Chapter 1

20. First we show that $A'$ is a $\sigma$-algebra. So suppose $A \in A'$. Then there is a countable subcollection $C_1$ of $C$ such that $A$ is in the $\sigma$-algebra $A_1$ generated by $C_1$. It follows that $\tilde{A} \in A_1$, so $\tilde{A} \in A'$. Next, suppose that $\langle A_n \rangle$ is a sequence of elements of $A'$. Then, for each $n$, there is a countable subcollection $C_n$ such that $A_n \in A_n$, the $\sigma$-algebra generated by $C_n$. Now write $A_\infty$ for the $\sigma$-algebra generated by $\bigcup_n C_n$. Since $A_\infty$ is a $\sigma$-algebra containing $\bigcup_n C_n$, it follows from Proposition 4 that $A_n \subset A_\infty$ for all $n$ and hence each $A_n$ is in $A_\infty$, so $\bigcup_n A_n \in A_\infty$ and therefore $\bigcup_n A_n \in A'$ because $\bigcup_n C_n$ is countable by Proposition 1.7 (from the text).

Because $A'$ is a $\sigma$-algebra, it follows from Proposition 4 that $A \subset A'$. Hence, any $E \in A$ is an element of $A'$, which means that $E$ is an element of a $\sigma$-algebra generated by some countable subcollection $C_0$ of $C$.

Chapter 2

21. (a) We prove the contrapositive: If $E$ is uncountable, then $\sum_{x \in E} x$ is infinite.

To prove this implication, we write

$$E_n = \{ x \in E : x > \frac{1}{n} \}$$

for $n \in \mathbb{N}$. Since $\bigcup_n E_n = E$ and $E$ is uncountable, some $E_n$ must be uncountable (by Proposition 1.7) and hence infinite. For a given $M > 0$ (and an $n$ such that $E_n$ is infinite), let $F$ be a subset of $E_n$ with $Mn$ elements. Then

$$\sum_{x \in F} x > Mn \left( \frac{1}{n} \right) = M,$$

so

$$\sum_{x \in E} x \geq \sum_{x \in F} x > M,$$

and hence the sum is infinite.

Here’s a direct proof. Let $\sum_{x \in E} x = S < \infty$ and define $E_n$ as before. Then

$$S \geq \sum_{x \in E_n} x,$$

so $E_n$ has at most $S/n$ elements, which means $E_n$ is finite for each $n$ and therefore $E$ is countable.
(b) We consider two cases. First, if
\[ \sum_{n=1}^{\infty} x_n = \infty, \]
then, for any \( M > 0 \), there is a positive integer \( N \) such that
\[ \sum_{n=1}^{N} x_n > M, \]
and hence \( \sum_{x \in E} x > M \), which implies that \( \sum_{x \in E} x = \infty \).

On the other hand, if
\[ \sum_{n=1}^{\infty} x_n = S < \infty, \]
then, for any \( \varepsilon > 0 \), there is a positive integer \( N \) such that
\[ \left| \sum_{n=1}^{m} x_n - S \right| < \varepsilon \]
as long as \( m > N \). It follows that
\[ \sum_{n=1}^{N} x_n > S - \varepsilon, \]
and hence
\[ \sum_{x \in E} x \geq S - \varepsilon. \]

Since \( \varepsilon \) is arbitrary, it follows that
\[ \sum_{x \in E} x \geq S. \]

Now, if \( F \) is finite, then there is an integer \( k \) such that
\[ F \subset \{ x_1, \ldots, x_k \}. \]
(Just let \( k = \text{sup}\{n : x_n \in F\} \).) Since the \( x_n \)'s are all positive, it follows that
\[ \sum_{x \in F} x \leq \sum_{n=1}^{k} x_n \leq S \]
and hence
\[ \sum_{x \in E} x \leq S. \]

Combining these last two inequalities yields
\[ \sum_{x \in E} x = S. \]
37. First, we show the “middle thirds” characterization of the Cantor set via an induction argument. So let \( x \not\in C \), and set

\[
x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.
\]

