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POSITIVE SEMIDEFINITE ZERO FORCING

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1 **Abstract.** The positive semidefinite zero forcing number $Z_+(G)$ of a graph G was introduced in [4]. We establish
2 a variety of properties of $Z_+(G)$: Any vertex of G can be in a minimum positive semidefinite zero forcing set (this
3 is not true for standard zero forcing). The graph parameters $\text{tw}(G)$ (tree-width), $Z_+(G)$, and $Z(G)$ (standard zero
4 forcing number) all satisfy the Graph Complement Conjecture (see [3]). Graphs having extreme values of the positive
5 semidefinite zero forcing number are characterized. The effect of various graph operations on positive semidefinite
6 zero forcing number and connections with other graph parameters are studied.

7 **Key words.** zero forcing number, maximum nullity, minimum rank, positive semidefinite, matrix, graph

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9 **1. Introduction.** Every graph discussed is simple (no loops or multiple edges), undirected,
10 and has a finite nonempty vertex set. In a graph G where some vertices S are colored black and the
11 remaining vertices are colored white, the *positive semidefinite color change rule* is: If W_1, \dots, W_k
12 are the sets of vertices of the k components of $G - S$ (note that it is possible that $k = 1$), $w \in W_i$,
13 $u \in S$, and w is the only white neighbor of u in the subgraph of G induced by $W_i \cup S$, then change
14 the color of w to black; in this case, we say u forces w and write $u \rightarrow w$. Given an initial set B
15 of black vertices, the *derived set* of B is the set of black vertices that results from applying the
16 positive semidefinite color change rule until no more changes are possible. A *positive semidefinite*
17 *zero forcing set* is an initial set B of vertices such that the derived set of B is all the vertices of G .
18 The *positive semidefinite zero forcing number* of a graph G , denoted $Z_+(G)$, is the minimum of $|B|$
19 over all positive semidefinite zero forcing sets $B \subseteq V(G)$. The positive semidefinite zero forcing
20 number is a variant of the (*standard*) *zero forcing number* $Z(G)$, which uses the same definition
21 with a different color change rule: If u is black and w is the only white neighbor of u , then change
22 the color of w to black. The (*standard*) zero forcing number was introduced in [1] as an upper
23 bound for maximum nullity, and the positive semidefinite zero forcing number was introduced in
24 [4] as an upper bound for positive semidefinite maximum nullity.

25 Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph*
26 of A , denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$.

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27 The *maximum positive semidefinite nullity* of G is

$$28 \quad M_+(G) = \max\{\text{null } A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

29 and *minimum positive semidefinite rank* of G is

$$30 \quad \text{mr}_+(G) = \min\{\text{rank } A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

31 The (standard) maximum nullity $M(G)$ and (standard) minimum rank $\text{mr}(G)$ use the same defini-
 32 tions omitting the requirement of positive semidefiniteness. It is clear that $\text{mr}_+(G) + M_+(G) = |G|$.
 33 In [4] it was shown that for every graph

$$34 \quad M_+(G) \leq Z_+(G).$$

35 It was also shown there that

$$36 \quad OS(G) + Z_+(G) = |G|$$

37 where $OS(G)$ is a graph parameter defined in [14], and in fact shown that the complement of an
 38 OS -set is a positive semidefinite zero forcing set and the complement of a positive semidefinite zero
 39 forcing set is an OS -set. The reader is referred to [14] for the definition of OS -set and $OS(G)$.

40 We establish a variety of properties of $Z_+(G)$. In Section 2 connections between zero forcing sets
 41 and OS -sets are applied to show that every vertex of G is in some minimum positive semidefinite
 42 zero forcing set (this is not true for standard zero forcing). It is also shown there that $T(G) \leq Z_+(G)$
 43 where $T(G)$ is the tree cover number of G , and cut-vertex reduction formulas for $TC(G)$ and $Z_+(G)$
 44 are established. In Section 3 it is shown that the graph parameters $\text{tw}(G)$ (tree-width), $Z_+(G)$,
 45 and $Z(G)$ (standard zero forcing number) all satisfy the Graph Complement Conjecture (see [3]).
 46 Graphs having extreme values of the positive semidefinite zero forcing number are characterized
 47 in Section 4. The effect of various graph operations on positive semidefinite zero forcing number
 48 and connections with other graph parameters are studied in Section 5.

49 There are a few more graph terms that we need to define. The subgraph $G[W]$ of $G = (V, E)$
 50 induced by $W \subseteq V$ is the subgraph with vertex set W and edge set $\{\{i, j\} \in E : i, j \in W\}$; $G - W$ is
 51 used to denote $G[V \setminus W]$. The graph $G - \{v\}$ is also denoted by $G - v$. The *complement* of a graph
 52 $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where \overline{E} consists of all two element sets from V that are not in
 53 E . The *union* of $G_i = (V_i, E_i)$ is $\bigcup_{i=1}^h G_i = (\bigcup_{i=1}^h V_i, \bigcup_{i=1}^h E_i)$. The *intersection* of $G_i = (V_i, E_i)$
 54 is $\bigcap_{i=1}^h G_i = (\bigcap_{i=1}^h V_i, \bigcap_{i=1}^h E_i)$ (provided the intersection of the vertices is nonempty). The *degree*
 55 of vertex v in graph G , $\deg_G v$, is the number of neighbors of v . A graph is *chordal* if it has no
 56 induced cycle of length 4 or more; clearly any induced subgraph of a chordal graph is chordal.

57 **2. Tree cover number, positive semidefinite zero forcing number, and maximum**
 58 **positive semidefinite nullity.** The *tree cover number* of a graph G , denoted $T(G)$, is defined as
 59 the minimum number of vertex disjoint trees occurring as induced subgraphs of G that cover all
 60 of the vertices of G , and was introduced by Barioli, Fallat, Mitchell, and Narayan in [5]. In that
 61 paper the authors show that for any outerplanar graph G , $M_+(G) = T(G)$ and if G is a chordal
 62 graph, then $T(G) \leq M_+(G)$. It is conjectured there that $T(G) \leq M_+(G)$ for every graph.

