Math 266, Section A
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December 14, 2009

SOLUTIONS FOR PRACTICE FINAL EXAM, PROBLEMS #4 AND #6, CORRECTED DEC. 14

4. Since this is a linear, second order equation, the solutions must have the form $y = e^{rt}$ with $r^6 - r^2 = 0$. Since
$$r^6 - r^2 = r^2(r-1)(r+1)(r-i)(r+i),$$
the general solution is
$$y = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 \cos t + c_6 sint.$$

6. From the eigenvalues and eigenvectors, a fundamental matrix for the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is
$$\mathbf{\Psi}(\mathbf{t}) = \begin{bmatrix} e^t & 5e^{-2t} \\ 2e^t & e^{-2t} \end{bmatrix}.$$
Then we look for the inverse to
$$\mathbf{\Psi}(\mathbf{0}) = \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix},$$
and we can get the inverse matrix by making an augmented matrix with the identity matrix on the right and then using Gaussian elimination to get an augmented matrix with the identity matrix on the left.
$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1 \begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & -9 & -2 & 1 \end{bmatrix} R_2 \to -R_2/9$$
$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & 2/9 & -1/9 \end{bmatrix} R_1 \to R_1 - 5R_2 \begin{bmatrix} 1 & 0 & -1/9 & 5/9 \\ 0 & 1 & 2/9 & -1/9 \end{bmatrix}.$$
Then
$$e^{\mathbf{At}} = \begin{bmatrix} e^t & 5e^{-2t} \\ 2e^t & e^{-2t} \end{bmatrix} \begin{bmatrix} -1/9 & 5/9 \\ 2/9 & -1/9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9}e^t + \frac{10}{9}e^{-2t} & \frac{5}{9}e^t - \frac{5}{9}e^{-2t} \\ -\frac{2}{9}e^t + \frac{2}{9}e^{-2t} & \frac{10}{9}e^t - \frac{1}{9}e^{-2t} \end{bmatrix}$$