(If \( x \) has two ternary expansions, we choose one of them.) Also, we define \( A_n \) inductively by \( A_0 = [0, 1] \), and \( A_{n+1} \) is obtained from \( A_n \) by deleting the (open) middle third of each interval of \( A_n \). In each \( A_n \), there are \( 2^n \) intervals, which we label as follows: number the intervals from left to right in binary notation with enough leading zeros to have \( n \) bits and then subtract 1 (so, for example, the fifth interval for \( n = 4 \) corresponds to 0100). If we now replace each 1 by a 2, we get the ternary expansion of the left endpoint of the interval and if we add infinitely many twos to the end of this ternary expansion, we obtain the right endpoint of the interval. Therefore, a number has the first \( n - 1 \) ternary terms equal to 0 or 2 if and only if that number is in \( A_n \), so the Cantor set is the intersection of the \( A_n \)'s.

Each \( A_n \) is the complement of a union of open intervals and hence it is closed. Since the Cantor set is the intersection of these \( A_n \)'s, it is also closed.

50. (a) \( \implies \). If \( f \) is lower semicontinuous at \( y \), let \( \varepsilon > 0 \) be given. Then there is a \( \delta > 0 \) such that

\[
\inf_{0 < |x-y| < \delta} f(x) \geq \sup_{\theta > 0} \inf_{0 < |x-y| < \theta} f(x) - \varepsilon,
\]

and therefore

\[
f(y) \leq \sup_{\theta > 0} \inf_{0 < |x-y| < \theta} f(x) \leq \inf_{0 < |x-y| < \delta} f(x) + \varepsilon
\]

and hence (because \( f(y) \leq f(y) \)), we have \( f(y) \leq f(x) + \varepsilon \) for all \( x \) with \( |x - y| < \delta \).

\( \Longleftarrow \). Set \( L = \sup_{\theta > 0} \inf_{0 < |x-y| < \theta} f(x) \) and let \( \varepsilon > 0 \) be given. Then there is \( \delta > 0 \) so that \( f(y) \leq f(x) + \varepsilon \) for all \( x \) such that \( |x - y| < \delta \). It follows that

\[
f(y) \leq \inf_{0 < |x-y| < \delta} f(x) + \varepsilon,
\]

so \( f(y) \leq L + \varepsilon \). Since \( \varepsilon \) is arbitrary, it follows that \( f(y) \leq L \).

(b) This follows easily from (a) and the \( \varepsilon-\delta \) definition of continuity or from part (c) of problem 49.

(c) \( \implies \). Given \( \lambda \in \mathbb{R} \) and \( x \) such that \( f(x) > \lambda \), set \( \varepsilon = (f(x) - \lambda)/2 \). From part (a), there is a number \( \delta \) such that \( f(x) \leq f(y) + \varepsilon \) for all \( y \) with \( |x - y| < \delta \). But this says that

\[
f(y) \geq f(x) - \varepsilon = \frac{f(x) + \lambda}{2} > \lambda
\]

so \( \{x : f(x) > \lambda \} \) is open.

\( \Longleftarrow \). Let \( \varepsilon > 0 \) and \( y \) be given and set \( \lambda = f(y) - \varepsilon \). Since \( \{x : f(x) > \lambda \} \) is open, there is a number \( \delta \) such that \( f(x) > \lambda \) for all \( x \) with \( |x - y| < \delta \). But this inequality says that \( f(x) > f(y) - \varepsilon \) which implies that \( f(y) \leq f(x) + \varepsilon \).

(d) For \( f \lor g \), given \( \varepsilon > 0 \) and \( y \), there are positive numbers \( \delta_1 \) and \( \delta_2 \) such that
$f(y) \leq f(x) + \varepsilon$ for $|x - y| < \delta_1$ and $g(y) \leq g(x) + \varepsilon$ for $|x - y| < \delta_2$. With
$\delta = \min\{\delta_1, \delta_2\}$, we see that

$$f(y) \leq f(x) + \varepsilon, \quad g(y) \leq g(x) + \varepsilon$$

for $|x - y| < \delta$. Hence,

$$f(y) \leq f \lor g(x) + \varepsilon, \quad g(y) \leq f \lor g(x) + \varepsilon$$

for $|x - y| < \delta$, and therefore $f \lor g$ is lower semicontinuous by part (a).