63 **2.1. Membership in a minimum positive semidefinite zero forcing set.** The next
 64 theorem is an interesting consequence of the connection between OS -number and Z_+ .

65 **THEOREM 2.1.** *If G is a graph and $v \in V(G)$, then there exist minimum positive semidefinite
 66 zero forcing sets B_1 and B_2 such that $v \in B_1$ and $v \notin B_2$.*

67 *Proof.* Let G be a graph and $v \in V(G)$. By Corollary 2.17 in [19], there exist OS -sets S_1 and
 68 S_2 with $|S_1| = |S_2| = OS(G)$, $v \notin S_1$ and $v \in S_2$. Then by [4, Theorem 3.6] $B_1 = \overline{S_1}$ and $B_2 = \overline{S_2}$
 69 are minimum positive semidefinite zero forcing sets, with $v \in B_1$ and $v \notin B_2$. \square

70 Note that the situation for positive semidefinite zero forcing as described by Theorem 2.1 is
 71 very different from (standard) zero forcing, where it is known that a graph can have a vertex that
 72 is not in any minimum zero forcing set. For example, a degree 2 vertex in a path P_n , $n \geq 3$ cannot
 73 be in a minimum zero forcing set for P_n . But we do have the extension to positive semidefinite of
 74 the property that no vertex is in every minimum zero forcing set.

75 **COROLLARY 2.2.** *If G is a connected graph of order greater than one, then*

$$76 \quad \bigcap_{B \in ZFS_+(G)} B = \emptyset,$$

77 *where $ZFS_+(G)$ is the set of all minimum positive semidefinite zero forcing sets of G .*

78 **2.2. Forcing trees.** Tree cover number can be viewed as a generalization of *path cover num-*
 79 *ber*, i.e., the minimum number of vertex disjoint paths occurring as induced subgraphs of G that
 80 cover all of the vertices of G . It is well known that path cover number $P(G)$ and maximum nullity
 81 $M(G)$ are noncomparable in general, but $P(G) \leq Z(G)$ for every graph G . The proof uses paths of
 82 forces, and we extend this to trees of positive semidefinite forces, thus showing that $T(G) \leq Z_+(G)$.
 83 Let G be a graph and B a positive semidefinite zero forcing set for G . Construct the derived set,
 84 listing the forces in the order in which they were performed. This list \mathcal{F} is a *chronological list of*
 85 *forces*. The terminology in the next definition will be justified in Theorem 2.4.

86 **DEFINITION 2.3.** Given a graph G , positive semidefinite zero forcing set B , chronological list
 87 of forces \mathcal{F} , and a vertex $b \in B$, define V_b to be the set of vertices w such that there is a sequence
 88 of forces $b = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$ in \mathcal{F} (the empty sequence of forces is permitted, i.e.,
 89 $b \in V_b$). The *forcing tree* T_b is the induced subgraph $T_b = G[V_b]$. The *forcing tree cover* (for the
 90 chronological list of forces \mathcal{F}) is $\mathcal{T} = \{T_b \mid b \in B\}$. An *optimal forcing tree cover* is a forcing tree
 91 cover from a chronological list of forces of a minimum positive semidefinite zero forcing set.

92 A graph with positive semidefinite zero forcing set with forces marked and the resulting forcing
 93 tree cover are shown in Figure 2.1.

94 **THEOREM 2.4.** *Assume G is a graph, B is a positive semidefinite zero forcing set of G , \mathcal{F} is
 95 a chronological list of forces of B , and $b \in B$. Then*

- 96 1. T_b is a tree.
- 97 2. The forcing tree cover $\mathcal{T} = \{T_b : b \in B\}$ is a tree cover of G .
- 98 3. $T(G) \leq Z_+(G)$.

99 *Proof.* The sets V_b of vertices forced by distinct $b \in B$ are disjoint because each vertex of G is
 100 forced only once. If a graph H is not a tree, then $Z_+(H) > 1$ (this follows from the result that H

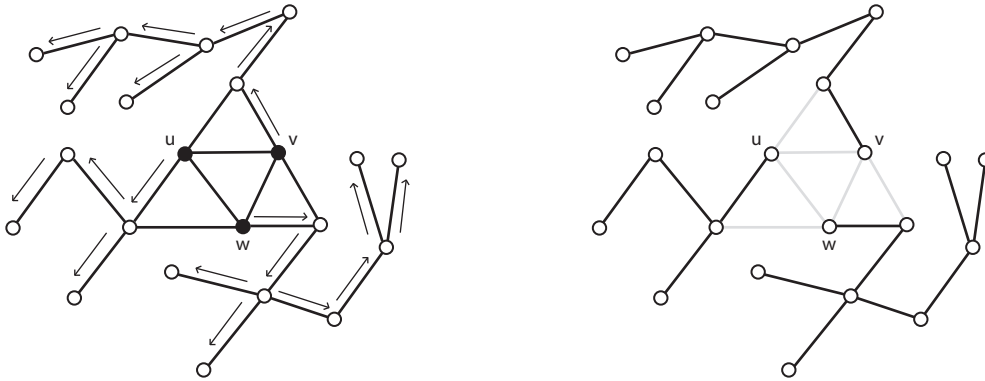


Fig. 2.1: A graph with forces marked, and the resulting forcing tree cover

101 not a tree implies $M_+(H) > 1$ [15]). So if $T_b = G[V_b]$ is not a tree, then there must exist a vertex
 102 $v \in V_b \setminus \{b\}$ such that either $v \in B$ or v was forced through a sequence of forces from some element
 103 of B not equal to b . In either case, this contradicts the fact that the sets V_b of vertices forced by
 104 different elements of B are disjoint. Thus T_b is a tree.

