HOMEWORK #4 SOLUTIONS

Chapter 4

9. Let $E$ be a measurable set, set $g_n = f_n \chi_E$ and $h_n = f_n(1 - \chi_E)$. Note that $g_n$ and $h_n$ are integrable and nonnegative with $g_n \to f\chi_E$ and $h_n \to f(1 - \chi_E)$, so Fatou’s Lemma tells us that

$$\int_E f = \int F \chi_E \leq \lim \int g_n = \lim \int_E f_n$$

and

$$\int f - \int_E f = \int f(1 - \chi_E) \leq \lim \int h_n = \lim \left(\int f_n - \int_E f_n\right)$$

$$= \lim \int f_n - \lim \int_E f_n = \int f - \lim \int_E f_n.$$  

The second chain of inequalities (and equalities) tells us that

$$\int_E f \geq \lim \int_E f_n,$$

so

$$\lim \int_E f_n \leq \int_E f \leq \lim \int_E f_n,$$

and therefore

$$\int_E f = \lim \int_E f_n.$$  

10. (a) If $f$ is integrable, then $f^-$ and $f^+$ are integrable. Hence so is $f^+ + f^- = |f|$. In addition, $-|f| \leq f \leq |f|$, so Proposition 3.15 (iii) implies that

$$-\int |f| = \int (-|f|) \leq \int f \leq \int |f|,$$

so

$$\left| \int f \right| \leq \int |f|.$$  

The integrability of $|f|$ doesn’t imply the integrability of $f$ because $f$ may not be measurable. For example, let $E$ be a nonmeasurable subset of $[0, 1]$ and define $f$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in [0, 1] \setminus E, \\ 0 & \text{if } x \notin [0, 1]. \end{cases}$$
Then $|f|$ is just the characteristic function of $[0, 1]$, so it’s integrable, but $f$ isn’t measurable, so it isn’t integrable.

Of course, if $f$ is measurable, and $|f|$ is integrable, then $f^+$ and $f^-$ are integrable (because $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$), so $f$ is integrable.

(b) Here’s a quick proof that $f$, defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not Lebesgue integrable on $[0, \infty)$ but the improper Riemann integral on $[0, \infty)$ exists. First, for any positive integer $n$, we have

$$\left| \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} \, dx \right| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} \, dx \geq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi} \, dx$$

$$= \frac{1}{(n+1)\pi} \left| \int_{n\pi}^{(n+1)\pi} \sin x \, dx \right| = \frac{2}{(n+1)\pi}.$$

Now, for any positive integer $N$, it follows that $|f| \geq |f|\chi_{[1,N]}$, so

$$\int |f| \geq \int_1^N |f| \geq \sum_{n=1}^{N-1} \frac{2}{(n+1)\pi}.$$  

Since $\sum 2/(n+1)$ is a divergent series, it follows that $\int |f| = \infty$, so $f$ is not integrable on $[0, \infty)$. On the other hand, an integration by parts gives

$$\int_1^b \frac{\sin x}{x} \, dx = \left. \frac{\cos x}{x} \right|_1^b - \int_1^b \frac{\cos x}{x^2} \, dx.$$  

Because $\cos x$ is bounded, it follows that $\lim_{b \to \infty} (\cos b)/b = 0$, and $|(\cos x)/x^2| \leq 1/x^2$, so the improper integral

$$\int_1^\infty \frac{\cos x}{x^2} \, dx$$

exists. It follows that the improper integral of $f$ exists. Suppose $f$ is integrable, and

$$R \int_0^\infty f = \lim_{b \to \infty} \int_0^b f$$

exists. We define $f_n = f\chi_{[0,n]}$ for $n \in \mathbb{N}$, and observe that $f_n \to f$ a.e. In addition, $|f_n| \leq |f|$, so the Lebesgue convergence theorem implies that $\int f_n \to \int f$. But

$$\lim \int f_n = \lim \int_0^n f = R \int_0^\infty f,$$

so the improper Riemann integral equals the Lebesgue integral in this case.
15. (a) Let $\varepsilon > 0$ be given. Since $f$ is integrable, so is $f^+$. Hence, from Problem 4, there is a nonnegative simple function $\psi_1$ such that $\psi_1 \leq f^+$, and $\int_E \psi_1 \geq \int_E f^+ - \varepsilon / 2$. It follows that

$$\int_E |f^+ - \psi_1| = \int_E f^+ - \psi_1 < \frac{\varepsilon}{2}.$$  

Similarly, there is a simple function $\psi_2$ such that

$$\int |f^- - \psi_2| < \frac{\varepsilon}{2}.$$  

If we define $\varphi = \psi_1 - \psi_2$, it follows that

$$\int_E |f - \varphi| = \int_E |(f^+ - \psi_1) + (\psi_2 - f^-)| \leq \int_E |f^+ - \psi_1| + \int_E |f^- - \psi_2| < \varepsilon.$$  