For $f + g$, we take $\delta_3$ and $\delta_4$ so that $f(y) \leq f(x) + \varepsilon/2$ for $|x - y| < \delta_3$ and $g(y) \leq g(x) + \varepsilon/2$ for $|x - y| < \delta_4$. Then, adding the inequalities gives $f(y) + g(y) \leq f(x) + g(x) + \varepsilon$ for $|x - y| < \delta$ and $\delta = \min\{\delta_3, \delta_4\}$.

(c) Let $\varepsilon > 0$ and $y$ be given. Then there is an index $n$ such that $f_n(y) \geq f(y) - \varepsilon/2$.

With $\delta > 0$ so that $f_n(y) \leq f_n(x) + \varepsilon/2$ for $|x - y| < \delta$, we see that

$$f(y) \leq f_n(y) + \varepsilon/2 \leq f_n(x) + \varepsilon \leq f(x) + \varepsilon$$

for $|x - y| < \varepsilon$. Hence $f$ is lower semicontinuous by part (a).

(f) $\implies$. Let $\varepsilon > 0$ be given and let $\delta$ be given from part (a). Then there are points $y_1 \in (x_{i-1}, x_i)$ and $y_2 \in (x_i, x_{i+1})$ such that $|x - y_1|, |x - y_2| < \delta$. Hence $f(x_i) \leq \min\{\varphi(y_1), \varphi(y_2)\} + \varepsilon$ by part (a). Since $\varepsilon$ is arbitrary, we have $\varphi(x_i) \leq \min\{\varphi(y_1), \varphi(y_2)\}$ as required.

$\Leftarrow$. From part (b), $\varphi$ is lower semicontinuous on each interval $(x_i, x_{i+1})$. At $x_i$, we take $\delta = \min\{x_{i} - x_{i-1}, x_{i+1} - x_i\}$ and note that $\varphi(x_i)$ is equal to the smaller of the two values assumed in $(x_{i-1}, x_i)$ and $(x_i, x_{i+1})$ if $0 < |x - x_i| < \delta$ and $\varphi(x) = \varphi(x_i)$ if $|x - x_i| = 0$. Hence $\varphi(x_i) \leq \varphi(x)$ if $|x - x_i| < \delta$, so $\varphi$ is also lower semicontinuous at each $x_i$. (Note that the choice of $\delta$ is independent of $\varepsilon$.)

(g) $\implies$. First, for each $n$, we define the partition

$$P_n = \{a = x_{n,0} < x_{n,1} < \cdots < x_{n,2^n} = b\}$$

by $x_{n,i} = a + 2^{-n}(b - a)$. We then define

$$\varphi_n(x) = \inf\{f(y) : y \in (x_{n,i}, x_{n,i+1})\}$$

if $x \in (x_{n,i}, x_{n,i+1})$ and $\varphi_n(x_i)$ is the minimum of the values assumed in $(x_{n,i-1}, x_{n,i})$ and $(x_{n,i}, x_{n,i+1})$. Also we take $\varphi_n(a)$ to be the value assumed in $(a, x_{n,1})$ and we take $\varphi_n(b)$ to be the value assumed in $(x_{n,2^n-1}, b)$. By parts (i) and (f), $\varphi_n$ is lower semicontinuous. It’s easy to see that $(\varphi_n(x))$ is an increasing sequence for each fixed $x$ since the infimum for $\varphi_n$ is taken over a smaller set than the infimum for $\varphi_{n-1}$. To see that $f = \lim \varphi_n$, we fix an $x$ and an $\varepsilon > 0$. By part (b), there is a $\delta > 0$ such that

$$f(x) \leq f(y) + \varepsilon$$

if $|x - y| < \delta$. By taking $n$ so large that $2^{-n}(b - a) < \delta$, we see that $\varphi_n(x) \geq f(x) - \varepsilon$ and hence $\varphi_n(x) \to f(x)$.

