14. (b) This is the same idea as for part (a). First, write $F_n$ for the set obtained by removing the intervals up to the $n$th step. Since $F_n$ is the complement of a finite union of open intervals, $F_n$ is closed. In addition, $F = \bigcap_{n=1}^{\infty} F_n$, so $F$ is an intersection of closed sets and hence closed. To show that $\tilde{F}$ is dense, we need to show that there is no (non-empty) open interval contained in $F$. This fact follows from the observation that, at the $n$th step, the remaining intervals have length no greater than $(1-\alpha)^{3^{-n}}$. Therefore, for any $n$, there is no interval of length greater than $(1-\alpha)^{3^{-n}}$ in $F$. It follows that there is no non-empty open interval in $F$, so $\tilde{F}$ is dense in $[0, 1]$.

Finally, we write $G_n$ for the union of the intervals removed at the $n$th step. Then $m(G_n) = (1-\alpha)^{3^{-n}}(3^{-n})^2 - 1$ because there are $2^{n-1}$ intervals remaining at the beginning of the $n$th step. It follows that $m(\bigcup G_n) = \sum mG_n = \alpha$.

Since $F = [0, 1] \setminus (\bigcup G_n)$, it follows that $mF = 1 - \alpha$.

15. Let $E_i$ be as in the hint. Then Lemma 12 implies that $E_i$ is measurable with $mE_i = mE$. Since $E_i \subset P_i$, it follows that $E_j \cap E_i = \emptyset$ if $i \neq j$. Therefore
\[\sum mE_i = m\left(\bigcup E_i\right)\]
by Proposition 3.13. Furthermore $\bigcup E_i \subset [0, 1]$, so $m(\bigcup E_i) \leq m[0, 1] = 1$. But
\[\sum mE_i = \sum mE = \infty \cdot mE.\]
If $mE > 0$, then $\infty \cdot mE = \infty$, so we must have $mE = 0$.

17. (a) Let $E_i = P_i$, the set from the construction of a nonmeasurable set. Each $P_i$ is nonmeasurable, so it follows from Lemma 12 that $m^*P_i > 0$. We next prove a variation on Lemma 20. Let $A$ be an arbitrary subset of $[0, 1)$ and let $y \in [0, 1)$. We now set $A_1 = A \cap [0, 1-y)$ and $A_2 = A \cap [1-y, 1)$. Since $[0, 1-y)$ is measurable, we have that
\[m^*A = m^* (A \cap [0, 1-y)) + m^*(A \sim [0, 1-y)) = m^*(A_1) + m^*(A_2).\]
(Note that $A \sim [0, 1-y) = A_2$.) From the translation invariance of Lebesgue outer measure, we conclude that
\[m^*A = m^*(A_1 + y) + m^*(A_2 + y).\]
Next, $A_1 \circ y = (A \circ y) \cap [y, 1)$ and $A_2 \circ y = (A \circ y) \sim [y, 1)$, so the measurability of $[y, 1)$ implies that

$$m^*(A \circ y) = m^*(A_1 \circ y) + m^*(A_2 \circ y).$$

Therefore $m^*A = m^*(A \circ y)$, so $m^* P_i = m^* P$ for all $i$. It follows that $\sum m^* E_i = \sum m^* P = \infty$ but $m^*(\bigcup E_i) \leq m^* [0, 1) \leq 1$.

(b) Now we take $E_i = [0, 1) \setminus \bigcup_{j=0}^{i} P_j$.

Since $P_{i+1} \subset E_i$, it follows that $m^* E_i \geq m^* P_{i+1} = m^* P > 0$ and therefore $\lim m^* E_i > 0$. On the other hand $\bigcap E_i = \emptyset$, so $m^*(\bigcap E_i) = 0$.

23. Parts of this problem were done in class, but I’ll try to include details here.

(a) For each positive integer $n$, set $A_n = \{ x : |f(x)| > n \}$. Then each $A_n$ is measurable and $\bigcap A_n = \{ x : f(x) = \infty \text{ or } f(x) = -\infty \}$, so $m(\bigcap A_n) = 0$. Since $A_{n+1} \subset A_n$, Proposition 21 tells us that $\lim m A_n = 0$, so given $\varepsilon > 0$, there is a positive integer $M$ such that $m A_n < \varepsilon/3$ if $n \geq M$. In particular, $m A_M < \varepsilon/3$.

(b) Given $\varepsilon > 0$ and $M \geq 0$, we choose a positive integer $N$ such that $N \varepsilon > M$, and we define

$$B_n = \{ x : -M + n \varepsilon \leq f(x) < -M + (n + 1) \varepsilon \}$$

for $n = 0, \ldots, 2N - 1$. We then take

$$\varphi = \sum_{n=0}^{2N-1} \left( -M + \left[ n + \frac{1}{2} \right] \varepsilon \right) \chi_{B_n}.$$

Then $|f(x) - \varphi(x)| \leq \varepsilon/2 < \varepsilon$ if $x \in \bigcup B_n$.