(b) From part (a), there is a simple function $\varphi$ such that

$$\int_E |f - \varphi| < \frac{\varepsilon}{4}.$$  

Since $\varphi$ is simple and integrable, there is a positive number $M$ such that $|\varphi| \leq M$, $m\{x : \varphi(x) \neq 0\} \leq M$, and

$$\int_{\mathbb{R}\setminus[-M,M]} \varphi < \frac{\varepsilon}{4}.$$  

Now Proposition 27 (applied to the function $\varphi$ on the interval $[-M,M]$) gives a step function $\psi$ such that $|\varphi - \psi| < \varepsilon/(8M)$ except on a set $B$ of measure less than $\varepsilon/(8M)$; and $|\psi| \leq M$. (Technically, $\psi$ is only defined on $[-M,M]$; we extend it to be zero outside of this interval.) It follows that

$$\int_{-M}^M |\psi - \varphi| = \int_{[-M,M] \setminus B} |\psi - \varphi| + \int_B |\psi - \varphi| \leq (2M) \frac{\varepsilon}{8M} + \frac{\varepsilon}{8M} (2M) = \frac{\varepsilon}{2}.$$  

Hence

$$\int_E |f - \psi| \leq \int_E |f - \varphi| + \int_E |\varphi - \psi| \leq \int_E |f - \varphi| + \int |\varphi - \psi| < \varepsilon.$$  

(c) Copy the proof of part (b) but use the part of Proposition 27 that gives a continuous function.

18. Let $(t_n)$ be a sequence which converges to 0. By Problem 2.49f, if we define $f_n$ by $f_n(x) = f(x, t_n)$, we have that $\lim f_n(x) = f(x)$ a.e. Since $|f_n| \leq g$, it follows from the Lebesgue Convergence Theorem that

$$\lim \int f_n = \int f.$$  

Another application of Problem 2.49f implies that

$$\lim_{t \to 0} \int f(x, t) \, dx = \int f(x) \, dx.$$
Suppose now that $f$ is continuous at some $t_0$. We define $F(x, t) = f(x, t - t_0)$. Then the first part of the problem shows that

$$\lim_{\tau \to 0} \int F(x, \tau) \, dx = \int F(x, 0) \, dx,$$

which means that

$$\lim_{t \to t_0} \int f(x, t) \, dx = \int f(x, t_0) \, dx.$$

Therefore $h$ is continuous.

21. First, we prove Fatou’s Lemma with ‘convergence a.e.’ replaced by ‘convergence in measure’. Suppose $(f_n)$ is a sequence of nonnegative measurable functions which converges in measure to some function $f$, and let $(f_n(k))$ be a subsequence of $(f_n)$ such that

$$\lim \int_E f_n(k) = \lim \int_E f_n.$$

From Problem 20, it follows that $(f_n(k))$ converges in measure to $f$, so Proposition 4.18 gives a further subsequence $(f_n(k(m)))$ which converges a.e. to $f$. Fatou’s Lemma then implies that

$$\int_E f \leq \lim \int_E f_n(k(m)).$$

Problem 2.12 then implies that

$$\lim \int_E f_n(k(m)) = \lim \int_E f_n(k),$$

and therefore

$$\int_E f \leq \lim \int_E f_n.$$

The Monotone Convergence Theorem is easier. If $(f_n)$ is an increasing sequence of nonnegative measurable functions that converges in measure to a function $f$, then there is a subsequence $(f_n(k))$ which converges a.e. to $f$ and hence

$$\int f = \lim \int f_n(k).$$

But the sequence $(I_n)$, defined by

$$I_n = \int f_n,$$

is an increasing sequence, so any subsequence has the same limit. Therefore

$$\int f = \int f_n.$$

For the Lebesgue Convergence Theorem, we follow the scheme for Fatou’s Lemma. First, let $n(k)$ be an increasing sequence. Then we extract a subsequence $(f_n(k(m)))$
which converges a.e to $f$. Since $|f_n| \geq g$ for all $n$, it follows from Proposition 4.18 that $|f_{n(k(m))}| \leq g$ for all $m$ and hence the Lebesgue Convergence Theorem gives

$$\int f = \lim \int f_{n(k(m))}.$$

Setting $I_n = \int f_n$, we see that, any subsequence of $(I_n)$ has a subsequence that converges to $\int f$, so Problem 2.12 implies that $\lim I_n = \int f$.

