SOLUTIONS FOR HOMEWORK ASSIGNMENT 2

A. Determine the type of each of the following differential equations and reduce them to canonical form:
   (a) Since $b^2 - 4ac = 0$, this equation is parabolic. To reduce to canonical form, we use the characteristic equation $(\xi_x)^2 + 2\xi_x\xi_y + (\xi_y)^2 = 0$, which says $\xi_x + \xi_y = 0$. If we take $\xi = x - y$, then we can choose $\eta$ however we want.
   For $\eta = y$, we have
   \[ u_x = u_\xi, \quad u_y = -u_\xi + u_\eta, \]
   \[ u_{xx} = u_{\xi\xi}, \quad u_{xy} = -u_{\xi\xi} + u_{\xi\eta}, \quad u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}, \]
   and the canonical form is $u_{\eta\eta} + 2u_\xi - u_\eta = 0$.
   
   For $\eta = x$, we have
   \[ u_x = u_\xi + u_\eta, \quad u_y = -u_\xi, \]
   \[ u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{xy} = -u_{\xi\xi} - u_{\xi\eta}, \quad u_{yy} = -u_{\xi\xi} - u_{\eta\eta}, \]
   and the canonical form is $u_{\eta\eta} + -u_\eta = 0$.

(b) Since $b^2 - 4ac = -16$, this equation is elliptic. In this case, we want
   \[ \xi_x \eta_x + 2(\xi_x \eta_y + \xi_y \eta_x) + 5\xi_y \eta_y = 0 \]
   and
   \[ \xi_x^2 + 2\xi_x \xi_y + 5\xi_y^2 = \eta_x^2 + 2\eta_x \eta_y + 5\eta_y^2. \]
   Now rearrange the second equation and then add $2i$ times the first equation to the rearranged second equation. We get
   \[ (\xi_x + i\eta_y)^2 + 2(\xi_x + i\eta_x)(\xi_y + i\eta_y) + 5(\xi_y + i\eta_y)^2 = 0, \]
   so
   \[ \frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = -1 \pm 2i. \]
   For simplicity, we choose the plus sign, multiply both sides by $\xi_y + i\eta_y$, and then equate real and imaginary parts. It follows that
   \[ \xi_x = \eta_x + 5\eta_y, \quad \xi_y = -\eta_x + \eta_y. \]
   Hence, we can choose $\eta_x$ and $\eta_y$ arbitrarily and then get $\xi_x$ and $\xi_y$. One choice is $\eta_x = \eta_y = 1$, which gives $\xi = 3x - y$ and $\eta = x + y$. For this choice, we have
   \[ u_x = 3u_\xi + u_\eta, \quad u_y = -u_\xi + u_\eta, \]
   \[ u_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta}, \quad u_{xy} = -3u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}, \]
   and the canonical form is $u_{\xi\xi} + u_{\eta\eta} + \frac{9}{8}u_\xi + \frac{3}{8}u_\eta + \frac{1}{8}u = 0$. 

(c) Since $b^2 - 4ac = 64$, this equation is hyperbolic. The characteristic equation is

$$3(\xi_x)^2 + 10\xi_x\xi_y + 3(\xi_y)^2 = 0,$$

so

$$\frac{\xi_x}{\xi_y} = -3, -\frac{1}{3}.$$ 

We take $\xi = 3x - y$ and $\eta = x - 3y$. In this case, we know that the canonical form is $u_{\xi\eta} = 0$.

B. The general solution is $f(\xi) + g(\eta)$, which is $f(3x - y) + g(x - 3y)$ in terms of the original variables.

C. First, make the change of variables $\tau = at$. Then the problem for $v(x, \tau) = u(x, t)$ is

$$v_{\tau\tau} = v_{xx} \text{ for } x \in \mathbb{R}, \tau > 0$$

$$v(x, 0) = 0, v_{\tau}(x, 0) = \frac{1}{a(1 + x^2)} \text{ for } x \in \mathbb{R}.$$ 

It follows from the d’Alembert solution that

$$v(x, \tau) = \frac{1}{2} 0 + \frac{1}{2} 0 + \frac{1}{2} \int_{x-\tau}^{x+\tau} \frac{1}{a(1 + \sigma^2)} d\sigma$$

$$= \frac{1}{2a} \left( \arctan(x + \tau) - \arctan(x - \tau) \right),$$ 

and hence

$$u(x, t) = \frac{1}{2a} \left( \arctan(x + at) - \arctan(x - at) \right).$$