105 Since each vertex $b \in B$ forces an induced subtree, the trees forced by distinct elements of B
 106 are disjoint, and B is a positive semidefinite zero forcing set, $\mathcal{T} = \{T_b : b \in B\}$ is a tree cover of
 107 G . Now suppose that B is a minimum positive semidefinite zero forcing set for G . Since \mathcal{T} is a
 108 tree cover of G , $T(G) \leq |\mathcal{T}| = |B| = Z_+(G)$. \square

109 **2.3. Cut-vertex reduction.** Cut-vertex reduction is a standard technique in the study of
 110 minimum rank. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected.
 111 Suppose $G_i, i = 1, \dots, h$ are graphs of order at least two, there is a vertex v such that for all
 112 $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$ (if $h \geq 2$, then clearly v is a cut-vertex of G). Then it is
 113 established in [16] that

$$114 \quad \text{mr}_+(G) = \sum_{i=1}^h \text{mr}_+(G_i).$$

115 Because $\text{mr}_+(G) + M_+(G) = |G|$, this is equivalent to

$$116 \quad M_+(G) = \left(\sum_{i=1}^h M_+(G_i) \right) - h + 1. \quad (2.1)$$

117 It is shown in [19] that

$$118 \quad OS(G) = \sum_{i=1}^h OS(G_i).$$

119 Because $OS(G) + Z_+(G) = |G|$ [4], this is equivalent to

$$120 \quad Z_+(G) = \left(\sum_{i=1}^h Z_+(G_i) \right) - h + 1. \quad (2.2)$$

121 An analogous reduction formula is valid for tree cover number.

122 PROPOSITION 2.5. *Suppose $G_i, i = 1, \dots, h$ are graphs, there is a vertex v such that for all*
 123 *$i \neq j, G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$. Then*

$$124 \quad T(G) = \left(\sum_{i=1}^h T(G_i) \right) - h + 1. \quad (2.3)$$

125 *Proof.* For each G_i , let \mathcal{T}_i be a tree cover of minimum cardinality. In each \mathcal{T}_i , there exists some
 126 T_i such that $v \in V(T_i)$. Define $T_v = \bigcup_{i=1}^h T_i$. Then $\mathcal{T} = \bigcup_{i=1}^h (\mathcal{T}_i \setminus \{T_i\}) \cup \{T_v\}$ is a tree cover for
 127 G . Therefore $T(G) \leq \left(\sum_{i=1}^h T(G_i) \right) - (h - 1)$.

128 Let \mathcal{T} be a minimum tree cover for G . Let T_v be the tree that includes v . For $i = 1, \dots, h$,
 129 define $T_{v,i} = T_v \cap G_i$. For each $T \in \mathcal{T}$ such that $v \notin V(T)$, T is a subgraph of some G_i . Define
 130 $\mathcal{T}_i = \{T_{v,i}\} \cup \{T \in \mathcal{T} : T \text{ is a subgraph of } G_i\}$. Since \mathcal{T}_i is a tree cover of G_i , $T(G_i) \leq |\mathcal{T}_i|$. Thus

$$131 \quad \sum_{i=1}^h T(G_i) \leq \sum_{i=1}^h |\mathcal{T}_i| = |\mathcal{T}| + h - 1 = T(G) + h - 1. \quad \square$$

132 We have the following immediate consequences of the cut-vertex reduction formulas (2.1),
 133 (2.2), and (2.3).

134 COROLLARY 2.6. *Suppose $G_i, i = 1, \dots, h$ are graphs, there is a vertex v such that for all*
 135 *$i \neq j, G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$.*

- 136 1. *If $M_+(G_i) = Z_+(G_i)$ for all $i = 1, \dots, h$, then $M_+(G) = Z_+(G)$.*
- 137 2. *If $T(G_i) = Z_+(G_i)$ for all $i = 1, \dots, h$, then $T(G) = Z_+(G)$.*
- 138 3. *If $M_+(G_i) = T(G_i)$ for all $i = 1, \dots, h$, then $M_+(G) = T(G)$.*

139 COROLLARY 2.7. *Suppose H is a graph, T is a tree, and H and T intersect in a single vertex.*
 140 *For $G = H \cup T$,*

- 141 1. $M_+(G) = M_+(H)$.
- 142 2. $Z_+(G) = Z_+(H)$.
- 143 3. $T(G) = T(H)$.

144 **3. Graph Complement Conjecture.** The *graph complement conjecture* or GCC (Conjec-
 145 ture 3.1 below) was stated at the 2006 American Institute of Mathematics workshop ‘‘Spectra of
 146 Families of Matrices described by Graphs, Digraphs, and Sign Patterns’’ [2].

147 CONJECTURE 3.1 (GCC). [10] *For any graph G ,*

$$148 \quad \text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2,$$

149 *or equivalently,*

$$150 \quad M(G) + M(\overline{G}) \geq |G| - 2. \quad (3.1)$$

151

152 The conjecture (3.1), which is a Nordhaus-Gaddum type problem, was generalized in [3] to a
 153 variety of graph parameters related to maximum nullity, including positive semidefinite maximum
 154 nullity. For a graph parameter β related to maximum nullity, the graph compliment conjecture for
 155 β , GCC_β , is

$$156 \quad \beta(G) + \beta(\overline{G}) \geq |G| - 2.$$

157 With this notation, GCC can be denoted GCC_M , and the graph compliment conjecture for positive
 158 semidefinite maximum nullity is denoted GCC_{M^+} . In this section we establish that GCC_{tw} , and
 159 hence GCC_{Z^+} and GCC_Z are true.

160 A *tree decomposition* of a graph G is a pair (T, \mathcal{W}) , where T is a tree and $\mathcal{W} = \{W_t : t \in V(T)\}$
 161 is a collection of subsets of $V(G)$ with the following properties:

- 162 1. $\bigcup_{t \in V(T)} W_t = V(G)$.
- 163 2. Every edge of G has both ends in some W_t .
- 164 3. If $t_1, t_2, t_3 \in V(T)$ and t_2 lies on a path from t_1 to t_3 , then $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$.

