Chapter 4

7. (a) Let \( \langle f_n \rangle \) be the sequence from the hint. Then
\[
\int f_n = 1,
\]
(with \( E = \mathbb{R} \)), but \( f_n \to 0 \) everywhere, so \( \int f = 0 \). Hence
\[
\int f = 0 < 1 = \lim \int f_n.
\]

(b) Let \( \langle f_n \rangle \) be the sequence from the hint. Then \( \lim f_n(x) = 0 \) for any \( x \) and \( f_n \geq f_{n+1} \), so \( \langle f_n \rangle \) is a decreasing sequence, so
\[
\int f = 0,
\]
but
\[
\lim \int f_n = \lim \infty = \infty.
\]

16. We proceed in steps. First, note that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \chi_{[a,b]}(x) \cos nx \, dx = \lim_{n \to \infty} \int_{a}^{b} \cos nx \, dx
= \lim_{n \to \infty} \frac{1}{n} (\sin na - \sin nb) = 0.
\]
By linearity, it follows that \( \lim_{n \to \infty} \int_{-\infty}^{\infty} \psi(x) \cos nx \, dx = 0 \) for any step function \( \psi \).

Now, let \( f \) be an integrable function, and let \( \varepsilon > 0 \) be given. Take \( \psi \), a step function from Problem 15(b), so that
\[
\int_{\mathbb{R}} |f - \psi| < \varepsilon/2,
\]
and then \( N \) (from our previous calculation) so that
\[
\left| \int_{-\infty}^{\infty} \psi(x) \cos nx \, dx \right| < \varepsilon/2
\]
for all \( n \geq N \). It follows (for any \( n \)) that
\[
\left| \int_{-\infty}^{\infty} f(x) \cos nx \, dx \right| \leq \left| \int_{-\infty}^{\infty} \psi(x) \cos nx \, dx \right| + \left| \int_{-\infty}^{\infty} [f(x) - \psi(x)] \cos nx \, dx \right|
\]
and \( |f(x) - \psi(x)| \cos nx \leq |f(x) - \psi(x)| \), so
\[
\left| \int_{-\infty}^{\infty} [f(x) - \psi(x)] \cos nx \, dx \right| \leq \varepsilon/2
\]
if \( n \geq N \) and hence
\[
\left| \int_{-\infty}^{\infty} f(x) \cos nx \, dx \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
for \( n \geq N \). Since \( \varepsilon > 0 \) is arbitrary, this gives the desired limit for \( f \).

Chapter 11

19. First, we prove the result for nonnegative functions. Let \( \langle \varphi_n \rangle \) be a sequence of nonnegative simple functions such that \( \varphi_n \leq f \) and \( \int \varphi_n \rightarrow \int f \). Let \( \varepsilon > 0 \) be given and choose \( n \) such that
\[
\int \varphi_n \geq \int f - \frac{\varepsilon}{2}.
\]
Now let \( M = \sup \varphi_n \) and set \( \delta = \varepsilon/(2M) \). If \( \mu E < \delta \), then
\[
\int_E f = \int_E (f - \varphi_n) + \int_E \varphi_n \leq \int (f - \varphi_n) + \int \varphi_n \leq \int f - \frac{\varepsilon}{2} + M \frac{\varepsilon}{2M} = \varepsilon.
\]

For general \( f \), we note that, for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for each measurable set \( E \) with \( \mu E < \delta \) we have
\[
\int_E |f| < \varepsilon.
\]
The desired result now follows by combining this inequality with the inequality
\[
\left| \int_E f \right| \leq \int_E |f|.
\]

20. (a) Fatou’s Lemma: Let \( \langle f_n \rangle \) be a sequence of nonnegative measurable functions which converge in measure to \( f \). Then, there is a subsequence \( f_{n_k} \) such that
\[
\lim \int f_{n_k} = \lim \int f_n.
\]
(by Problem 2.9a). Set \( g_k = f_{n_k} \). Then \( \langle g_k \rangle \) converges to \( f \) in measure also, and, by Problem 11.13a, there is a subsequence \( \langle g_{k_m} \rangle \) which converges almost everywhere to \( f \). It follows that
\[
\int_E f \leq \lim \int_E g_{k_m}.
\]
But
\[
\lim \int_E g_{k_m} = \lim \int_E g_{k_m} = \lim \int_E g_k = \lim \int_E f_n.
\]
Combining these two inequalities yields
\[
\int_E f \leq \lim \int_E f_n.
\]
(b) The Monotone Convergence Theorem and the Lebesgue Dominated Convergence Theorem follow from Fatou’s Lemma just as before.
21. (a) Let \( E_n = \{ x : f(x) \geq 1/n \} \), and set
\[
\varphi_n(x) = \frac{1}{n} \chi_{E_n}(x).
\]
Then \( \varphi_n \leq f \), so
\[
\int f \geq \int \varphi = \frac{1}{n} \mu(E_n).
\]
Hence \( \mu(E_n) < \infty \) for each positive \( n \). On the other hand
\[
\{ x : f(x) \neq 0 \} = \bigcup_{n=1}^{\infty} E_n,
\]
so this set is \( \sigma \)-finite.
(b) From part (a), the set \( S = \{ x : f(x) \neq 0 \} \) is \( \sigma \)-finite, so Proposition 32 gives an increasing sequence of simple functions \( \langle \psi_n \rangle \) on \( S \) each of which vanishes outside a set of finite measure. The proof is completed by defining
\[
\varphi_n(x) = \begin{cases} 
\psi_n(x) & \text{if } x \in S \\
0 & \text{if } x \notin S.
\end{cases}
\]
(c) Let \( \langle \varphi_n \rangle \) and \( \langle \psi_n \rangle \) be the sequences from part (b) corresponding to \( f^+ \) and \( f^- \) respectively. From the Monotone Convergence Theorem we have that
\[
\int f^+ = \lim \int \varphi_n, \quad \int f^- = \lim \int \psi_n.
\]
Given \( \varepsilon \), choose \( N \) so that
\[
\int f^+ - \int \varphi_n < \varepsilon/2, \quad \int f^- - \int \psi_n < \varepsilon/2
\]
for all \( n \geq N \), and set \( \varphi = \varphi_N - \psi_N \). Then write \( A = \{ x : f(x) \geq 0 \} \) and \( B = \{ x : f(x) < 0 \} \). Note that \( \varphi = \varphi_n \) on \( A \) and \( \varphi = -\psi_N \) on \( B \). Hence
\[
\int |f - \varphi| = \int_A |f^+ - \varphi_N| + \int_B |f^- - \psi_N| = \int f^+ - \varphi_N + \int f^- - \psi_N < \varepsilon.
\]