Fermat No sol $x^4+y^4=z^4$ x,y,z pos integers

(0) WLOG no common divisor of x,y,z
$\Rightarrow$ pairwise rel prime

(1) must have $\overset{x}{e}+\overset{y}{o}=\overset{z}{o}$ e = even, o = odd

(2) For any such Pythagorean triple, $\exists\, p,q$
$x=2pq$ $y=p^2-q^2$ $z=p^2+q^2$. Note one of p,q odd other even else p^2+q^2, p^2-q^2 both even

(3) use inf descent to show $\nexists\, x,y,z$ $x^2+y^2=z^2$ and $\frac{1}{2}xy=w^2$

Assume x,y,z exist, with x even y z odd
Then $\exists\, p,q$ (opp parity) $x=2pq$ $y=p^2-q^2$ $z=p^2+q^2$
If $\frac{1}{2}xy$ is square then
$pq(p^2-q^2)$ is square
p,q,p^2-q^2 rel prime since x,y are
$\therefore$ each is a square $\leftarrow$ where d,f have opposite parity since p,q do - see Note
$\exists\, c,d,f$ $p=d^2$ $q=f^2$ $p^2-q^2=c^2=d^4-f^4$
$c^2=(d^2+f^2)(d^2-f^2)$, d,f opposite parity and no odd common divisor
because if no odd prime $P \mid d^2+f^2$ and $P \mid d^2-f^2$ then $P \mid d,f$
so d^2+f^2, d^2-f^2 are squares
$\exists\, g,h$ $g^2=d^2+f^2$, $h^2=d^2-f^2$ both odd & rel prime
$g^2-h^2=2f^2=(g+h)(g-h)$
g,h both odd & rel prime $\Rightarrow g+h, g-h$ both not div by any odd prime
$g=2s+1$ $h=2t+1$
$g+h=2(s+t)+2$ $g-h=2(s-t)$
$=2(s+t+1)$

If $s+t+1$ is even then $s-t=s+t+1-2t-1$ is odd
If $s+t+1$ is odd then $s-t$ is even
Thus $g+h=2m_1$ $g-h=n_1$ (1)
or $g+h=n_1$ $g-h=2m_1$ (2) } In both cases m_1 odd, n_1 even, m_1,n_1 rel prime

Say (2) $g+h=n_1 \quad g-h=2m_1$

$2f^2 = g^2-h^2 = (g+h)(g-h) = n_1 \cdot 2m_1$

$f^2 = n_1 m_1 \qquad n_1, m_1$ rel prime

$\therefore \exists\, n, m \quad n^2 = n_1 \quad m^2 = m_1$

$f^2 = n^2 m^2$

$g = \frac{1}{2}((g+h)+(g-h)) = (n^2+2m^2)\frac{1}{2} = \frac{n^2}{2} + m^2$

$h = \frac{1}{2}((g+h)-(g-h)) = (n^2-2m)\frac{1}{2} = \frac{n^2}{2} - m^2$

$d^2 = \frac{1}{2}((d^2+f^2)+(d^2-f^2)) = \frac{1}{2}(g^2+h^2)$

$= \frac{1}{2}\left(\left(\frac{n^2}{2}+m^2\right)^2 + \left(\left(\frac{n^2}{2}\right)-m^2\right)^2\right)$

$d^2 = \left(\frac{n^2}{2}\right)^2 + (m^2)^2$

So $\frac{n}{2}$ & m are sides of a positive integer rt Δ w/ area

$\frac{1}{2}\left(\frac{n^2}{2}\right)(m^2) = \frac{n^2m^2}{4}$ is $\left(\frac{n}{2}\right)^2 m$ is a square

and smaller

$(p = d^2 \quad x = 2pq > d^2 > \left(\frac{n^2}{2}\right)^2 \geq \frac{n^2}{2} \leftarrow$ even side of new Δ

If one could find $d^4 - f^4 = c^2$

then let $p = d^2 \quad q = f^2$

$x = 2pq = 2d^2f^2 \quad y = p^2 - q^2 = d^4 - f^4 = c^2 \quad z = p^2 + q^2$

$\frac{1}{2}xy = \frac{1}{2} \cdot 2 \cdot d^2 f^2 c^2 = (dfc)^2$

contradicting previous.

If $x^4 + y^4 = z^4$

then $(x^2)^2 = z^4 - y^4$.