If we know that $m \leq f \leq M$, then we choose the positive integer $N$ such that $N \varepsilon > M - m$ and we define

$$B_n = \{ x : m + n \varepsilon \leq f(x) < m + (n + 1) \varepsilon \}$$

for $n = 0, \ldots, N - 1$, and we define $\varphi$ as before.

(c) We write the canonical representation of the simple function:

$$\varphi = \sum_{i=1}^{k} a_i \chi_{A_i}.$$ 

Then, for each $i$, there is a finite collection of intervals $(I_{i,j})$ such that

$$m^* \left( \left( \bigcup_j I_{i,j} \right) \Delta A_i \right) < \frac{1}{3k} \varepsilon.$$ 

Note that we may assume that $I_{i,j} \cap I_{i,m} = \emptyset$ if $j \neq m$. We then define $U_i = \bigcup_j I_{i,j}$ and take

$$g = \sum_{i=1}^{k} a_i \chi_{U_i}.$$
Then $g$ is a step function and $g = \varphi$ except on some subset of $A = \bigcup_{i=1}^{k} U_i \Delta A_i$. Since $m^*(U_i \Delta A_i) < \varepsilon/(3k)$, it follows (from countable subadditivity) that

$$mA \leq \sum_{i=1}^{k} m(U_i \Delta A_i) < \frac{\varepsilon}{3}.$$ 

Hence (by monotonicity of outer measure), $g = \varphi$ except on a set of measure less than $\varepsilon/3$.

If $m \leq \varphi \leq M$, then $m \leq g(x) \leq M$ except on a subset of $A$. More specifically, this inequality is true as long as $x$ is in only one $U_i$. For $x$ in the intersection of two or more $U_i$’s, we redefine $g(x) = m$. This gives a function which agrees with $\varphi$ outside of $A$, so this $g$ agrees with $\varphi$ except on a set of measure less than $\varepsilon/3$ and it satisfies $m \leq g \leq M$.

(d) Let $(x_i)_{i=0}^{n}$ be the points of discontinuity of $g$ (with $x_0 = a$ and $x_n = b$), written in increasing order and choose $\delta > 0$ so that

$$2\delta < \min_i x_i - x_{i-1}, \quad n\delta < \frac{\varepsilon}{3}.$$ 

(The minimum is over $i = 1, \ldots, n$.) We then define

$$h(x) = \begin{cases} g(x) & \text{if } x_0 \leq x \leq x_1 - \delta \text{ or } x_{n-1} + \delta \leq x \leq x_n, \\ g(x) & \text{if } x_{i-1} + \delta < x < x_i - \delta \text{ for some } i \in \{2, \ldots, n-1\}, \\ A_i(x - x_i) + B_i & \text{if } x_i - \delta < x < x_i + \delta \text{ for some } i \in \{1, \ldots, n-1\}, \end{cases}$$

where

$$A_i = \frac{g(x_i + \delta) - g(x_i - \delta)}{2\delta}, \quad B_i = \frac{g(x_i + \delta) + g(x_i - \delta)}{2}.$$ 

28. (a) From Exercise 2.48, $f_1$ is continuous and increasing, so $f$ is continuous and strictly increasing. (To see that $f$ is strictly increasing, we note that $x > y$ implies that $f(x) = f_1(x) + x \geq f_1(y) + x > f_1(y) + y = f(y)$.) Since $f(0) = 0$ and $f(1) = 2$, it follows from the intermediate value theorem that $f$ is onto, and $f$ is one-to-one because it’s strictly increasing. Exercise 2.46 also tells us that $f$ is a homeomorphism.

(b) Let $\langle I_n \rangle$ be the sequence of “middle third” intervals from the definition of $C$. Then

$$[0, 2] = f(C) \cup \bigcup_{n=1}^{\infty} f(I_n)$$

and this is a disjoint union because $f$ is one-to-one and $C \cap I_n = I_m \cap I_n = \emptyset$ if $m \neq n$. Hence

$$2 = m(f(C)) + \sum_{n=1}^{\infty} m(f(I_n)).$$
But \( f(I_n) \) is an interval with the same length as \( I_n \) (because \( f_1 \) is constant on each of these intervals), so

\[
\sum_{n=1}^{\infty} m(f(I_n)) = \sum_{n=1}^{\infty} l(I_n) = 1.
\]

Combining the two displayed equations yields \( m(f(C)) = 1 \).

