Chapter 6

7. (a) To simplify the notation, I will write $\xi$ for the sequence $\langle \xi_\nu \rangle$ and $\eta$ for the sequence $\langle \eta_\nu \rangle$. We imitate the proof of Minkowski’s inequality for integrals. First, if $p = \infty$, we note that, for any positive integer $n$, $|\xi_n| \leq ||\xi||_\infty$ and $|\eta_n| \leq ||\eta||_\infty$. The triangle inequality for numbers gives $|\xi_n + \eta_n| \leq |\xi_n| + |\eta_n|$, and therefore $|\xi_n + \eta_n| \leq ||\xi||_\infty + ||\eta||_\infty$. Hence $||\xi + \eta||_\infty \leq ||\xi||_\infty + ||\eta||_\infty$.

If $1 \leq p < \infty$, we first note that if $\xi = 0$ or $\eta = 0$, then the inequality is immediate. If $\xi \neq 0$ and $\eta \neq 0$, we set $\alpha = ||\xi||$ and $\beta = ||\eta||$. We also define sequences $F = \langle F_\nu \rangle$ and $G = \langle G_\nu \rangle$ by $F_\nu = |\xi_\nu|/\alpha$ and $G_\nu = |\eta_\nu|/\beta$, and we set $\lambda = \alpha/(\alpha + \beta)$. Then

$$|\xi_\nu + \eta_\nu|^p \leq (|\xi_\nu| + |\eta_\nu|)^p = [\alpha F_\nu + \beta G_\nu]^p$$

$$= (\alpha + \beta)^p [\lambda F_\nu + (1 - \lambda) G_\nu]^p$$

$$\leq (\alpha + \beta)^p [\lambda F_\nu^p + (1 - \lambda) G_\nu^p]$$

by the convexity of $\varphi$ defined by $\varphi(t) = t^p$. We now sum on $\nu$ to see that

$$\sum |\xi_\nu + \eta_\nu|^p \leq (\alpha + \beta)^p [\lambda \sum F_\nu^p + (1 - \lambda) \sum G_\nu^p] = (\alpha + \beta)^p$$

because $\|F\| = \|G\| = 1$. Therefore

$$||\xi + \eta||_p^p \leq (\alpha + \beta)^p = (||\xi|| + ||\eta||)^p.$$

(b) Now we imitate the proof of Hölder’s inequality for integrals. For $p = 1$, we have

$$\sum |\xi_\nu \eta_\nu| \leq \sum |\xi_\nu| \sup |\eta_\nu| = ||\xi||_1 ||\eta||_\infty,$$

and a similar argument works for $p = \infty$.

For $1 < p < \infty$, we set $F = |\xi|$ and $G = |\eta|$. Then

$$\sum |\xi_\nu \eta_\nu| = \sum F_\nu G_\nu.$$

We now define $H$ by $H_\nu = G_\nu^{q-1}$. Then, for any $t \in (0, 1)$, we have

$$pt F_\nu G_\nu = pt F_\nu H_\nu^{p-1} \leq (H_\nu + tF_\nu)^p - H_\nu^p$$

by Lemma 102. Summing on $\nu$ gives

$$pt \sum F_\nu G_\nu \leq \sum (H_\nu + tF_\nu)^p = \sum H_\nu^p = ||H + tF||_p^p + ||H||_p^p.$$

From Minkowski’s inequality, we have $||H + tF||_p \leq ||H||_p + ||tF||_p$, and the properties of norms tell us that $||tF||_p = t ||F||_p$. Therefore

$$pt \sum F_\nu G_\nu \leq (||H||_p + p + t ||F||_p)^p - ||H||_p^p.$$
We now divide by \( pt \) and take the limit as \( t \to 0^+ \) to see that
\[
\sum F_\nu G_\nu \leq \| F \|_p \| H \|_p^{p-1} = \| \xi \|_p \| \eta \|_q.
\]

10. If \( f_n \to f \) in \( L^\infty \), then for any positive integer \( k \), there is a positive integer \( N(k) \) such that \( \| f_n - f \|_\infty \leq \frac{1}{k} \) if \( n \geq N(k) \). Hence, for each \( n \geq N(k) \), there is a set \( A(n, k) \) such that \( mA(n, k) = 0 \) and \( |f_n - f| \leq \frac{1}{k} \) on \( E \setminus A(n, k) \). Now set \( E = \bigcup A(n, k) \), where the union is taken over all positive integers \( k \) and, for each \( k \), over all integers \( n \geq N(k) \). Since the integers are countable, it follows that \( E \) is a countable union of sets of measure zero and hence \( mE = 0 \). On \( \tilde{E} \), we see that, for any positive integer \( k \) and any integer \( n \geq N(k) \), we have \( |f_n - f| < 1/k \). Hence \( f_n \to f \) uniformly on \( \tilde{E} \).

If \( f_n \to f \) uniformly on \( \tilde{E} \) with \( mE = 0 \), it follows that \( \| f_n - f \|_\infty \leq \sup_{\tilde{E}} |f_n - f|, \) so \( \| f_n - f \| \to 0 \).

