y)g(y)\ dy$ is defined for almost all x and $||f||_p \le ||k||_r||g||_q$ by Proposition 21.

(*) Proof of Proposition 26

***38.** Let g, h and k be functions on $\mathbb{R}^n$ of class L^q, L^p and L^r with $1/p + 1/q + 1/r \le 2$.

12.6 Inner measure

39a. Suppose $\mu(X) < \infty$. By definition, $\mu_*(E) \ge \mu(X) - \mu^*(E^c)$. Conversely, since E and E^c are disjoint, we have $\mu_*(E) + \mu^*(E^c) \le \mu^*(E \cup E^c) = \mu(X)$. Hence $\mu_*(E) = \mu(X) - \mu^*(E^c)$.

39b. Suppose $\mathfrak{A}$ is a σ-algebra. If $A \in \mathfrak{A}$ and $E \subset A$, then $\mu^*(E) \le \mu(A)$. Thus $\mu^*(E) \le \inf\{\mu(A) : E \subset A, A \in \mathfrak{A}\}$. Conversely, for any sequence $\langle A_i\rangle$ of sets in $\mathfrak{A}$ covering E, we have $\bigcup A_i \in \mathfrak{A}$ so $\inf\{\mu(A) : E \subset A, A \in \mathfrak{A}\} \le \mu(\bigcup A_i) \le \sum \mu(A_i)$. Thus $\inf\{\mu(A) : E \subset A, A \in \mathfrak{A}\} \le \mu^*(E)$. Hence $\mu^*(E) = \inf\{\mu(A) : E \subset A, A \in \mathfrak{A}\}$.

If $A \in \mathfrak{A}$ and $A \subset E$, then $\mu(A) = \mu(A) - \mu^*(A \setminus E)$. Thus $\sup\{\mu(A) : A \subset E, A \in \mathfrak{A}\} \le \mu_*(E)$.

Conversely, if $A \in \mathfrak{A}$ and $\mu^*(A \setminus E) < \infty$, then for any $\varepsilon > 0$, there is a sequence $\langle A_i \rangle$ of sets in $\mathfrak{A}$ such that $A \setminus E \subset \bigcup A_i$ and $\sum \mu(A_i) < \mu^*(A \setminus E) + \varepsilon$. Let $B = \bigcup A_i$. Then $B \in \mathfrak{A}$ and $\mu(B) < \mu^*(A \setminus E) + \varepsilon$. Also, $A \setminus B \in \mathfrak{A}$ and $A \setminus B \subset E$. Thus $\mu(A) - \mu^*(A \setminus E) - \varepsilon = \mu(A) - \mu(B) = \mu(A \setminus B) \leq \sup\{\mu(A) : A \subset E, A \in \mathfrak{A}\}$. It follows that $\mu_*(E) \leq \sup\{\mu(A) : A \subset E, A \in \mathfrak{A}\}$. Hence $\mu_*(E) = \sup\{\mu(A) : A \subset E, A \in \mathfrak{A}\}$.

39c. Let μ be Lebesgue measure on $\mathbb{R}$. By part (c), we have $\mu_*(E) = \sup\{\mu(A) : A \subset E, A \text{ measurable}\}$. If A is measurable and $A \subset E$, then given $\varepsilon > 0$, there is a closed set $F \subset A$ with $\mu^*(A \setminus F) < \varepsilon$. Thus $\mu(A) = \mu(F) + \mu(A \setminus F) < \mu(F) + \varepsilon$. It follows that $\mu_*(E) \leq \sup\{\mu(F) : F \subset E, F \text{ closed}\}$. Conversely, if $F \subset E$ and F is closed, then F is measurable so $\mu(F) \leq \mu_*(E)$. Thus $\sup\{\mu(F) : F \subset E, F \text{ closed}\} \leq \mu_*(E)$. Hence $\mu_*(E) = \sup\{\mu(F) : F \subset E, F \text{ closed}\}$.