165 The *bags* of the tree decomposition are the subsets W_t . The *width* of a tree decomposition is
 166 $\max\{|W_t| - 1 : t \in V(T)\}$, and the *tree-width* $\text{tw}(G)$ of G is the minimum width of any tree
 167 decomposition of G . Tree-width can be characterized in terms of the clique number of chordal
 168 graphs and in terms of partial k -trees. The greatest integer r such that $K_r \subseteq G$ is the *clique*
 169 *number* $\omega(G)$. It follows from [11, Corollary 12.3.12] that

$$170 \quad \text{tw}(G) = \min\{\omega(H) - 1 : V(H) = V(G), G \subseteq H, \text{ and } H \text{ is chordal}\} \quad (3.2)$$

171 Note that in [11, Corollary 12.3.12], the minimum is taken over all chordal supergraphs; however,
 172 if $H \supseteq G$ is chordal, then $H[V(G)] \supseteq G$, $H[V(G)]$ is chordal, and $\omega(H[V(G)]) \leq \omega(H)$ and so
 173 we may take the minimum over only those chordal supergraphs with the same vertex set. For a
 174 positive integer k , a *k-tree* is constructed inductively by starting with a complete simple graph on
 175 $k + 1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. A
 176 *partial k-tree* is a subgraph of a k -tree. Then $\text{tw}(G)$ is the least positive integer k such that G is a
 177 partial k -tree [9, F12, p. 111].

178 A graph is *co-chordal* if its complement is chordal. A *triangulation* of a graph G is a chordal
 179 graph that is obtained from G by adding edges. A graph G is a *split graph* if there is a nonempty
 180 set $S \subset V(G)$ such that S is an independent set in G and $G - S$ is a clique. This definition of
 181 split graph differs slightly from the definition given in [17], where neither $S \neq \emptyset$ nor $S \neq V(G)$
 182 is required. However, the two definitions are equivalent for graphs of order at least two: In case
 183 $S = V(G)$ is independent, then for any vertex $v \in V(G)$, $S' = S \setminus \{v\}$ is independent and $G - S'$
 184 is an (order 1) clique. In case $S = \emptyset$ (so G is a clique), then for any vertex $v \in V(G)$, $S' = \{v\}$ is
 185 independent and $G - S'$ is a clique.

186 **THEOREM 3.2.** *Let $G = (V, E)$ be a graph of order at least two. Let H be a chordal supergraph*
 187 *of G and F be a co-chordal subgraph of G with $V(G) = V(H) = V(F)$. Then for some clique of*
 188 *H and some clique of \overline{F} , the union of their vertex sets is all of V .*

189 *Proof.* Since $F \subseteq G \subseteq H$ and H is chordal, H is a triangulation of F . Let $\Gamma \subseteq H$ be a minimal
 190 triangulation of F . Since F is co-chordal, it is $2K_2$ free (see, for example [17, Fact 2]), so by [17,

191 Corollary 7], Γ is a split graph. Let S be an independent set of vertices such that $\Gamma - S$ is a clique.
 192 Since S is independent, $\overline{\Gamma[S]} = \overline{\Gamma}[S]$ is also a clique. Since $\Gamma \subseteq H$, $\Gamma - S \subseteq H$ and since $F \subseteq \Gamma$
 193 with the same vertex set, $\overline{\Gamma} \subseteq \overline{F}$ and so $\overline{\Gamma}[S] \subseteq \overline{F}$. Finally, it is obvious that $(V \setminus S) \cup S = V$. \square

194 THEOREM 3.3. GCC_{tw} is true, i.e., $\text{tw}(G) + \text{tw}(\overline{G}) \geq |G| - 2$.

195 *Proof.* Let G be a graph. By (3.2), we can choose chordal graphs $H \supseteq G$ and $H' \supseteq \overline{G}$ such
 196 that $\omega(H) = \text{tw}(G) + 1$, $\omega(H') = \text{tw}(\overline{G}) + 1$, and $V(G) = V(H) = V(H')$. Observe that Theorem
 197 3.2 can be applied with H as H and $\overline{H'}$ as F in the theorem. So there exist cliques $K_r \subseteq H$ and
 198 $K_{r'} \subseteq H'$ such that $V(G) = V(K_r) \cup V(K_{r'})$. Therefore,

$$199 \quad |G| = |V(K_r) \cup V(K_{r'})| \leq |K_r| + |K_{r'}| \leq \omega(H) + \omega(H') = \text{tw}(G) + \text{tw}(\overline{G}) + 2. \quad \square$$

200 Since for every graph G , $\text{tw}(G) \leq Z_+(G) \leq Z(G)$, we have the following corollary.

201 COROLLARY 3.4. GCC_{Z_+} and GCC_Z are true, i.e.,

$$202 \quad Z_+(G) + Z_+(\overline{G}) \geq |G| - 2 \quad \text{and} \quad Z(G) + Z(\overline{G}) \geq |G| - 2.$$

203

204 Note that GCC_{Z_+} also follows from [18, Proposition 9].

205 **4. Graphs with extreme positive semidefinite zero forcing number.** In this section
 206 we show that for graphs having very low or very high maximum positive semidefinite nullity or
 207 positive semidefinite zero forcing number, these two parameters are equal. Since characterizations
 208 of graphs having very low or very high maximum positive semidefinite nullity are known, these
 209 extend to graphs having very low or very high positive semidefinite zero forcing number.

210 It is well known that $M_+(G) = 1$ if and only if G is a tree if and only if $Z_+(G) = 1$ (the first
 211 equivalence is established in [15], and the latter follows from $M_+(G) \leq Z_+(G)$ and the fact that any
 212 one vertex is a positive semidefinite zero forcing set for a tree). Graphs that have $M_+(G) = 2$ are
 213 characterized in [15] (note that here a graph is required to be simple whereas in [15] multigraphs
 214 are considered).

215 A connected graph is *nonseparable* if it does not have a cut-vertex. A *block* of a graph is a
 216 maximal nonseparable subgraph.

217 THEOREM 4.1. *Let G be a graph. The following are equivalent.*

- 218 1. $Z_+(G) = 2$,
- 219 2. $M_+(G) = 2$,
- 220 3. *Either*
 - 221 (a) G is the disjoint union of two trees, or
 - 222 (b) G is connected, exactly one block of G has a cycle, and G does not have a K_4 or T_3
 223 minor.