Chapter 5

3. (a) First, $D_- f(c) \leq D^- f(c)$ and $D_+ f(c) \leq D^+ f(c)$ because the liminf of any set is always less than or equal to its limsup. At a maximum, we have $f(c+h) \leq f(c)$ for $h > 0$, so

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

for $h > 0$, and therefore $D^+ f(c) \leq 0$. Similarly, $f(c) \geq f(c-h)$ for $h \geq 0$, so

$$\frac{f(c)-f(c-h)}{h} \geq 0$$

for $h > 0$, and therefore $D_-(f(c)) \geq 0$. Note that these inequalities are the reverse of the ones in the book.

(b) If $f$ has a local maximum at $a$, then the derivates $D^- f(a)$ and $D_+ f(a)$ are undefined but this argument still gives $D_+ f(a) \leq D^+ f(a) \leq 0$. If $f$ has a local maximum at $b$, then the derivates $D^+ f(b)$ and $D_+ f(b)$ are undefined but this argument still gives $0 \leq D_-(f(b)) \leq D^- f(b)$.

5. We begin with some simple observations. Let $g_1$ and $g_2$ be two functions defined on a set $A$. Then $g_1(x) \leq \sup_A g_1$ and $g_2(x) \leq \sup_A g_2$ for any $x \in A$. Therefore

$$\sup_A (g_1 + g_2) \leq \sup_A g_1 + \sup_A g_2.$$

It follows that

$$\limsup_{x \to y} (g_1(x) + g_2(x)) \leq \limsup_{x \to y} g_1(x) + \limsup_{x \to y} g_2(x).$$

Similarly,

$$\liminf_{x \to y} (g_1(x) + g_2(x)) \geq \liminf_{x \to y} g_1(x) + \liminf_{x \to y} g_2(x).$$

(a) We use our first observation with $g_1(h) = (f(x+h)-f(x))/h$ and $g_2(h) = (g(x+h)-g(x))/h$.

(b) From our first observation, we also conclude that $D^- (f+g) \leq D^- f + D^- g$. The second observation gives $D_+ (f+g) \geq D_+ f + D_+ g$ and $D^- (f+g) \geq D^- f + D_+ g$.

(c) We first consider some possibilities that lead to trouble. If $f(c) = g(c) = 0$, then the inequality need not be true. For example, suppose $\alpha$ and $\beta$ are positive numbers and set

$$f(x) = (x^+)^{\alpha}, \quad g(x) = (x^+)^{\beta}.$$
It’s easy to check that

$$D^+ f(0) = \begin{cases} 
\infty & \text{if } \alpha < 1, \\
0 & \text{if } \alpha = 1, \\
0 & \text{if } \alpha > 0,
\end{cases}$$

so the inequality is true if $\alpha + \beta > 1$ and false if $\alpha + \beta \leq 1$.

On the other hand, if $f(c) > 0$, $g(c) > 0$, $D^+ f(c) = \infty$ and $D^+ g(x) = -\infty$, then the right hand side of the inequality is undefined, so the inequality isn’t true here, either. (The same remark applies if $f(c) > 0$, $g(c) > 0$, $D^+ f(c) = -\infty$ and $D^+ g(x) = \infty$.)

In all other cases, the inequality is true. We divide into the following cases:

(i) $f(c) \geq 0$, $g(c) > 0$ and $D^+ f(c)$ is finite.

(ii) $f(c) = 0$, $g(c) > 0$ and $D^+ f(c) = \infty$.

(iii) $f(c) = 0$, $g(c) > 0$ and $D^+ f(c) = -\infty$.

(iv) $f(c) > 0$, $g(c) > 0$, $D^+ f(c) = \infty$, and $D^+ g(c) \neq -\infty$.

(v) $f(c) > 0$, $g(c) > 0$, $D^+ f(c) = -\infty$, and $D^+ g(c) \neq \infty$.