40. Let $\langle B_i \rangle$ be a sequence of disjoint sets in $\mathfrak{B}$. Then $\bar{\mu}(\bigcup B_i) = \mu^*(\bigcup B_i \cap E) + \mu_*(\bigcup B_i \cap E^c) = \sum \mu^*(B_i \cap E) + \sum \mu_*(B_i \cap E^c) = \sum \bar{\mu}(B_i)$. Also, $\underline{\mu}(\bigcup B_i) = \mu_*(\bigcup B_i \cap E) + \mu^*(\bigcup B_i \cap E^c) = \sum \mu_*(B_i \cap E) + \sum \mu^*(B_i \cap E^c) = \sum \underline{\mu}(B_i)$. Hence the measures $\bar{\mu}$ and $\underline{\mu}$ in Theorem 38 are countably additive on $\mathfrak{B}$.

41. Let μ be a measure on an algebra $\mathfrak{A}$, and let E be a μ^*-measurable set. If $B \in \mathfrak{B}$, then B is of the form $(A \cap E) \cup (A' \cap E^c)$ where $A, A' \in \mathfrak{A}$. Thus $\bar{\mu}(B) = \mu^*(A \cap E) + \mu_*(A' \cap E^c)$ and $\mu^*(B) = \mu^*(A \cap E) + \mu^*(A' \cap E^c)$. If $\mu_*(A' \cap E^c) = \infty$, then $\mu^*(A' \cap E^c) = \infty$ and $\bar{\mu}(B) = \mu^*(B)$. If $\mu_*(A' \cap E^c) < \infty$, then since $A' \cap E^c$ is μ^*-measurable, $\mu_*(A' \cap E^c) = \bar{\bar{\mu}}(A' \cap E^c) = \mu^*(A' \cap E^c)$ where $\bar{\bar{\mu}}$ is the measure induced by μ^*. Thus $\bar{\mu}(B) = \mu^*(B)$.

42a. Let G and H be two measurable kernels for E so $G \subset E$, $H \subset E$ and $\mu_*(E \setminus G) = 0 = \mu_*(E \setminus H)$. Then $\mu_*(G \setminus H) = 0 = \mu_*(H \setminus G)$ since $G \setminus H \subset E \setminus H$ and $H \setminus G \subset E \setminus G$. In particular, $G \Delta H$ is measurable and $\mu(G \Delta H) = 0$. Let G' and H' be two measurable covers for E so $G' \supset E$, $H' \supset E$ and $\mu_*(G' \setminus E) = 0 = \mu_*(H' \setminus E)$. Then $\mu_*(G' \setminus H') = 0 = \mu_*(H' \setminus G')$ since $G' \setminus H' \subset G' \setminus E$ and $H' \setminus G' \subset H' \setminus E$. In particular, $G' \Delta H'$ is measurable and $\mu(G' \Delta H') = 0$.

42b. Suppose E is a set of σ-finite outer measure. Then $E = \bigcup E_n$ where each E_n is μ^*-measurable and $\mu^*(E_n) < \infty$. Then $\mu_*(E_n) < \infty$ so there exist $G_n \in \mathfrak{A}_{\delta\sigma}$ such that $G_n \subset E_n$ and $\bar{\mu}(G_n) = \mu_*(E_n) = \mu^*(E_n)$. Let $G = \bigcup G_n$. Then G is measurable, $G \subset E$ and $\mu_*(E \setminus G) \leq \mu^*(E \setminus G) \leq \sum \mu^*(E_n \setminus H_n) = 0$. Hence $\mu_*(E \setminus G) = 0$ and G is a measurable kernel for E.

Also, there exist $H_n \in \mathfrak{A}_{\sigma\delta}$ such that $E_n \subset H_n$ and $\mu^*(H_n) = \mu^*(E_n)$. Let $H = \bigcup H_n$. Then H is measurable, $E \subset H$ and $\mu_*(H \setminus E) \leq \mu^*(H \setminus E) \leq \sum \mu^*(H_n \setminus E_n) = 0$. Hence $\mu_*(H \setminus E) = 0$ and H is a measurable cover for E.

43a. Let P be the nonmeasurable set in Section 3.4. Note that $m_*(P) \leq m^*(P) \leq 1$. By Q3.15, for any measurable set E with $E \subset P$, we have $m(E) = 0$. Hence $m_*(P) = \sup\{m(E) : E \subset P, E \text{ measurable}, m(E) < \infty\} = 0$.