224 *Proof.* (2) \Leftrightarrow (3): This follows from Theorems 4.3 and 2.2 in [15] and the fact that $M_+(G) = 1$
 225 if and only if G is a tree.

226 (1) \Rightarrow (2) because $M_+(G) \leq Z_+(G)$ and $M_+(G) = 1 \Leftrightarrow Z_+(G) = 1$.

227 (3) \Rightarrow (1): By hypothesis, G has no K_4 minor, so $\text{tw}(G) \leq 2$ (see [9, F31, p. 112]). It is shown
 228 in [12] that if $\text{tw}(G) \leq 2$, then $Z_+(G) = M_+(G)$. (Note that [12] defines tree-width in terms of
 229 partial k -trees, but as noted in Section 3, that definition is equivalent to the standard definition
 230 used here.) \square

231 **COROLLARY 4.2.** *If $Z_+(G) \leq 3$, then $Z_+(G) = M_+(G)$.*

232 *Proof.* If $Z_+(G) = 3$, then $M_+(G) \leq 3$, but $M_+(G) \leq 2$ would imply $Z_+(G) \leq 2$ by Theorem
 233 4.1 and the fact that $M_+(G) = 1 \Leftrightarrow Z_+(G) = 1$. \square

234 Observe that $Z_+(V_8) = 4$ but $M_+(V_8) = 3$, so for $Z_+(G) \geq 4$ there is no result analogous to
 235 Corollary 4.2.

236 Theorem 4.4 below, which characterizes high positive semidefinite zero forcing number, follows
 237 from the characterization of graphs having $\text{mr}_+(G) \leq 2$ in [7], using the parameter mz_+ and the
 238 next proposition. Define $\text{mz}_+(G) = |G| - Z_+(G)$. Since $M_+(G) \leq Z_+(G)$, $\text{mz}_+(G) \leq \text{mr}_+(G)$.
 239 The proof of Proposition 4.3 below is the same as the proof of Proposition 4.4 in [1].

240 **PROPOSITION 4.3.** *If H is an induced subgraph of G , then $\text{mz}_+(H) \leq \text{mz}_+(G)$.*

241 **THEOREM 4.4.** *Let G be a graph. The following are equivalent.*

- 242 1. $Z_+(G) \geq |G| - 2$,
- 243 2. $M_+(G) \geq |G| - 2$,
- 244 3. G has no induced $P_4, K_{1,3}, P_3 \dot{\cup} K_2, 3K_2$

245 *Proof.* (1) \Rightarrow (3) by Proposition 4.3, because $\text{mz}_+(H) = 3$ for $H = P_4, K_{1,3}, P_3 \dot{\cup} K_2$, or $3K_2$.
 246 (3) \Rightarrow (2) by Theorem 8 in [7]. (2) \Rightarrow (1) since $M_+(G) \leq Z_+(G)$. \square

247 It is clear that $M_+(G) = |G|$ if and only if G has no edges, and the same is true for $Z_+(G)$.
 248 Similarly, $M_+(G) = |G| - 1 \Leftrightarrow G = K_r \cup sK_1 \Leftrightarrow Z_+(G) = |G| - 1$. The next corollary is analogous
 249 to Corollary 4.2.

250 **COROLLARY 4.5.** *If $M_+(G) \geq |G| - 3$, then $M_+(G) = Z_+(G)$.*

251 5. Effects of graph operations on Z_+ .

252 We examine the effect of various graph operations, including vertex deletion, edge deletion,
 253 edge subdivision, and edge contraction on positive semidefinite zero forcing number.

254 **5.1. Vertex deletion.** The effect of vertex deletion (and edge deletion) on the (standard)
 255 zero forcing number was established in [13], where this was described using the language of spreads,
 256 i.e., the difference between the parameter evaluated on G and on G with a vertex or edge deleted.
 257 In this section we examine the effect of vertex deletion on positive semidefinite zero forcing number.

258 **DEFINITION 5.1.** *Let G be a graph and v be a vertex in G .*

- 259 1. *The positive semidefinite rank spread of v is $r_v^+(G) = \text{mr}_+(G) - \text{mr}_+(G - v)$.*
- 260 2. *The positive semidefinite null spread of v is $n_v^+(G) = M_+(G) - M_+(G - v)$.*
- 261 3. *The positive semidefinite zero spread of v is $z_v^+(G) = Z_+(G) - Z_+(G - v)$.*

262 OBSERVATION 5.2. For any graph G and vertex v ,

- 263 1. $0 \leq r_v^+(G)$.
- 264 2. $n_v^+(G) \leq 1$.
- 265 3. $r_v^+(G) + n_v^+(G) = 1$.

266 The proof of the next proposition is the same as part of the proof of Theorem 2.3 in [13].

267 PROPOSITION 5.3. Let G be a graph and v be a vertex in G . Then $Z_+(G - v) \geq Z_+(G) - 1$,
 268 so $z_v^+(G) \leq 1$.

269 However, there is no upper bound for $r_v^+(G)$ and no lower bound for $n_v^+(G)$ and $z_v^+(G)$ as
 270 exhibited in the following example.

271 EXAMPLE 5.4. The complete bipartite graph $K_{1,s}$ with $s \geq 2$ has $\text{mr}_+(K_{1,s}) = s$ and
 272 $M_+(K_{1,s}) = 1 = Z_+(K_{1,s})$. However if v is the cut-vertex, then $K_{1,s} - v$ has no edges and
 273 thus $\text{mr}_+(K_{1,s} - v) = 0$ and $M_+(K_{1,s} - v) = s = Z_+(K_{1,s} - v)$. Thus $r_v^+(K_{1,s}) = s$ and
 274 $n_v^+(K_{1,s}) = 1 - s = z_v^+(K_{1,s})$.

275 As is the case with (standard) zero forcing number and maximum nullity [13], the parameters
 276 $n_v^+(G)$ and $z_v^+(G)$ are not comparable.

