

Spectrally and Inertially Arbitrary Sign Patterns

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Definitions

$n \times n$ sign pattern matrix

$$S \equiv [s_{ij}] \text{ with entries } s_{ij} \text{ in } \{+, -, 0\}$$

defines a sign pattern class of real matrices

$$Q(S) = \{A \equiv [a_{ij}] : \text{sign}(a_{ij}) = s_{ij} \forall i, j\}$$

\hat{S} is a superpattern of S if $\hat{s}_{ij} = s_{ij}$ when $s_{ij} \neq 0$

\hat{S} is a subpattern of S if $\hat{s}_{ij} = s_{ij}$ when $s_{ij} = 0$

inertia of real matrix A

$$i(A) = (i_+(A), i_-(A), i_0(A))$$

number of eigenvalues with positive, negative, zero real parts

matrix A is (negative) stable if $i(A) = (0, n, 0)$

inertia of pattern S

$$i(S) = \{i(A) : A \in Q(S)\}$$

pattern S is

sign stable if $i(S) = \{(0, n, 0)\}$

potentially stable if $(0, n, 0) \in i(S)$

pattern S is

inertially arbitrary (IAP)

if $(n_1, n_2, n_3) \in i(S) \forall n_1 + n_2 + n_3 = n$, with $n_i \geq 0$

spectrally arbitrary (SAP)

if for each real monic polynomial $r(x)$ with degree n , $\exists A \in Q(S)$

with characteristic polynomial $r(x)$

(that is, any self-conjugate set of complex numbers is the spectrum of some $A \in Q(S)$)

Note: If S is a SAP, then S is an IAP

minimal IAP / SAP

no proper subpattern has this property

EXAMPLES

$S_1 = \begin{bmatrix} - & + \\ + & - \end{bmatrix}$ is potentially stable, not sign stable

For example:

$\begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$ is negative stable

$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ has inertia $(1, 1, 0)$

is not an IAP or SAP: cannot have $(2, 0, 0) \in i(S_1)$

$$S_2 = \begin{bmatrix} - & + \\ - & + \end{bmatrix} \text{ is a minimal SAP}$$

(and thus is potentially stable and an IAP):

there exist $a, b, c > 0$ such that the characteristic polynomial

of $\begin{bmatrix} -a & 1 \\ -b & c \end{bmatrix}$ is any given real polynomial $x^2 + \alpha x + \beta$

(choose $c > 0$ s.t. $a = c + \alpha > 0$ and $b = \beta + c(c + \alpha) > 0$)

SAPs and IAPs are preserved under

negation, transposition, permutation similarity, signature
similarity

Tools Introduced

- (1) To show that S is a SAP: Nilpotent – Jacobian method
- find a nilpotent matrix $\hat{A} \in Q(S)$
 - let X be the matrix obtained from \hat{A} by replacing n nonzero entries of \hat{A} , say $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$, by variables x_1, x_2, \dots, x_n
 - show that the Jacobian of the coefficients of the characteristic polynomial w.r.t. the variables x_i is nonzero at $(x_1, \dots, x_n) = (\hat{x}_1, \dots, \hat{x}_n)$

Then the Implicit Function Theorem can be used to show that $\exists A \in Q(S)$ having any monic characteristic polynomial with coefficients sufficiently close to 0.

(2) To show that any superpattern of such a SAP is also a SAP:

For any pattern S that can be shown to be a SAP by (1), the Jacobian of the coefficients of the characteristic polynomial of any superpattern can be made arbitrarily close to that of S , and thus nonzero; again use the Implicit Function Theorem

Difficulty of extending the proof for T_n to $n > 7$:

need to find nilpotent $\hat{A} \in Q(T_n)$ and show that the Jacobian is nonzero

Related Results

Elsner, Olesky, vdD – LAA 2003

T_n is a SAP for $8 \leq n \leq 16$

(used MAPLE to find nilpotent matrices and to evaluate the Jacobian)

Elsner and Hershkowitz – LAA 2003

there exists $A \in Q(T_n)$ with spectrum arbitrarily close to any given self-conjugate set of complex numbers

use (close to-) Schwarz matrices

Miao and Li – LAA 2002

give some inertia results for the $(2r - 1)$ -diagonal matrix

$$S_{n,r} = \begin{bmatrix} - & + & + & \cdots & + & & & 0 \\ - & 0 & + & + & \cdots & + & & \\ - & - & 0 & + & + & & \ddots & \\ \vdots & - & - & \ddots & + & \ddots & & + \\ - & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & - & & \ddots & \ddots & 0 & \ddots & + \\ & & \ddots & & - & - & 0 & + \\ 0 & & & - & \cdots & - & - & + \end{bmatrix}$$