43b. Let $E = [0,1] \setminus P$. Let $\langle I_n \rangle$ be a sequence of open intervals such that $E \subset \bigcup I_n$. We may assume that $I_n \subset [0,1]$ for all n. Then $[0,1] \setminus \bigcup I_n \subset P$ so $m([0,1] \setminus \bigcup I_n) = 0$. Thus $m(\bigcup I_n) = 1$. Hence $m^*(E) = 1$. Suppose $A \cap [0,1]$ is a measurable set. Note that $m^*(A \cap E) \leq m(A \cap [0,1])$ and $m^*(E \setminus A) \leq m([0,1] \setminus A)$. Thus $1 = m^*(E) = m^*(A \cap E) + m^*(E \setminus A) \leq m(A \cap [0,1]) + m([0,1] \setminus A) = 1$ so $m^*(A \cap E) = m(A \cap [0,1])$.

***43c.**

***44.** Let μ be a measure on an algebra $\mathfrak{A}$ and E a set with $\mu^*(E) < \infty$. Let β be a real number with $\mu_*(E) \leq \beta \leq \mu^*(E)$.

***45a.**

***45b.**

46a. Let $\mathfrak{A}$ be the algebra of finite unions of half-open intervals of $\mathbb{R}$ and let $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ for $A \neq \emptyset$. If $E \neq \emptyset$ and $\langle A_i \rangle$ is a sequence of sets in $\mathfrak{A}$ with $E \subset \bigcup A_i$, then $A_i \neq \emptyset$ for some i and $\mu(A_i) = \infty$. Thus $\sum \mu(A_i) = \infty$ so $\mu^*(E) = \infty$.

46b. If E contains no interval, then the only $A \in \mathfrak{A}$ with $m^*(A \setminus E) < \infty$ is $\emptyset$. Thus $\mu_*(E) = 0$. If E contains an interval I, then $\mu_*(E) \geq \mu(I) - \mu^*(I \setminus E) = \mu(I) = \infty$.

46c. Note that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, $\mu_*(\mathbb{R}) = \infty$, $\mu_*(\mathbb{Q}) = \mu_*(\mathbb{Q}^c) = 0$. Thus μ_* restricted to $\mathfrak{B}$ is not a measure and there is no smallest extension of μ to $\mathfrak{B}$.

46d. The counting measure is a measure on $\mathfrak{B}$ and counting measure restricted to $\mathfrak{A}$ equals μ. Hence the counting measure on $\mathfrak{B}$ is an extension of μ to $\mathfrak{B}$.

46e. Let E be an interval. Then $\mu_*(E) = \infty$ but $\mu_*(E \cap \mathbb{Q}) = 0$ and $\mu_*(E \cap \mathbb{Q}^c) = 0$. Hence Lemma 37 fails if we replace "sets in $\mathfrak{A}$" by "measurable sets".

47a. Let $X = \{a, b, c\}$ and set $\mu^*(X) = 2$, $\mu^*(\emptyset) = 0$ and $\mu^*(E) = 1$ if E is not X or $\emptyset$. Setting $\mu_*(E) = \mu^*(X) - \mu^*(E^c)$, we have $\mu_*(X) = 2$, $\mu_*(\emptyset) = 0$ and $\mu_*(E) = 1$ if E is not X or $\emptyset$.

47b. $\emptyset$ and X are the only measurable subsets of X.

47c. $\mu_*(E) = \mu^*(E)$ for all subsets E of X but all subsets except $\emptyset$ and X are nonmeasurable.

47d. By taking $E = \{a\}$ and $F = \{b\}$, we have $\mu_*(E) + \mu_*(F) = \mu_*(E) + \mu^*(F) = 2$ but $\mu_*(E \cup F) = \mu^*(E \cup F) = 1$. Hence the first and third inequalities of Theorem 35 fail.

(*) If μ^* is not a regular outer measure (i.e. it does not come from a measure on an algebra), then we do not get a reasonable theory of inner measure by setting $\mu_*(E) = \mu^*(X) - \mu^*(E^c)$.