$\Leftarrow$. The limit of the $\varphi_n$’s is equal to the supremum, so part (e) implies that this limit is lower semicontinuous.

(h) $\implies$. With $\varphi_n$ from part (g), we define $\psi_n$ separately on each interval. So, on $[x_{n,i}, x_{n,i+1}]$, we fix points $y_{n,i} < z_{n,i}$ in this interval so that the intervals $|x_{n,i} - y_{n,i}| =
Then \( |y_{n,i} - z_{n,i}| = |z_{n,i} - x_{n,i+1}| \). Then \( \psi_n \) is linear on each of these intervals and agrees with \( \varphi_n \) at \( x_{n,i}, y_{n,i}, z_{n,i} \), and \( x_{n,i+1} \). Again, it’s easy to see that \( (\psi_n) \) is an increasing sequence that converges to \( f \).

\( \Leftarrow \). The argument in part (g) shows that \( f \) is lower semicontinuous.

(i) Let \( \langle x_n \rangle \) be a sequence such that \( f(x_n) \) converges to the infimum \( I \) (note that the case of the infimum being \( -\infty \) has not yet been excluded). From the Bolzano-Weierstrass theorem, there is a convergent subsequence \( \langle x_{n_k} \rangle \) with limit \( x_0 \in [a, b] \). It follows that \( f(x_{n_k}) \) also converges to \( I \) and part (a) implies that \( f(x_0) \leq I \). Since \( f(x_0) \leq I \), it follows that \( f(x_0) = I \), so the infimum is achieved and hence finite.

53. For each positive integer \( n \), write \( A_n \) for the set of all \( x \in \mathbb{R} \) such that there is an open interval \( I \) with \( x \in I \) and \( \sup I f - \inf I f \leq 1/n \). Then, if \( x \in A_n \), there is a number \( \delta > 0 \) such that \( (x - \delta, x + \delta) \subset I \). Hence there is a number \( \delta > 0 \) such that \( (x - \delta, x + \delta) \subset I \). If \( |x - y| < \delta \), then \( y \in I \), so \( y \in A_n \). Therefore \( A_n \) is open, so \( A = \cap A_n \) is a \( G_\delta \) set.

If \( x \in A \), then, given \( \varepsilon > 0 \), there is a positive integer \( n \) such that \( 1/n < \varepsilon \). Therefore, there is an open interval \( I \) with \( x \in I \) and \( \sup I f - \inf I f \leq 1/n \). Hence there is a number \( \delta > 0 \) such that \( (x - \delta, x + \delta) \subset I \). If \( |x - y| < \delta \), then \( y \in I \), so

\[
 f(y) - f(x) \leq \sup_I f - \inf_I f \leq \frac{1}{n} < \varepsilon,
\]

and

\[
 f(y) - f(x) \geq \inf_I f - \sup_I f \geq -\frac{1}{n} > -\varepsilon.
\]

Hence \( |x - y| < \delta \) implies that \( |f(x) - f(y)| < \varepsilon \), so \( f \) is continuous at each point of \( A \).

Conversely, if \( f \) is continuous at \( x \), then, for each positive integer \( n \), there is a number \( \delta > 0 \) such that \( |x - y| < \delta \) implies that \( |f(x) - f(y)| < 1/(2n) \). If we now let \( I \) be the interval \( (x - \delta, x + \delta) \), it follows that

\[
 \sup_I f \leq f(x) + \frac{1}{2n}
\]

and

\[
 \inf_I f \geq f(x) - \frac{1}{2n},
\]

so \( \sup_I f - \inf_I f \leq 1/n \). Hence \( x \in A_n \), so the set of points of continuity of \( f \) is exactly the set \( A \), which we have already shown to be a \( G_\delta \) set.