16. Suppose first that \( f_n \to f \) in \( L^p \). Then set \( g_n = |f_n|^p \) and \( h_n = 2^p|f_n - f|^p + |f|^p \). Then \( g_n \to |f|^p \) and \( h_n \to 2^p|f|^p \) almost everywhere with \( |g_n| \leq h_n \) and \( \int g_n \to \int |f|^p \), so \( \| f_n \|_p \to \| f \|_p \).

Conversely, if \( \| f_n \| \to \| f \| \), we set \( g_n = (|f| + |f_n|)^p \) and \( h_n = 2^p(|f|^p + |f_n|^p) \). Then \( g_n \to 2^p|f|^p \) and \( h_n \to 4^p|f|^p \) almost everywhere. In addition, \( |g_n| \leq h_n \). Since
\[
\| f_n \| \to \| f \|,
\]
we conclude that
\[
\int h_n \to \int h.
\]

Now we use the generalized Lebesgue convergence theorem to conclude that
\[
\int g_n \to \int g.
\]

Finally, we set \( k_n = |f_n - f|^p \). Then \( k_n \to 0 \) almost everywhere and \( |k_n| \leq g_n \).

It follows from the generalized Lebesgue convergence theorem that \( \int k_n \to 0 \) and therefore \( f_n \to f \) in \( L^p \).

19. First, we compute
\[
\| T_\Delta f \|_p^p = \int |T_\Delta f|^p = \sum_{k=1}^n (\xi_{k+1} - \xi_k) \left( \frac{1}{\xi_{k+1} - \xi_k} \right)^p \left| \int_{\xi_k}^{\xi_{k+1}} f(t) \, dt \right|^p
\]
because \( T_\Delta f \) is a step function. Now we do some estimation on the interval \([a, b] \):
\[
\left| \int_a^b f \right| \leq \int_a^b |f| \leq \left( \int_a^b |f|^p \right)^{1/p} \left( \int_a^b 1^q \right)^{1/q},
\]
where \( q = p/(p-1) \). Then
\[
\left( \int_a^b 1^q \right)^{1/q} = (b-a)^{1/q} = (b-a)^{(p-1)/p},
\]
so
\[ \|T\Delta f\|_p^p \leq \sum (\xi_{k+1} - \xi_k)(\xi_{k+1} - \xi_k)^{-p}(\xi_{k+1} - \xi_k)^{p-1} \int_{\xi_k}^{\xi_{k+1}} |f|^p \]
\[ = \sum \int_{\xi_k}^{\xi_{k+1}} |f|^p = \int |f|^p = \|f\|^p. \]

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2. Let \((h_n)\) be a sequence of positive numbers with \(h_n \to 0\). We then define
\[ f_n(x) = \frac{1}{h_n} \int_x^{x+h_n} f(t) \, dt. \]
Our goal is to show that \(f_n \to f\) in \(L^1\). (To get the full limit as \(h \to 0^+\), we use problem 2.49f.) First, we suppose that \(f \geq 0\), and we observe that \(f_n \to f\) a.e. by Theorem 5.10. Since \(f \in L^1(\mathbb{R})\), we have that \(F(x) = \int_x^f f(t) \, dt\) is absolutely continuous and \(\lim_{x \to \infty} F(x) = f\). For any finite numbers \(a < b\), we then have
\[ \int_a^b f_n = \frac{1}{h_n} \int_a^b F(x + h_n) - f(x) \, dx = \frac{1}{h_n} \left( \int_a^{b+h_n} F - \int_a^{a+h_n} F \right). \]
Next, we set
\[ g(s) = \frac{1}{h_n} \int_s^{s+h_n} F. \]
Because \(f \geq 0\), it follows that \(F\) is increasing and hence so is \(g\). Now, given any \(\varepsilon > 0\), there is a constant \(B(\varepsilon)\) such that \(\int f \geq F(x) \geq \int f - \varepsilon\) if \(x \geq B\). For \(x \geq B\), we infer that
\[ \int f \geq g(x) \geq \int f - \frac{\varepsilon}{h_n}, \]
so \(\lim_{x \to \infty} g(x) = \int f\) and a similar argument shows that \(\lim_{x \to -\infty} g(x) = 0\). Sending \(b \to \infty\) and \(a \to -\infty\) gives \(f_n \in L^1\), with \(\int f_n = \int f\). We now set \(g_n = f_n + f\) and \(h_n = |f_n - f|\). We have \(|h_n| \leq g_n\) a.e., \(h_n \to 0\) a.e., and \(\int g_n \to 2\int f\). It follows from the generalized Lebesgue convergence theorem that \(\int h_n \to \int 0 = 0\). Therefore
\[ \lim \int |f_n - f| = 0. \]
To get the result for \(f\) not necessarily nonnegative, we note that \(f_n = (f^+)_n - (f^-)_n\) with
\[ (f^\pm)_n = \frac{1}{h_n} \int_x^{x+h_n} f^\pm(t) \, dt. \]