Directions: Work one problem from questions 1, 2, 3; one problem from questions 4, 5, 6, and two problems from the list of homework questions. You may use the result of any problem from this exam to work a later problem, and you may use the result of any homework problem (assigned or not) to solve any problem from the list of homework questions, as long as the problem you are working is later in the book than the one you want to use. If you have any questions about which problems are acceptable to use, ask me. In addition, the two homework solutions you turn in must be for problems from different chapters. Unlike the case for homework, you are NOT to consult with other people for help on this exam, but you may ask me for help. Also, do not use the symbols ⇒, ∃, ∀, ∴, or ∵; use English words instead.

Turn in your solutions by 12 noon, Friday, October 29, 2010.

1. A subset $S$ of $\mathbb{R}$ is called open if, for every $x \in S$, there is a positive number $\delta$ such that $(x-\delta, x+\delta) \subseteq S$. (In particular, $\mathbb{R}$ and the empty set are open.) Show that the union of any collection of open sets is open and that the intersection of finitely many open sets is open, but the intersection of infinitely many open sets does not have to be open.

2. A subset $S$ of $\mathbb{R}$ is called closed if $\mathbb{R} \setminus S$ is open. (See question 1 for the definition of open.) (In particular, $\mathbb{R}$ and the empty set are closed.) Show that the intersection of any collection of closed sets is closed and that the union of finitely many closed sets is closed, but the union of infinitely many closed sets does not have to be closed.

3. A subset $S$ of $\mathbb{R}$ is called sequentially compact if any convergent sequence in $S$ has its limit in $S$. Show that a subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

4. Show that any infinite set has a countable subset. (Several people used this fact without proof on a homework problem.)

5. Show that, for any set $S$, there is no one-to-one map from $S$ onto the set of all subsets of $S$. (If $S$ is finite, this can be done very easily, but I want a proof that works for any set, finite or infinite.)

6. A function $f$ is called piecewise monotone on an interval $[a, b]$ if there are finitely many numbers $x_1, \ldots, x_n$ with $a = x_1 < x_2 < \cdots < x_n = b$ such that $f$ is either increasing or decreasing on each interval $(x_i, x_{i+1})$. Show that any function which is piecewise monotone and satisfies the intermediate value property on an interval $[a, b]$ is continuous on that interval.
7. List of homework problems from the fourth edition: 1.4.8, 1.6.6, 2.3.10, 2.5.5, 3.1.9, 3.3.6, 3.3.11.

8. Some of the questions in the fourth edition aren’t in the third edition. They are included as separate parts of this problem.
   (a) List of homework problems from the third edition: 1.4.10, 2.3.11, 2.5.5, 3.3.5
   (b) (From Chapter 1) Prove that
   \[ 2^n > \frac{n(n-1)(n-2)}{6} \]
   for \( n \in \mathbb{N} \).
   (c) (From Chapter 3) Suppose that \( a \in \mathbb{R} \) and \( I \) is an open interval which contains \( a \). If \( f : I \to \mathbb{R} \) satisfies \( f(x) \to f(a) \), as \( x \to a \), and if there exist numbers \( M \) and \( m \) such that \( m < f(a) < M \), prove that there exist positive numbers \( \varepsilon \) and \( \delta \) such that
   \[ m + \varepsilon < f(x) < M - \varepsilon \]
   for all \( x \)’s which satisfy \( |x - a| < \delta \).
   (d) (From Chapter 3) Let \( a > 1 \). Assume that \( a^{p+q} = a^p a^q \) for all \( p, q \in \mathbb{Q} \), and that \( a^p < a^q \) for all \( p, q \in \mathbb{Q} \) which satisfy \( p < q \). (This is easy, but tedious, to prove using algebra, induction and the definitions \( a^0 = 1 \), \( a^{-n} = 1/a^n \), and \( a^{m/n} = \sqrt[n]{a^m} \) for \( n \in \mathbb{N} \) and \( m \in \mathbb{Z} \). The hard part is proving that \( \sqrt[n]{a^m} \) exists, and this requires the Completeness Axiom—see Appendix A.10 of the fourth edition.) For each \( x \in \mathbb{R} \), define
   \[ A(x) := \sup\{a^q : q \in \mathbb{Q} \text{ and } q \leq x\} \].
   i. Prove that \( A(x) \) exists and is finite for all \( x \in \mathbb{R} \), and that \( A(p) = a^p \) for all \( p \in \mathbb{Q} \). Thus \( a^x = A(x) \) extends the “power of \( a \)” function from \( \mathbb{Q} \) to \( \mathbb{R} \).
   ii. If \( x, y \in \mathbb{R} \) with \( x < y \), prove that \( a^x < a^y \).
   iii. Use Example 2.21 to prove that the function \( a^x \) is continuous on \( \mathbb{R} \).
   iv. Prove that \( a^{x+y} = a^x a^y \), \((a^x)^y = a^{xy}\), and \( a^{-x} = 1/a^x \) for all \( x, y \in \mathbb{R} \).
   v. For \( 0 < b < 1 \), define \( b^x = (1/b)^{-x} \). Prove that iii and iv hold for \( b \) in place of \( a \). State and prove an analogue of ii for \( b^x \) and \( b^y \) in place of \( a^x \) and \( a^y \).