(c) Let \( P \) be a nonmeasurable subset of \( F \) (from Exercise 3.16) and set \( A = f^{-1}(P) \). Then \( A \) is a subset of \( C \) and hence measurable because \( mC = 0 \) and Lebesgue measure is complete. However, \( g^{-1}[A] = f(A) = P \) is not measurable.

(d) Take \( g = f^{-1} \) and \( h = \chi_A \). Then \( (h \circ g)^{-1}(1/2, \infty) = g^{-1}(A) \) is not measurable, so \( h \circ g \) is not measurable.

(e) The set \( A \) from part (c) is measurable but it isn’t a Borel set because of Exercise 3.26.

Chapter 4

2. (a) First, from Problem 2.50(c), problem 2.51(b), and Proposition 22, it follows that \( h \) is measurable. (Actually Problem 2.50(c) is only stated for lower semicontinuous functions, but the a similar proof shows that \( h \) is upper semicontinuous if and only if the set \( \{ x : f(x) < \lambda \} \) is open for all \( \lambda \).)

Next, let \( \varphi \) be a step function such that \( \varphi \geq f \) and suppose \( a = x_0 < x_1 < \cdots < x_n = b \) is a partition such that, for each \( i \), \( \varphi \) assumes only one value in the interval \( (x_i, x_{i+1}) \). We write \( \bar{\varphi} \) for the step function defined by \( \bar{\varphi}(x) = \varphi(x) \) for \( x \in (x_i, x_{i+1}) \) and \( \bar{\varphi}(x_i) \) is the maximum of \( \varphi(x_i) \) and the values assumed in \( (x_{i-1}, x_i) \) and \( (x_i, x_{i+1}) \). It follows that \( \bar{\varphi} \) is an upper semicontinuous step function which is greater than or equal to \( f \), so Problem 2.51(c) implies that \( \varphi \geq \bar{\varphi} \). Hence \( \varphi \geq h \) except possibly at \( x_1, \ldots, x_{n-1} \), that is, at a finite number of points. Hence

\[
\int_a^b h \leq \int_a^b \bar{\varphi} = R \int_a^b \varphi.
\]

Taking the supremum over all step functions \( \varphi \leq f \) gives us

\[
\int_a^b h \leq R \int_a^b f.
\]

From Problem 2.51(g), there is a decreasing sequence \( (\varphi_n) \) of upper semicontinuous step functions which converges to \( h \). Since \( \varphi \geq f \), it follows that

\[
R \int_a^b f \leq \int_a^b \varphi_n,
\]

and sending \( n \to \infty \) gives

\[
R \int_a^b f \leq \int_a^b h.
\]
(b) Write \( h \) for the upper envelope of \( f \) and \( g \) for the lower envelope of \( f \). From part (a), we know that \( R\int_a^b f = \int_a^b h \), and a similar argument shows that \( R\int_a^b f = \int_a^b g \). Hence \( f \) is Riemann integrable if and only if \( \int_a^b h = \int_a^b g \).

If \( f \) is Riemann integrable, then \( \int_a^b h = \int_a^b g \). Now set \( k = h - g \), so \( k \) is a bounded measurable function with \( \int_a^b k = 0 \) and \( k \geq 0 \). Now let \( H = \{x : k(x) > 0\} \) and \( H_n = \{x \in H : k(x) > 1/n\} \). Then \( H = \bigcup H_n \), so \( mH \leq \sum mH_n \). Since \( \int_a^b k \geq \int_a^b (1/n) \chi_{H_n} \) (by part (iii) of Proposition 4.8) and \( 0 = \int_a^b k \), it follows that

\[
0 = \int_a^b \frac{1}{n} \chi_{H_n} = \frac{1}{n} mH_n.
\]

Therefore \( mH_n = 0 \) for all \( n \), so \( k = 0 \) a.e. so \( h = g \) a.e. Problem 2.51(a) then tells us that \( f \) is continuous wherever \( h = g \) and therefore the set of points at which \( f \) is discontinuous has measure zero.

Conversely, if the set of points at which \( f \) is discontinuous has measure zero, then \( h = g \) a.e. (by part (a) of Problem 2.51), so \( \int_a^b h = \inf \int_a^b g \) and hence \( f \) is Riemann integrable.

8. We define a new sequence of functions \( (g_m) \) by the expression \( g_m(x) = \inf \{f_n(x) : m \geq n\} \). Then \( g_m \) is measurable by Theorem 26, and \( \lim f_n = \lim g_m \). It follows that

\[
\int \lim f_n = \int \lim g_m \leq \lim \int g_m.
\]

But \( g_m \leq f_m \), so

\[
\int \lim f_n \leq \lim \int f_m.
\]