(extensions of cases (i), (ii) and (iii)). In all cases, we write

$$D^+(fg)(c) = \lim_{h \to 0^+} \frac{f(c+h)g(c+h) - f(c)g(c)}{h}.$$ 

Using the same algebra as for the product rule for derivatives, we have

$$\frac{f(c+h)g(c+h) - f(c)g(c)}{h} = g(c+h)\frac{f(c+h) - f(c)}{h} + f(c)\frac{g(c+h) - g(c)}{h}.$$ 

It follows that

$$D^+(fg)(c) \leq \lim_{h \to 0^+} g(c+h)\frac{f(c+h) - f(c)}{h} + \lim_{h \to 0^+} f(c)\frac{g(c+h) - g(c)}{h}.$$ 

The second limsup is easy. Because $f(c) \geq 0$, it follows that

$$\lim_{h \to 0^+} f(c)\frac{g(c+h) - g(c)}{h} = f(c)\lim_{h \to 0^+} g(c+h) - g(c) = f(c)D^+ g(c).$$

In case (i), for the first limsup, let $\varepsilon > 0$ be given and choose $\delta_0$ so that $|g(c+h) - g(c)| < \varepsilon$ if $|h| < \delta_0$. We now examine three cases. Suppose first that $D^+ f(c) > 0$. Then there is $\delta_1 \in (0, \delta_0]$ such that $0 \leq h \leq \delta_1$ implies that $f(c+h) - f(c) \geq 0$. If $h \in (0, \delta_1)$, it follows that

$$g(c+h)\frac{f(c+h) - f(c)}{h} \leq [g(c) + \varepsilon]\frac{f(c+h) - f(c)}{h},$$

so $\delta \in (0, \delta_1)$ implies that

$$\sup_{0 < h < \delta} g(c+h)\frac{f(c+h) - f(c)}{h} \leq [g(c) + \varepsilon]\sup_{0 < h < \delta} \frac{f(c+h) - f(c)}{h}.$$
Therefore
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq [g(c) + \varepsilon] \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} = [g(c) + \varepsilon]D^+ f(c). \]

Because \( \varepsilon > 0 \) is arbitrary, we conclude that
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq g(c)D^+ f(c). \]

If \( D^+ f(c) < 0 \), then there is \( \delta_2 \in (0, \delta_0) \) such that \( 0 \leq h \leq \delta_2 \) implies that \( f(c + h) - f(c) \leq 0 \). If \( h \in (0, \delta_2) \), it follows that
\[ g(c + h) \frac{f(c + h) - f(c)}{h} \leq [g(c) - \varepsilon] \frac{f(c + h) - f(c)}{h}, \]
so \( \delta \in (0, \delta_1) \) implies that
\[ \sup_{0 < h < \delta} g(c + h) \frac{f(c + h) - f(c)}{h} \leq [g(c) - \varepsilon] \sup_{0 < h < \delta} \frac{f(c + h) - f(c)}{h}. \]

Therefore
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq [g(c) - \varepsilon] \sup_{h \to 0^+} \frac{f(c + h) - f(c)}{h} = [g(c) - \varepsilon]D^+ f(c). \]

Because \( \varepsilon > 0 \) is arbitrary, we conclude that
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq g(c)D^+ f(c). \]

Finally, if \( D^+ f(c) = 0 \), then there is \( \delta_3 \in (0, \delta_0) \) such that \( h \in [0, \delta_3] \) implies that
\[ \frac{f(c + h) - f(c)}{h} \leq \frac{\varepsilon}{g(c) + \varepsilon}, \]
and, hence, that
\[ g(c + h) \frac{f(c + h) - f(c)}{h} \leq g(c + h)\varepsilon g(c) + \varepsilon \leq \varepsilon. \]

Therefore
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that
\[ \lim_{h \to 0^+} g(c + h) \frac{f(c + h) - f(c)}{h} \leq 0 = g(c)D^+ f(c). \]

In case (ii), there is a number \( \delta_0 > 0 \) such that \( g(c + h) \geq g(c) / 2 \) if \( 0 \leq h \leq \delta_0 \). Now let \( M > 0 \) be given. Then there is a number \( \delta_1 \in (0, \delta_0) \) such that
\[ \frac{f(c + h) - f(c)}{h} \geq \frac{2M}{g(c)}. \]
Hence, for $h \in (0, \delta_1)$, we have
\[
g(c + h) \frac{f(c + h) - f(c)}{h} \geq M.
\]
Therefore
\[
\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} = \infty = g(c) D^+ f(c).
\]
Since
\[
\lim_{h \to 0^+} \frac{f(c) g(c + h) - g(c)}{h} = f(c) D^+ g(c) = 0,
\]
it follows that $D^+ (fg)(c) \leq f(c) D^+ g(c) + g(c) D^f(c)$. Case (iii) is handled similarly.
In case (iv), the argument from case (ii) shows that $f(c) D^+ g(c) + g(c) D^f(c) = \infty$, so the inequality is true in this case.
In case (v), an argument similar to the one from case (ii) shows that
\[
\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} = -\infty = g(c) D^+ f(c),
\]
so the inequality is true in this case.