277 EXAMPLE 5.5. The graph V_8 (also known as the Möbius ladder on 8 vertices) shown in
 278 Figure 5.1a has $M_+(G) = 3$ and $Z_+(G) = 4$ [19, 4]. Since $\{1, 2, 3\}$ is a positive semidefinite
 279 zero forcing set for $V_8 - 8$, $Z_+(V_8 - 8) \leq 3$. Then by Corollary 4.2, $M_+(V_8 - 8) = Z_+(V_8 - 8)$,
 280 so $n_8^+(V_8) < z_8^+(V_8)$. (It is not difficult to find a matrix $A \in \mathcal{S}_+(V_8 - 8)$ with $\text{rank } A = 4$, so
 281 $M_+(V_8 - 8) \geq 3$, $M_+(V_8 - 8) = Z_+(V_8 - 8) = 3$, and $n_8^+(V_8) = 0$ and $z_8^+(V_8) = 1$.)

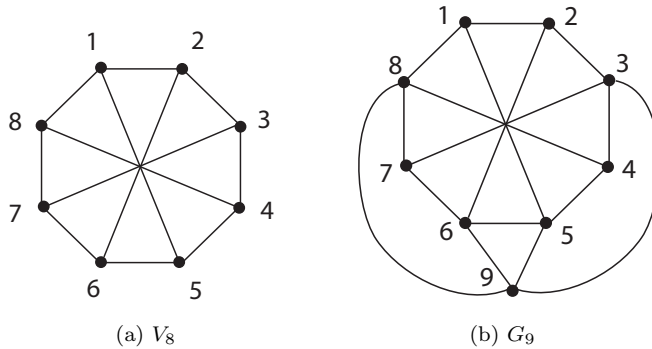


Fig. 5.1: The graphs V_8 and G_9

282 EXAMPLE 5.6. The graph G_9 in Figure 5.1b has a positive semidefinite zero forcing set

283 $\{3, 4, 7, 8\}$ so $Z_+(G_9) \leq 4$. Since

$$284 \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 3 & -5 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & -4 & 1 & 0 & -1 \end{bmatrix}$$

285 is an orthogonal representation of G_9 in \mathbb{R}^5 (i.e., $B^T B \in \mathcal{S}_+(G_9)$), $M_+(G_9) \geq 4$. Thus $Z_+(G_9) =$
 286 $M_+(G_9) = 4$. Since $G_9 - 9 = V_8$, $z_9^+(G_9) < n_9^+(G_9)$ (in fact, $z_9^+(G_9) = 0$ and $n_9^+(G_9) = 1$).

287 As in [13], we have the following observation.

288 **OBSERVATION 5.7.** *Let G be a graph such that $M_+(G) = Z_+(G)$ and let v be a vertex of G .*

- 289 1. $n_v^+(G) \geq z_v^+(G)$.
- 290 2. If $z_v^+(G) = 1$, then $n_v^+(G) = 1$.

291 In the case of standard maximum nullity and zero forcing number, $M(G) = Z(G)$ and $n_v(G) =$
 292 -1 imply $z_v(G) = -1$. However, since there are no lower bounds on $z_v^+(G)$ and $n_v^+(G)$, we do not
 293 have any bound based on $n_v^+(G) = -1$, as the next example shows.

294 **EXAMPLE 5.8.** Let H be the graph obtained from G_9 in Example 5.6 by appending two leaves
 295 to vertex 9. Then by cut-vertex reduction (2.1) and (2.2), $M_+(H) = 4 + 1 + 1 - 3 + 1 = Z_+(G)$.
 296 Since $H - 9 = V_8 \dot{\cup} 2K_1$, $M_+(H) = 5$ and $Z_+(H) = 6$. Thus $n_9^+(H) = -1$ and $z_9^+(H) = -2$.

297 A tree cover \mathcal{T} of G contains a vertex v as a *singleton* if $\{v\}$ (with no other vertices and no
 298 edges) is one of the trees in \mathcal{T} . The proof of the next proposition is the same as the proof of
 299 Theorem 2.7 in [13].

300 **PROPOSITION 5.9.** *Let G be a graph and $v \in V(G)$. Then there exists an optimal forcing tree*
 301 *cover of G that contains v as a singleton if and only if $z_v^+(G) = 1$.*

302 **REMARK 5.10.** For the (standard) zero forcing number Z , we know that if G is a graph,
 303 $v \in V(G)$, B is a minimum zero forcing set, and $v \in B$, then $z_v(G) \geq 0$. However, this is not the
 304 case for $z_v^+(G)$, because for any vertex v , there is a minimum positive semidefinite zero forcing set
 305 containing v by Theorem 2.1, yet there are vertices that have negative spread (such as in Example
 306 5.4).

307 For a graph G , the *neighborhood* of $v \in V(G)$ is $N_G(v) = \{w \in V(G) : w \text{ is adjacent to } v\}$.
 308 Vertices v and w of G are called *duplicate vertices* if $N_G(v) \cup \{v\} = N_G(w) \cup \{w\}$. Observe that
 309 duplicate vertices are necessarily adjacent. It was shown in [8] that if v is a duplicate vertex in a
 310 connected graph G of order at least three, then $\text{mr}_+(G-v) = \text{mr}_+(G)$, so $M_+(G-v) = M_+(G) - 1$.

311 **PROPOSITION 5.11.** *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$,*
 312 *then $Z_+(G-v) = Z_+(G) - 1$, or equivalently, $z_v^+(G) = 1$.*

313 *Proof.* Choose a minimum positive semidefinite zero forcing set B that contains v . We show
 314 that $B - \{v\}$ is a positive semidefinite zero forcing set for $G - v$. Proposition 5.3 then implies that
 315 $B - \{v\}$ is a minimum positive semidefinite zero forcing set for $G - v$.

316 Observe that in G , unless v forces w , v cannot perform a force until w is black. If v does not

317 force w in G , then either $w \in B$ or there is a $u \in N_G(w)$ such that $u \rightarrow w$. In the latter case, u also
318 forces w in $G - v$ starting with black vertices $B - \{v\}$. Then in $G - v$, w can perform any forces
319 that v had performed in G . So if v does not force w in G , then $B - \{v\}$ is a positive semidefinite
320 zero forcing set for $G - v$.