... and show that $S_{n,n-1}$ is an IAP for $n \geq 3$ (note $T_n = S_{n,2}$)

Gao and Shao – LaMA 2001

show that

$$\begin{bmatrix} - & + & + & \cdots & + \\ - & 0 & + & & + \\ - & - & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & + \\ - & - & \cdots & - & + \end{bmatrix}$$

is an IAP for all $n \geq 2$

Both papers above give explicit constructions for several cases and continuity arguments

**Relevant papers not directly considering
IAPs and SAPs:**

Hall, Li and Wang – LAA 2001

- characterize symmetric sign patterns that require unique inertia

Hall and Li – Numerical Math J. Chinese Univ. 2001

- specify possible inertias of certain symmetric sign pattern classes

Shao, Sun and Gao – LAA 2004

- characterize possible inertias of real symmetric matrices having symmetric star and nonnegative tridiagonal sign patterns

$$\begin{bmatrix} * & + & + & \cdots & + \\ + & * & & & \\ + & & * & & \\ \vdots & & & \ddots & \\ + & & & & * \end{bmatrix}$$

where $* \in \{+, -, 0\}$

$$\begin{bmatrix} * & + & & & & \\ + & * & + & & & \\ & + & * & \ddots & & \\ & & \ddots & * & \ddots & \\ & & & \ddots & * & + \\ & & & & + & * \end{bmatrix}$$

where $* \in \{+, 0\}$

- give explicit constructions, can use interlacing

Further SAP results

McDonald, Olesky, Tsatsomeros, vdD – LaMA 2003

- given any self-conjugate set of $n - 1$ complex numbers and a sufficiently large positive number, there exists an $n \times n$ normal matrix $A > 0$ with that spectrum
 - explicit construction using an orthogonal Soules matrix
 - the all $+$ sign pattern S can have any inertia with $i_+(S) \geq 1$
- any sign pattern with each column either all $+$ or all $-$ (at least one of each) is a SAP: called striped patterns
 - explicit construction using the above result to obtain principal submatrices that are all $+$ or all $-$
 - this was the first known SAP for all $n \geq 2$

Britz, McDonald, Olesky, van den Driessche – SIMAX 2004

- the first $n \times n$ classes of minimal SAPs are determined

e.g. class $W_n(k)$, where $n \geq 3$ and $0 \leq k \leq n - 2$, with $2n + k$ nonzeros
(Nilpotent–Jacobian method)

$$\begin{bmatrix}
 + & + & & & & & & & - \\
 + & & + & & & & & & - \\
 \vdots & & & \ddots & & & & & \vdots \\
 + & & & & + & & & & - \\
 + & & & & & - & & & 0 \\
 + & & & & & & - & & \vdots \\
 \vdots & & & & & & & \ddots & 0 \\
 + & & & & & & & & - \\
 + & & & & & & & & -
 \end{bmatrix}
 \left. \vphantom{\begin{bmatrix}
 + & + & & & & & & & - \\
 + & & + & & & & & & - \\
 \vdots & & & \ddots & & & & & \vdots \\
 + & & & & + & & & & - \\
 + & & & & & - & & & 0 \\
 + & & & & & & - & & \vdots \\
 \vdots & & & & & & & \ddots & 0 \\
 + & & & & & & & & - \\
 + & & & & & & & & -
 \end{bmatrix}} \right\} \begin{array}{l} k \\ \text{entries} \end{array}$$

Corollary: Any striped pattern (McDonald et al.) is a SAP.