48a. Let $X = \mathbb{R}^2$ and $\mathfrak{A}$ the algebra consisting of all disjoint unions of vertical intervals of the form $I = \{\langle x, y\rangle : a < y \le b\}$. Let $\mu(A)$ be the sum of the lengths of the intervals of which A is composed. Then μ is a measure on $\mathfrak{A}$. Let $E = \{\langle x, y\rangle : y = 0\}$. If $E \subset \bigcup A_n$ where $\langle A_n\rangle$ is a sequence in $\mathfrak{A}$, then some A_n must be an uncountable union of vertical intervals so $\sum \mu(A_n) = \infty$. Thus $\mu^*(E) = \infty$. If $A \in \mathfrak{A}$ with $\mu^*(A \setminus E) < \infty$, then $\mu^*(A \setminus E) = \mu(A)$ so $\mu_*(E) = 0$.

48b. Let $E' \subset E$ and let $A_n = \bigcup_{\{x:\langle x,0\rangle \in E'\}} \{\langle x, y\rangle : -1/n < y \le 1/n\}$. Then $E' = \bigcap A_n$ so E' is an $\mathfrak{A}_\delta$.

***48c.**

12.7 Extension by sets of measure zero

49. Let $\mathfrak{A}$ be a σ-algebra on X and $\mathfrak{M}$ a collection of subsets of X which is closed under countable unions and which has the property that each subset of a set in $\mathfrak{M}$ is in $\mathfrak{M}$. Consider $\mathfrak{B} = \{B : B = A\Delta M, A \in \mathfrak{A}, M \in \mathfrak{M}\}$. Clearly $\emptyset \in \mathfrak{B}$. If $B = A\Delta M \in \mathfrak{B}$, then $B^c = (A \setminus M)^c \cap (M \setminus A)^c = (A^c \cup M) \cap (M^c \cup A) = (A \cup M)^c \cup (A \cap M) = (A^c \setminus M) \cup (M \setminus A^c) = A^c \Delta M \in \mathfrak{B}$. Suppose $\langle B_i\rangle$ is a sequence in $\mathfrak{B}$ with $B_i = A_i \Delta M_i$. Then $\bigcup B_i = \bigcup (A_i \setminus M_i) \cup \bigcup (M_i \setminus A_i) = (\bigcup A_i \setminus \bigcup M_i) \cup (\bigcup M_i \setminus \bigcup A_i) = \bigcup A_i \Delta \bigcup M_i \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a σ-algebra.

***50.**

12.8 Carathéodory outer measure

51. Let (X, ρ) be a metric space and let μ^* be an outer measure on X with the property that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$. Let Γ be the set of functions φ of the form $\varphi(x) = \rho(x, E)$. Suppose A and B are separated by some $\varphi \in \Gamma$. Then there are numbers a and b with $a > b$, $\rho(x, E) > a$ on A and $\rho(x, E) < b$ on B. Thus $\rho(A, B) > 0$ so $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ and μ^* is a Carathéodory outer measure with respect to Γ. Now for a closed set F, we have $F = \{x : \rho(x, F) \le 0\}$, which is measurable since $\varphi(x) = \rho(x, F)$ is μ^*-measurable. Thus every closed set (and hence every Borel set) is measurable with respect to μ^*.

(*) Proof of Proposition 41

12.9 Hausdorff measures

52. Suppose $E \subset \bigcup E_n$. If $\varepsilon > 0$ and $\langle B_{i,n}\rangle_i$ is a sequence of balls covering E_n with radii $r_{i,n} < \varepsilon$, then $\langle B_{i,n}\rangle_{i,n}$ is a sequence of balls covering E so $\lambda_\alpha^{(\varepsilon)}(E) \le \sum_{i,n} r_{i,n}^\alpha = \sum_n \sum_i r_{i,n}^\alpha$. Thus $\lambda_\alpha^{(\varepsilon)}(E) \le \sum_n \lambda_\alpha^{(\varepsilon)}(E_n)$. Letting $\varepsilon \to 0$, we have $m_\alpha^*(E) \le \sum_n m_\alpha^*(E_n)$ so m_α^* is countably subadditive.

53a. If E is a Borel set and $\langle B_i\rangle$ is a sequence of balls covering E with radii $r_i < \varepsilon$, then $\langle B_i + y\rangle$ is a sequence of balls covering $E + y$ with radii r_i. Conversely, if $\langle B_i\rangle$ is a sequence of balls covering $E + y$ with radii $r_i < \varepsilon$, then $\langle B_i - y\rangle$ is a sequence of balls covering E with radii r_i. It follows that $\lambda_\alpha^{(\varepsilon)}(E + y) = \lambda_\alpha^{(\varepsilon)}(E)$. Letting $\varepsilon \to 0$, we have $m_\alpha(E + y) = m_\alpha(E)$.