321 So assume v forces w , then at the stage at which $v \rightarrow w$, all vertices in $N_G(v) - \{w\}$ are
322 black. So in $G - v$, $B - \{v\}$ can still force all the vertices in $N_{G-v}(w)$. Since $|G| \geq 3$ and G is
323 connected, $N_{G-v}(w) \neq \emptyset$, and any $u \in N_{G-v}(w)$ can force w (since w is an isolated vertex after
324 all the currently black vertices are deleted from $G - v$). As before, all remaining forces can then
325 be performed. Therefore $B - \{v\}$ is a positive semidefinite zero forcing set. \square

326 **5.2. Edge deletion.** If e is an edge of G , then $G - e$ is the graph obtained from G by
327 deleting e . In this section we examine the effect of edge deletion on positive semidefinite zero
328 forcing number, using spread terminology.

329 **DEFINITION 5.12.** *Let G be a graph and e be an edge in G .*

- 330 1. *The positive semidefinite rank edge spread of e is $r_e^+ = \text{mr}_+(G) - \text{mr}_+(G - e)$.*
- 331 2. *The positive semidefinite null edge spread of e is $n_e^+(G) = M_+(G) - M_+(G - e)$.*
- 332 3. *The positive semidefinite zero edge spread of e is $z_e^+(G) = Z_+(G) - Z_+(G - e)$.*

333 **OBSERVATION 5.13.** *For any graph G and edge e of G , $r_e^+(G) + n_e^+(G) = 0$.*

334 **PROPOSITION 5.14.** *Let G be a graph and $e = \{i, j\}$ be an edge in G . Then*

- 335 1. $-1 \leq r_e^+(G) \leq 1$,
- 336 2. $-1 \leq n_e^+(G) \leq 1$,
- 337 3. $-1 \leq z_e^+(G) \leq 1$.

338 *Proof.* Nylen [20] established that the (standard) rank edge spread is between -1 and 1 , and
339 the same proof establishes that $r_e^+(G) \leq 1$. For the other inequality in part (1), choose a matrix
340 $A \in \mathcal{S}_+(G)$ having rank $A = \text{mr}_+(G)$, and let \mathbf{e}_k denote the k th standard basis vector in \mathbb{R}^n . Define
341 $A' = A + (\mathbf{e}_i - a_{ij}\mathbf{e}_j)(\mathbf{e}_i - a_{ij}\mathbf{e}_j)^T$. Then $A' \in \mathcal{S}_+(G - e)$ and $\text{rank } A' \leq \text{rank } A + 1 = \text{mr}_+(G) + 1$,
342 so $r_e^+(G) \geq -1$. Part (2) follows from part (1) and Observation 5.13. Part (3) can be proven by the
343 same method used to prove Theorem 2.17 in [13] (although Theorem 2.1 could be used to simplify
344 the proof). \square

345 As is the case with (standard) zero forcing number and maximum nullity [13], the parameters
346 $n_e^+(G)$ and $z_e^+(G)$ are not comparable.

347 **EXAMPLE 5.15.** The graph V_8 has $M_+(V_8) = 3$ and $Z_+(V_8) = 4$ [19, 4]. Consider the edge
348 $e = \{1, 8\}$. Since $\{1, 2, 3\}$ is a positive semidefinite zero forcing set for $V_8 - e$, $Z_+(V_8 - e) \leq 3$.
349 Then by Corollary 4.2, $M_+(V_8 - e) = Z_+(V_8 - e)$, so $n_8^+(V_8) < z_8^+(V_8)$.

350 **EXAMPLE 5.16.** In Example 5.6 it was shown that the graph G_9 has $Z_+(G_9) = M_+(G_9) = 4$.
351 Let $e_1 = \{3, 9\}$, $e_2 = \{5, 9\}$, $e_3 = \{6, 9\}$, $e_4 = \{8, 9\}$. Define $H_0 = G_9$ and $H_k = G_9 - \{e_1, \dots, e_k\}$

352 for $k = 1, \dots, 4$. Note that $H_4 = V_8 \dot{\cup} K_1$, so $Z_+(H_4) = 5$ and $M_+(H_4) = 4$. Since

353
$$-1 = Z_+(H_0) - Z_+(H_4) = z_{e_1}^+(H_0) + z_{e_2}^+(H_1) + z_{e_3}^+(H_2) + z_{e_4}^+(H_3), \text{ and}$$

354
$$0 = M_+(H_0) - M_+(H_4) = n_{e_1}^+(H_0) + n_{e_2}^+(H_1) + n_{e_3}^+(H_2) + n_{e_4}^+(H_3),$$

355 necessarily there exists a $k \in \{1, 2, 3, 4\}$ such that $z_{e_k}^+(H_{k-1}) < n_{e_k}^+(H_{k-1})$.

356 **OBSERVATION 5.17.** *Let G be a graph such that $M_+(G) = Z_+(G)$ and let e be an edge of G .*

- 357 1. $n_e^+(G) \geq z_e^+(G)$.
 358 2. If $z_e^+(G) = 1$, then $n_e^+(G) = 1$.
 359 3. If $n_e^+(G) = -1$, then $z_e^+(G) = -1$.

