PRACTICE FINAL EXAM

Directions: Work any four problems. Indicate clearly the problem that is not to be counted for credit. Each problem is worth 25 points. (The actual final exam will have more questions, including some covering topics not on this exam, but you will only need to work four of them.)

1. If \( f : [0, \infty) \to \mathbb{R} \) is continuous and \( \lim_{x \to \infty} f(x) \) exists, show that \( f \) is uniformly continuous.

2. Let \( f \) be continuous on \((-1, 1)\) and suppose it is known that \( f \) is differentiable at each \( x \neq 0 \) in \((-1, 1)\). If \( \lim_{x \to 0} f'(x) = 5 \), show that \( f \) is differentiable at 0.

3. Suppose \( f \) is integrable on \([a, b]\) and \( f(x) = g(x) \) except for a finite number of points in \([a, b]\). Is \( g \) integrable on \([a, b]\)? If so, prove it. If not, give a counterexample. (You may choose the interval for your counterexample.)

4. Prove that the sequence

\[
x_n = \frac{(n^2 + 20n + 35) \sin(n^3)}{n^2 + n + 1}
\]

has a convergent subsequence.

5. Let \( \alpha \in (0, 1] \) and let \( E \) be a non-empty subset of \( \mathbb{R} \). We say that a real-valued function \( f \) defined on \( E \) is Hölder continuous of order \( \alpha \) if there is a positive constant \( M \) such that

\[
|f(x) - f(y)| \leq M|x - y|^{\alpha}
\]

for all \( x \) and \( y \) in \( E \).

(a) Show that, if \( f \) is Hölder continuous of order \( \alpha \), then \( f \) is uniformly continuous on \( E \).

(b) Show that \( f(x) = x^{\alpha} \) is Hölder continuous of order \( \alpha \) on \([0, 1]\).

(c) Show that, if \( f \) and \( g \) are Hölder continuous of order \( \alpha \), then so is \( f + g \).