53b. Since m_α is invariant under translations, it suffices to consider rotations about 0. Let T denote rotation about 0. If $\langle B_i\rangle$ is a sequence of balls covering E with radii $r_i < \varepsilon$, then $\langle T(B_i)\rangle$ is a sequence of balls covering $T(E)$ with radii r_i. It follows that $\lambda_\alpha^{(\varepsilon)}(E) = \lambda_\alpha^{(\varepsilon)}(T(E))$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have $m_\alpha(E) = m_\alpha(T(E))$.

***54.**

***55a.** Let E be a Borel subset of some metric space X. Suppose $m_\alpha(E)$ is finite for some α.

55b. Suppose $m_\alpha(E) > 0$ for some α. If $m_\beta(E) < \infty$ for some $\beta > \alpha$, then by part (a), $m_\alpha(E) = 0$. Contradiction. Hence $m_\beta(E) = \infty$ for all $\beta > \alpha$.

55c. Let $I = \inf\{\alpha : m_\alpha(E) = \infty\}$. If $\alpha > I$, then $m_\alpha(E) = \infty$ by part (a). Thus I is an upper bound for $\{\beta : m_\beta(E) = 0\}$. If $U < I$, then there exists β such that $U < \beta < I$ and $m_\beta(E) = 0$ by part (b). Hence $I = \sup\{\beta : m_\beta(E) = 0\}$.

***55d.** Let $\alpha = \log 2/\log 3$. Given $\varepsilon > 0$, choose n such that $3^{-n} < \varepsilon$. Then since the Cantor ternary set C can be covered by 2^n intervals of length 3^{-n}, we have $\lambda_\alpha^{(\varepsilon)}(C) \leq 2^n(3^{-n})^\alpha = 1$. Thus $m_\alpha^*(C) \leq 1$.

Conversely, given $\varepsilon > 0$, if $\langle I_n \rangle$ is any sequence of open intervals covering C and with lengths less than ε, then we may enlarge each interval slightly and use compactness of C to reduce to the case of a finite collection of closed intervals. We may further take each I to be the smallest interval containing some pair of intervals J, J' from the construction of C. If J, J' are the largest such intervals, then there is an interval $K \subset C^c$ between them. Now $l(I)^s \geq (l(J)+l(K)+l(J'))^s \geq (\frac{3}{2}(l(J)+l(J')))^s = 2(\frac{1}{2}(l(J))^s + \frac{1}{2}(l(J'))^s) \geq (l(J))^s + (l(J'))^s$. Proceed in this way until, after a finite number of steps, we reach a covering of C by equal intervals of length 3^{-j}. This must include all intervals of length 3^{-j} in the construction of C. It follows that $\sum l(I_n) \geq 1$. Thus $\lambda_\alpha^{(\varepsilon)}(C) \geq 1$ for all $\varepsilon > 0$ and $m_\alpha^*(C) \geq 1$.

Since $m_\alpha^*(C) = 1$, by parts (a) and (b), we have $m_\beta(C) = 0$ for $0 < \beta < \alpha$ and $m_\beta(C) = \infty$ for $\beta > \alpha$. Hence the Hausdorff dimension of C is $\alpha = \log 2/\log 3$.

13 Measure and Topology

13.1 Baire sets and Borel sets

1.

[Incomplete: Q5.20c, Q5.22c, d, Q7.42b, d, Q7.46a, Q8.17b, Q8.29, Q8.40b, c, Q9.38, Q9.49, Q9.50, Q10.47, Q10.48b, Q10.49g, Q11.5d, Q11.6c, Q11.8d, Q11.9d, e, Q11.37b, c, d, e, Q11.46b, Q11.47a, b, Q11.48, Q12.9b, Q12.11b, Q12.18, Q12.27, Q12.33, Q12.34, Q12.38, Q12.43c, Q12.44, Q12.45a, b, Q12.48c, Q12.50, Q12.54, Q12.55a]