360 The proof of the next proposition is the same as the proof of Theorem 2.21 in [13].

361 **PROPOSITION 5.18.** *Let G be a graph and $e \in E(G)$. If $z_e^+(G) = -1$, then for every optimal*
 362 *forcing tree cover of G , e is an edge in some forcing tree. Equivalently, if there is an optimal*
 363 *forcing tree cover of G such that e is not an edge in any tree, then $z_e^+(G) \geq 0$.*

364 **QUESTION 5.19.** *Is the converse of Proposition 5.18 true? That is, if G is a graph, e is an*
 365 *edge of G , and $z_e^+(G) \geq 0$, must there exist an optimal forcing tree cover \mathcal{T} of G such that e is not*
 366 *an edge of any tree in \mathcal{T} ?*

367 **PROPOSITION 5.20.** *Let G be a graph and $e = \{v, w\}$ be an edge of G . If $z_e^+(G) = 1$, then*
 368 *there exists an optimal forcing tree cover \mathcal{T} , such that e is not an edge of any tree in \mathcal{T} .*

369 *Proof.* Let G be a graph and $e = \{v, w\}$ be an edge of G with $z_e^+(G) = 1$. Since $z_e^+(G) = 1$
 370 we know that $Z_+(G) = Z_+(G - e) + 1$. Let B be a minimum positive semidefinite zero forcing
 371 set for $G - e$ such that $v \in B$. Note that B is not a positive semidefinite zero forcing set for G
 372 since $|B| < Z_+(G)$. Furthermore, $w \notin B$ because if it were, then adding the edge e back into our
 373 graph would not change what v and w could force, implying that B would force G . Now we let
 374 $B' = B \cup \{w\}$. Then B' forces G and $|B'| = Z_+(G)$, so B' is a minimum positive semidefinite zero
 375 forcing set for G and e is not in the forcing tree cover of any chronological forces of B' . \square

376 The converse of Proposition 5.20 is false.

377 **EXAMPLE 5.21.** For the edge e of the graph G shown in Figure 5.2, $Z_+(G) = Z_+(G - e) = 2$,
 378 so $z_e^+(G) = 0$, but e is not in any tree in the forcing tree cover of the chronological list of forces
 379 shown in Figure 5.2.

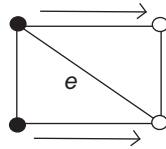


Fig. 5.2: A chronological list of forces in the graph G that does not contain edge e

380 **5.3. Edge subdivision and edge contraction.** The effect of edge contraction and edge
 381 subdivision on the (standard) zero forcing number was established in [21]. The *contraction* of edge

382 $e = \{u, v\}$ of G , denoted G/e , is obtained from G by identifying the vertices u and v , deleting any
 383 loops that arise in this process, and replacing any multiple edges by a single edge. In [21] it is
 384 shown that $Z(G) - 1 \leq Z(G/e) \leq Z(G) + 1$. The first inequality remains true but the second does
 385 not.

386 PROPOSITION 5.22. *Let G be a graph and $e = \{u, v\} \in E(G)$. Then $Z_+(G) - 1 \leq Z_+(G/e)$.*

387 *Proof.* Let w be the vertex of G/e obtained by identifying u and v . Choose a minimum positive
 388 semidefinite zero forcing set B' of G/e that contains w (this is possible by Theorem 2.1). Then
 389 $B = B' \setminus \{w\} \cup \{u, v\}$ is a positive semidefinite zero forcing set for G , so $Z_+(G) \leq Z_+(G/e) + 1$. \square

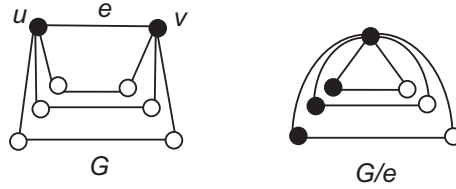


Fig. 5.3: A graph G with $Z_+(G/e) = Z_+(G) + 2$.

390 EXAMPLE 5.23. Let G be the graph obtained from k copies of C_4 by identifying a common
 391 edge $e = \{u, v\}$ as shown shown on the left in Figure 5.3 for $k = 3$; G/e as shown on the right in
 392 Figure 5.3, and the black vertices are minimum zero forcing sets for each of the graphs G and G/e .
 393 Then $Z_+(G) = 2$ and $Z_+(G/e) = k + 1$, so $Z_+(G/e) = Z_+(G) + (k - 1)$.

394 The *subdivision* of edge $e = \{u, v\}$ of G , denoted G_e , is the graph from G obtained by deleting
 395 e and inserting a new vertex w adjacent exactly to u and v . In the case of contraction, the result
 396 for positive semidefinite zero forcing was the same as for (standard) zero forcing. It was shown in
 397 [21] that $Z(G) \leq Z(G_e) \leq Z(G) + 1$, and each of the inequalities can be equality, but the case of
 398 positive semidefinite zero forcing is simpler.

399 THEOREM 5.24. *Let G be a graph and $e = \{u, v\} \in E(G)$. Then $Z_+(G_e) = Z_+(G)$ and any
 400 positive semidefinite zero forcing set for G is a positive semidefinite zero forcing set for G_e .*

401 *Proof.* In G_e , denote the vertex added to G in the subdivision by w . Let B be a positive
 402 semidefinite zero forcing set for G and \mathcal{F} a chronological list of forces. Without loss of generality,
 403 either $u \rightarrow v$ or neither forces the other in \mathcal{F} . In G_e , color the vertices in B black. If $u \rightarrow v$ in
 404 \mathcal{F} , replace this by $u \rightarrow w \rightarrow v$ and otherwise perform the same forces as in \mathcal{F} . If neither u nor v
 405 forces the other in \mathcal{F} , then $u \rightarrow w$ after all the forces in \mathcal{F} have been performed in G_e . In either
 406 case, if $u \rightarrow x \neq v$ when v is white, then x and v are in different components of $G - S$ (where S
 407 is the set of black vertices at this stage). Then x and w are in different components of $G_e - S$,
 408 and the forcing can continue as before. A similar argument holds for $v \rightarrow x \neq u$ when u is white.
 409 Thus B is a positive semidefinite zero forcing set for G_e . By choosing B so that $|B| = Z_+(G)$,
 410 $Z_+(G_e) \leq Z_+(G)$.

411 Now let B be a minimum positive semidefinite zero forcing set for G_e with $u \in B$. If $w \in B$,
 412 then set $B' = B \setminus \{w\} \cup \{v\}$; otherwise set $B' = B$. Then B' is a positive semidefinite zero forcing
 413 set for G . Since $|B'| = |B|$, $Z_+(G) \leq Z_+(G_e)$. \square

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416

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