Conjecture ($2n$ Conjecture)

An $n \times n$ SAP has at least $2n$ nonzero entries

Britz et al. prove that an irreducible $n \times n$ SAP has at least $2n - 1$ nonzero entries

MacGillivray, Tifenbach, vdD – LAA 2005

Characterize all minimal SAPs whose graph is a star (SAP = IAP)

Cavers, Kim, Shader and Vander Meulen – EJLA 2005

Find a SAP family $K_{n,r}$ with $2n$ nonzero entries using the N-J method, but not explicitly finding a nilpotent $A \in Q(K_{n,r})$

Bingham, Olesky, vdD – LAA Find a SAP family for n even with digraph having n 1-cycles and a cycle of each even length (about $5n/2$ nonzeros)

Cavers and Vander Meulen – LAA 2005

- determine some new minimal SAPs

e.g. the $n \times n$ class $D_{n,r}$ where $2 \leq r \leq n$ with $2n$ nonzero entries

$$\left. \begin{array}{l} r \\ \text{entries} \end{array} \right\} \left[\begin{array}{cccccccc} - & + & & & & & & \\ - & & + & & & & & \\ - & & & + & & & & \\ \vdots & & & & & & & \\ - & & & & + & & & \\ & & & & & \ddots & & \\ & - & & & & & + & \\ & & - & & & & & + \\ & & & \ddots & & & & + \\ & & & & - & & & + \end{array} \right]$$

For $r \geq 3$:

- minimal SAP if $\left\lceil \frac{n+1}{2} \right\rceil \leq r \leq n$

(use Nilpotent – Jacobian method)

Proves superpatterns $S_{n,n-1}$ of Miao and Li and patterns of Gao and Shao are SAPs

- not a SAP if $r = \left\lceil \frac{n-1}{2} \right\rceil$

(not potentially nilpotent)

- unknown if $3 \leq r \leq \left\lceil \frac{n-3}{2} \right\rceil$ or if $r = 2$ ($D_{n,2} = T_n$)

- give the first known example of a pattern that is an IAP and potentially nilpotent but not a SAP:

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & - & - \\ + & + & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}$$

(examine the characteristic polynomial)

- give a 7×7 reducible IAP that is not potentially nilpotent: it is the direct sum of an irreducible 2×2 IAP and a 5×5 sign pattern that is not an IAP

Kim, Olesky, vdD – LAA 2006

- give a family \mathcal{G}_{2k+1} for $k \geq 2$ of irreducible IAPs that are not potentially nilpotent (so not SAPs).

\mathcal{G}_{2k+1} has about $5n/2$ nonzero entries.

For $i_0 \geq 1$, a realization of \mathcal{G}_{2k+1} is identified with $2k - 1$ zero eigenvalues and a pair of pure imaginary eigenvalues, then the Implicit Function Theorem is used.

For $i_0 = 0$, matrices in \mathcal{G}_{2k+1} are constructed by recursion

- \mathcal{G}_5 and \mathcal{G}_7 are minimal IAPs
is \mathcal{G}_{2k+1} a minimal IAP for $k \geq 4$?
- For $n \geq 4$ even, find a family of irreducible IAPs that are not potentially nilpotent

Zero-nonzero Patterns

Matrix \mathcal{A} has entries in $\{*, 0\}$, where $*$ denotes a nonzero real number

Corpuz, McDonald – LaMA 2006

characterize all 4×4 zero-nonzero SAPs

Kim, McDonald, Olesky, vdD – LaMA

$$\mathcal{M} = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}$$

is not a zero-nonzero IAP: $(1, 0, 3), (2, 0, 2) \notin i(\mathcal{M})$

If \mathcal{A} is an $m \times m$ IAP, then $\mathcal{A} \oplus \mathcal{M}$ is an IAP (not a SAP)

For $n \geq 6$, $\exists n \times n$ IAP with $2n - 1$ nonzero entries:

$\bigoplus_m (\mathcal{A} \oplus \mathcal{M})$ order n is an IAP with $2n - m$ nonzero entries

Some conjectures and open problems

1. Conjecture: T_n is a SAP for all n . Proved by Nilpotent-Jacobian method for $n \leq 16$
2. $2n$ Conjecture: an $n \times n$ SAP has at least $2n$ nonzero entries
Proved for $n = 2$; $n = 3$ as all SAPs classified by Britz et al. and by Cavers VM (SAP=IAP); $n = 4$ as all zero-nonzero SAPs classified by Corpuz, McDonald
3. Is the Cavers, VM pattern $D_{n,r}$ a SAP if $3 \leq r \leq \left\lceil \frac{n-3}{2} \right\rceil$?
4. If S is a SAP or IAP, what are the properties of the cycles of the digraph of S ?
5. Spectrally arbitrary zero-nonzero patterns:
Interesting open questions in the recent work of Corpuz, McDonald, eg Does every zero-nonzero SAP have a signing that is a sign pattern SAP? True for $n \leq 4$. But a minimal SAP may be a signing of a zero-nonzero SAP that is not minimal)